P424 Assignment 1 Solutions

1) Natural units

(a) The neutral B mesons oscillate between particle and anti-particle states. The probability of a particle initially produced as a B^0 to be detected as a $\overline{B}^0$ is usually written as (using "natural" units)

$$P(B^0 \to \overline{B}^0) = \frac{e^{-t/\tau}}{2}\,(1 - \cos(\Delta m\, t))$$

and Δm is quoted to be $0.502\,\mathrm{ps}^{-1}$, i.e. is treated as a frequency. Express Δm in units of mass and give the numerical value in an appropriate system of units, like GeV/c^2.

(b) A pion has invariant mass $0.1395\,\mathrm{GeV}/c^2$. What is its Compton wavelength in Fermi (fm)?

(a) From $E = \hbar\omega$, 0.502 ps^{-1} = 0.502 x 10^{12} x 1.05 x 10^{-34} Joules =
0.502 x 10^{12} x 1.05 x 10^{-34} / (1.6 x 10^{-19}) eV/c^2 =
0.502 x 10^{12} x 1.05 x 10^{-34} x 10^{-9} / (1.6 x 10^{-19}) GeV/c^2 = 3.3 x 10^{-13} GeV/c^2.

(b) 1.395 x 10^8 eV/c^2 = 1.395 x 10^8 x 1.6 x 10-^{19} Joules =
(again from $E = \hbar\omega$) 1.395 x 10^8 x 1.6 x 10^{-19} / (1.05 x 10^{-34}) Hz =
1.395 x 10^8 x 1.6 x 10^{-19} / (3 x 10^8 x 1.05 x 10^{-34}) m^{-1} = k (wave number).
The Compton wavelength $\lambda_c = 2\pi/k \Rightarrow$
λ_c = 2π x 3 x 10^8 x 1.05 x 10^{-34} / (1.395 x 10^8 x 1.6 x 10^{-19}) m = 8.87 fm.

2) Scattering kinematics

A 2 GeV electron beam is incident on a thin Carbon target. Assuming the electrons scatter dominantly from individual protons, calculate

(a) the invariant mass of an e^--p pair in the CM frame

(b) the energy of the scattered (final state) electron as a function of its angle w.r.t. the electron beam for elastic collisions (i.e. when the proton does not get broken apart).

(c) Does your answer to (b) change if you scatter polarized electrons from a polarized target?

(a) The mass is invariant, so we can calculate it in the lab frame instead of the CM frame. We can neglect the mass of the electron (0.000511 GeV), so the total 4-vector is

$$\begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.938 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.938 \\ 2 \\ 0 \\ 0 \end{pmatrix},$$ so the invariant mass is $\sqrt{2.938^2 - 2^2} = 2.15$ GeV.

(b) We have, by conservation of 4-momentum, $$\begin{pmatrix} 2.938 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p_e \\ p_e \cos\phi \\ p_e \sin\phi \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{p_p^2 + 0.938^2} \\ p_p \cos\theta \\ p_p \sin\theta \\ 0 \end{pmatrix}.$$

As this is 3 equations with 4 unknown variables, it of course does not have a unique numerical solution, but we *can* solve for p_e as a *function* of ϕ, as the question asks for.

From energy conservation, $\sqrt{p_p^2 + 0.938^2} = 2.938 - p_e \quad \Rightarrow \quad p_p^2 = p_e^2 - 5.876 p_e + 7.752.$

Then, from *y*-momentum conservation, $p_e \sin\phi = p_p \sin\theta \;\; \Rightarrow$

$\theta = \sin^{-1}\left(\dfrac{p_e \sin\phi}{\sqrt{p_e^2 - 5.876 p_e + 7.752}} \right)$. Then, from x-momentum conservation,

$p_e \cos\phi = 2 - p_p \cos\theta \;\; \Rightarrow \;\; p_e = \dfrac{5.876 + \sqrt{4.511 - 45.024 \cos^2\phi}}{4 + 6\cos^2\phi}$ (after a fair amount of algebra, which I omit).

