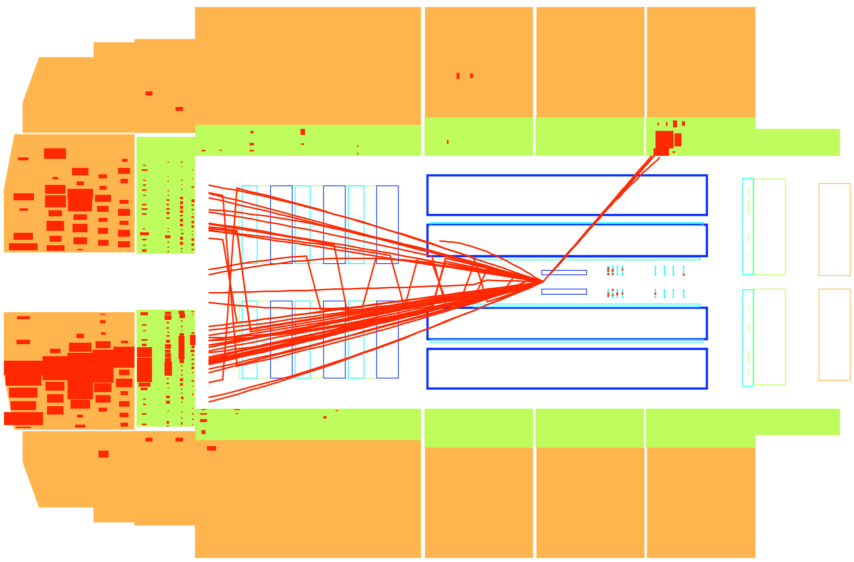
P424 Assignment 7 SOLUTIONS

1. The figure below shows a deep-inelastic scattering event $e^+p \to e^+X$ recorded by the H1 experiment at the HERA collider. The positron beam, of energy $E_1 = 27.5\,\mathrm{GeV}$, enters from the left and the proton beam, of energy $E_2 = 820\,\mathrm{GeV}$, enters from the right. The energy of the outgoing positron is measured to be $E_3 = 31\,\mathrm{GeV}$.



a) Show that the Bjorken scaling variable x is given by

$$x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right]$$

where θ is the angle through which the positron has scattered.

The variable x is defined as $x \equiv -\dfrac{q^2}{2q\cdot p}$, thus we have $q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1p_3 =$ $\cancel{2m_e^2} - 2E_1E_3(1-\cos\theta)$ and $q\cdot p = (p_1 - p_3)\cdot p_2 = E_1E_2 + E_1|\vec{p}_2| - E_2E_3 - |\vec{p}_2|E_3\cos\theta \approx$ $2E_1E_2 - E_2E_3(1+\cos\theta) = E_1E_2[2-(E_3/E_1)(1+\cos\theta)]$. Thus we have

$$x = \frac{\cancel{2}\cancel{E_1}E_3(1-\cos\theta)}{\cancel{2}\cancel{E_1}E_2[2-(E_3/E_1)(1+\cos\theta)]} = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right].$$

b) Estimate the values of Q^2, x and y for this event.

$Q^2 \equiv -q^2 = 2E_1E_3(1-\cos\theta)$. $\theta \approx 45° \quad \Rightarrow \quad Q^2 \approx 2(27.5)(31)(0.2929) \approx 500\ \text{GeV}^2$.

$$x = \frac{E_3}{E_2}\left[\frac{1-\cos\theta}{2-(E_3/E_1)(1+\cos\theta)}\right] \approx \frac{31}{820}\left[\frac{1-\sqrt{2}/2}{2-(31/27.5)(1+\sqrt{2}/2)}\right] = 0.0378(0.2929/0.0756) = 0.146.$$

$$y \equiv \frac{p_2\cdot q}{p_2\cdot p_1} = \frac{E_1E_2[2-(E_3/E_1)(1+\cos\theta)]}{2E_1E_2} = 1 - \frac{E_3}{2E_1}(1+\cos\theta) \approx 0.038.$$

