P424 Assignment 5 SOLUTIONS

1) Fun with Dirac delta functions

The Dirac delta function is a convenient way to enforce constraints. Work through the following simple example. The quantity x is distributed according to $f(x) = 1$ for $0 < x < 1$ and is elsewhere zero, and the quantity y is distributed according to $g(y) = 2y$ for $0 < y < 1$ and is elsewhere zero. How is the quantity $z = x + y$ distributed? You can determine this by solving the equation

$$h(z) = \int dx\, f(x) \int dy\, g(y)\, \delta\left(z - (x + y)\right)$$

where the integrations are over all values of x and y and the delta function neatly enforces the relation between x, y and z.

$$\begin{aligned} h(z) &= \int dx f(x) \int dy g(y) \delta(z - (x + y)) \\ &= \int dx f(x) g(z - x) \\ &= \int dx (1)[2(z - x)], \text{ if } 0 < x < 1 \;\;\&\;\; 0 < z - x < 1 \\ &\qquad\qquad\qquad\qquad \Rightarrow \max(0, z - 1) < x < \min(1, z). \end{aligned}$$

If $0 < z < 1$, then

$$h(z) = \int_0^z dx 2(z - x) = 2zx - z^2\Big|_0^z = z^2.$$

If $1 < z < 2$, then

$$h(z) = \int_{z-1}^1 dx 2(z - x) = 2zx - x^2\Big|_{z-1}^1 = 2z - z^2.$$

Thus, $h(z) = \begin{cases} z^2 & \text{for } 0 < z < 1 \\ 2z - z^2 & \text{for } 1 < z < 2 \\ 0 & \text{otherwise} \end{cases}.$

2) Consider the elastic scattering reaction $A + B \to A + B$ in the lab frame (B initially at rest) and assume that the initial energy E_1 of the incoming A particle satisfies $E_1 \ll m_B$ so that the recoil of the target can be neglected.

(a) Use the Golden Rule for scattering to show that the differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{(8\pi m_B)^2}$$

(b) Write down the lowest order diagram(s) for this scattering process in ABC theory

(c) Calculate the decay amplitude using the Feynman rules for ABC theory (express your result using the Mandelstam variables s, t and/or u as relevant).

(d) Combine the results from (a) and (c) to obtain the differential cross section (in the limit $E_1 \ll m_B$ and assuming that m_A and m_C are tiny compared to m_B).

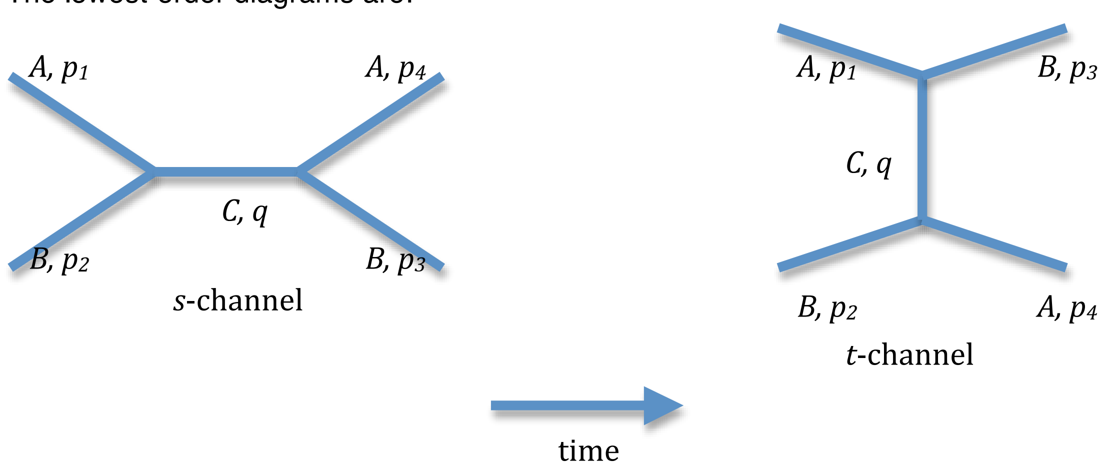
(e) Show that the total cross-section is

$$\sigma = \frac{g^4}{4\pi m_B^6}$$

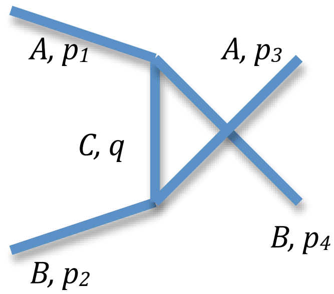
under the conditions stated in part (d).

(a) For 2 → 2 scattering we showed in class that $\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\frac{S|\mathcal{M}|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}$ in the c.m. frame. In the c.m. frame, since $m_B >> E_1$, the *B* particle is moving very slowly before and after the collision, so the above expression is valid in the lab frame too. *S* = 1, $E_1 << m_B \Rightarrow E_1 + E_2 \approx m_B$, and $|\vec{p}_f| \approx |\vec{p}_i| \quad \Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{(8\pi m_B)^2}$.

(b) The lowest-order diagrams are:



The "*u*-channel" diagram:



is the same as the *t*-channel (as the final state and initial state particles are both different) so we don't need to include it. (Some people used p_3 for the outgoing *A* and thus said it is the *u*-channel, not the *t*-channel – this is purely a matter of convention and doesn't affect the final result.)

(c) For the *s*-channel, the amplitude is found from

$$(-ig)^2 \int \frac{i}{q^2 - m_C^2} (2\pi)^8 \delta^4(p_1 + p_2 - q) \delta^4(q - p_3 - p_4) \frac{d^4 q}{(2\pi)^4} =$$

$\frac{-ig^2}{(p_1 + p_2)^2 - m_C^2} (2\pi)^4 \delta^4(q - (p_1 + p_2))$. Taking $s = (p_1 + p_2)^2$, and removing the extra delta function term, we get that the amplitude is $\frac{g^2}{s - m_C^2}$.

For the *t*-channel, we obtain the same result with the substitution $p_2 \to -p_3 \Rightarrow s \to t \Rightarrow$ the t-channel amplitude is $\frac{g^2}{t - m_C^2}$. Thus the total amplitude is $\frac{g^2}{s - m_C^2} + \frac{g^2}{t - m_C^2}$.

(d) In the limits $E_1 << m_B$ and $m_A, m_C << m_B$, we have $s = (p_1 + p_2)^2 = (E_1 + m_B)^2 - |\vec{p}_1|^2$ $= m_A^2 + m_B^2 + 2E_1 m_B \approx m_B^2$ and $s = (p_1 - p_3)^2 = (E_1 - m_B)^2 - |\vec{p}_1|^2 = m_A^2 + m_B^2 - 2E_1 m_B \approx m_B^2$, thus the total amplitude is approximately $\frac{2g^2 m_B^2}{m_B^4} = \frac{2g^2}{m_B^2}$. Thus $\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{(8\pi m_B)^2} = \frac{g^4}{16\pi^2 m_B^2}$.

(e) $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega \quad = \quad 4\pi \frac{g^4}{16\pi^2 m_B^6} \quad = \quad \frac{g^4}{4\pi m_B^6}$.