3) Decay kinematics

Consider the decay $K^+ \to \pi^+\pi^0\pi^0$, where $m_K^+ = 495$ MeV, $m_{\pi^+} = 139$ MeV and $m_{\pi^0} = 135$ MeV.

(a) What is the maximum momentum of the π^+ in the K^+ rest frame?

(b) Suppose the experimental apparatus cannot record decays where the invariant mass of the π^0s, $m_{\pi^0\pi^0}$, is less than 320 MeV. What are the allowed momenta for the π^+ under this restriction?

(c) What is the maximum allowed $m_{\pi^+\pi^0}$ under the conditions of part (b)?

(d) One often finds that random combinatorial background (combinations of particles that did not come from the decay of a single parent particle) populates the area near the boundary of a Dalitz plot. What characterstic do the decays that lie near the boundary of the plot share?

(a) The maximum possible π^+ momentum will be when the two π^0's share the same momentum and fly opposite the π^+. Thus we have

$$\begin{pmatrix} 0.495 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{0.139^2 + p^2} \\ p \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{(2*0.135)^2 + p^2} \\ -p \\ 0 \\ 0 \end{pmatrix} \Rightarrow p_{\max} = 134 \text{ MeV/c.}$$

(b) The maximum momentum would then be

$$\begin{pmatrix} 0.495 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{0.139^2 + p^2} \\ p \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{0.320^2 + p^2} \\ -p \\ 0 \\ 0 \end{pmatrix} \Rightarrow p_{\max} = 86 \text{ MeV/c.}$$

(The minimum momentum would be zero.)

(c) The π^0 momentum that will maximize $m_{\pi^+\pi^0}$ occurs when the chosen π^0 flies directly opposite the π^+ and the other π^0 is at rest. This is given by the negative value of p for which $0.320 = 0.135 + \sqrt{0.135^2 + p^2} \Rightarrow p = -126$ MeV/c. Thus we have

$$\begin{pmatrix} \sqrt{0.139^2 + 0.086^2} \\ 0.086 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{0.135^2 + 0.126^2} \\ -0.126 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.349 \\ -0.040 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{m^2_{\pi^+\pi^0} + 0.040^2} \\ -0.040 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$m_{\pi^+\pi^0} = 347$ MeV/c.

(d) Near the edge of the Dalitz plot, at least one of the particles in the decay is near the minimum momentum allowed by the reconstruction. Since background tracks typically occur more often at low momentum, such events usually tend to have more combinatoric background.

4) Interaction of particles in matter

(a) The amount of energy a 1.5 GeV particle loses when traversing material depends on the particle type. Assume the particles e, μ, π, K and p, each with 1.5 GeV momentum, traverse a medium of thickness $0.01X_0$. Which, on average, loses the most energy (order them from largest to smallest energy loss)?

(b) The particles from (a) enter a quartz bar with index of refraction $n = 1.5$. Which particles will emit Cherenkov radiation?

(c) Suppose you measure K^- and π^- in your detector. You measure the track momentum with high precision. In addition, you measure the amount of time required for the particle to travel the 1 m between the production vertex and a time-of-flight (TOF) detector. What time resolution σ_t must the TOF detector have in order to achieve a 3 standard deviation separation between π^- and K^- at 1 GeV momentum?

(d) Consider an electromagnetic shower initiated by a 1 GeV positron in Lead. If we model the shower process as a series of bremsstrahlung events and subsequent pair production from the photons, estimate (crudely) how many positrons we expect to produce. (Hint: the critical energy in Lead is about 1 MeV.)

(c) At 1 GeV momentum, the speed of the kaon is 0.896c and the speed of the pion is 0.990c. Thus the kaon will arrive at the TOF at approximately 3.72 ns and the pion will arrive at 3.37 ns, a time separation of approximately 350 ps. Thus the TOF must achieve an approximately 115 ps time resolution to achieve 3 sigma separation. (That is difficult but doable for plastic scintillator + phototubes + very good timing electronics.)