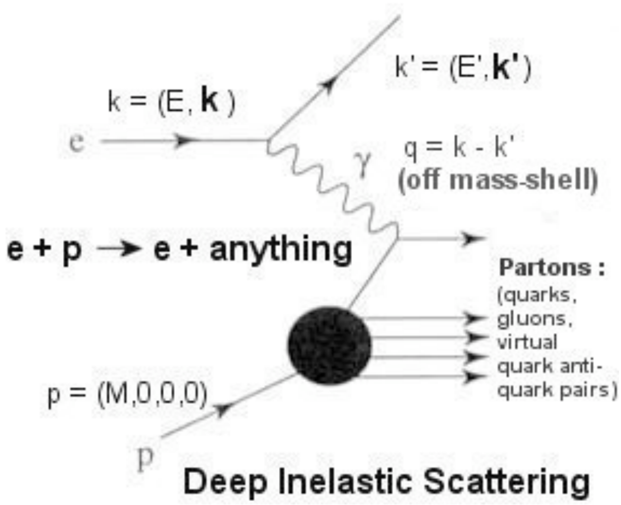
c) Estimate the invariant mass M_X of the final state hadronic system.

$$M_X = \sqrt{(p_2+p_1-p_3)^2} \approx \sqrt{-2E_1E_3(1-\cos\theta) - 2E_2E_3(1+\cos\theta) + 4E_1E_2} =$$
$$\sqrt{-500 - 86789 + 90200}\ \text{GeV} \approx 54\ \text{GeV}.$$

d) Draw quark level diagrams to illustrate the possible origins of this event. Using the plot overleaf of the parton distribution functions $xu_V(x)$, $xd_V(x)$, $x\overline{u}(x)$ and $x\overline{d}(x)$, estimate the relative probabilities of the various possible quark-level processes for the event.

[Neglect contributions from the heavier quarks s, c, b, t.]

The event originates in the deep inelastic scattering class of diagrams, i.e.

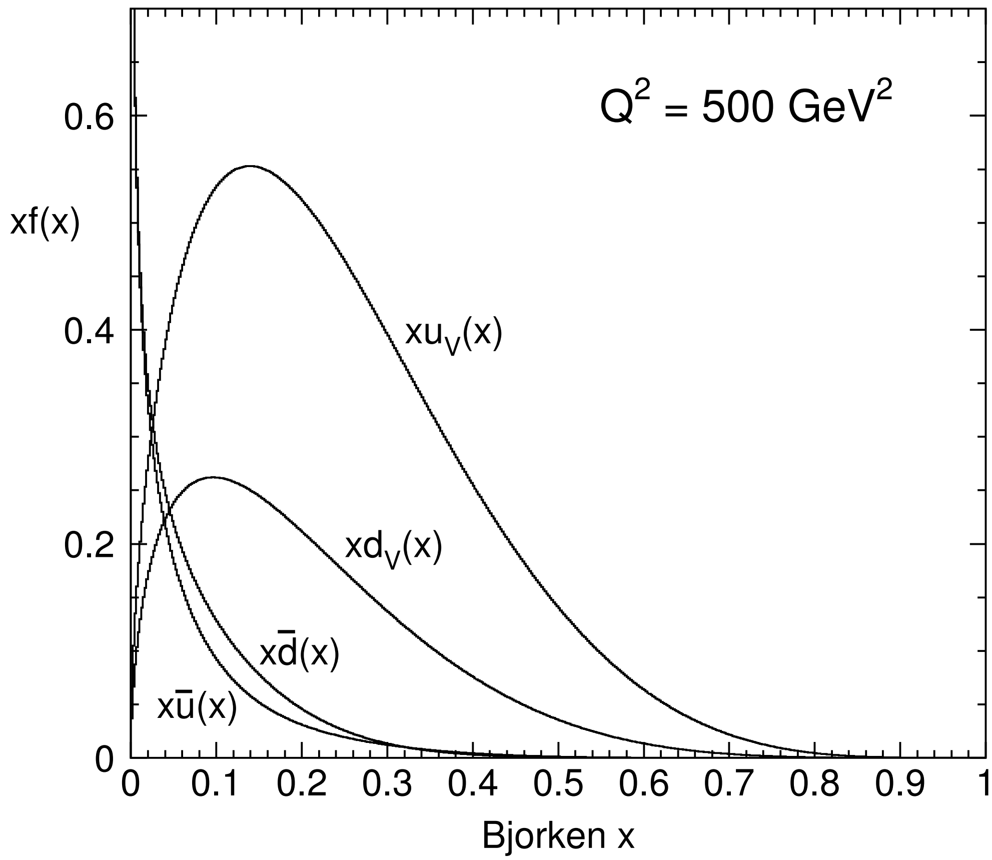


Since x ~= 0.15, we have the relative probabilities

$$\left.\begin{aligned} xu_V(x) &\approx 0.55 \quad \Rightarrow \quad u_V(x) \approx 3.67 \\ xd_V(x) &\approx 0.25 \quad \Rightarrow \quad d_V(x) \approx 1.67 \\ x\overline{u}(x) &\approx 0.05 \quad \Rightarrow \quad \overline{u}(x) \approx 0.33 \\ x\overline{d}(x) &\approx 0.07 \quad \Rightarrow \quad \overline{d}(x) \approx 0.47 \end{aligned}\right\} \Sigma = 6.14$$

so the actual probabilities are approximately

$u_V \approx 3.67/6.14 = 0.60$; $\quad d_V \approx 1.67/6.14 = 0.27$; $\quad \overline{u} \approx 0.33/6.14 = 0.05$; $\quad \overline{d} \approx 0.47/6.14 = 0.08$.



2. Pseudo-scalar meson leptonic decay.

a) Calculate the decay rate for $K^+ \to \mu^+ \nu_\mu$. We now know that neutrinos have mass, so use m_ν as the mass of the neutrino (i.e. don't set it to zero).

This proceeds very similarly to the calculation of the pion decay rate in the lecture notes (electroweak #1), however we must be sure not to make the assumption that the neutrino is massless. The first time that assumption is made is in the second line of p. 30, i.e. this should be changed to

$$\left\langle |\mathcal{M}|^2 \right\rangle = \left(\frac{g_w^2 f_K}{8 M_W^2}\right)^2 p_{1\mu} p_{1\nu} Tr\left[\gamma^\mu (1-\gamma^5)(\not{p}_2 + m_\nu)\gamma^\mu (1-\gamma^5)(\not{p}_3 + m_\ell)\right]$$

$$= \left(\frac{g_w^2 f_K}{8 M_W^2}\right)^2 p_{1\mu} p_{1\nu} Tr\left[\gamma^\mu (1-\gamma^5)^2 (\not{p}_2 + m_\nu)\gamma^\mu (\not{p}_3 + m_\ell)\right]$$

$$= \left(\frac{g_w^2 f_K}{8 M_W^2}\right)^2 p_{1\mu} p_{1\nu} 8\left[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - (p_2 \cdot p_3) g^{\mu\nu} + m_\nu m_\ell g^{\mu\nu}\right]$$

$= \frac{1}{8}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 \left[2(p_1 \cdot p_2)(p_1 \cdot p_3) + m_K^2(-p_2 \cdot p_3 + m_\nu m_\ell)\right]$. We have the dot products:

$p_1 = p_2 + p_3 \Rightarrow p_1^2 = p_2^2 + p_3^2 + 2(p_2 \cdot p_3) \Rightarrow m_K^2 = m_\ell^2 + m_\nu^2 + 2(p_2 \cdot p_3) \Rightarrow (p_2 \cdot p_3) = \tfrac{1}{2}(m_K^2 - m_\ell^2 - m_\nu^2)$

Similarly, $(p_1 \cdot p_2) = \tfrac{1}{2}(m_K^2 - m_\ell^2 + m_\nu^2)$ and $(p_1 \cdot p_3) = \tfrac{1}{2}(m_K^2 + m_\ell^2 - m_\nu^2)$. Thus, we have, after inserting the dot products in the above (and a few lines of algebra),

$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{16}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 \left[m_K^2(m_\ell^2 + m_\nu^2) - (m_\ell^2 - m_\nu^2)^2\right]$. To get the decay rate, as usual we use Fermi's Golden Rule: $\Gamma = \frac{|\vec{p}_f| |\mathcal{M}|^2}{8\pi m_K^2}$. For a decay at rest we have $p_1 = (m_K, 0)$ and $p_3 = (E_\ell, \vec{p}_f)$, thus $p_1 \cdot p_3 = E_\ell m_K \quad \Rightarrow \quad E_\ell = \frac{p_1 \cdot p_3}{m_K} =$ (from above) $\frac{m_K^2 + m_\ell^2 - m_\nu^2}{2m_K}$. We have $|\vec{p}_f|^2 = E_\ell^2 - m_\ell^2 = \frac{(m_K^2 + m_\ell^2 - m_\nu^2)}{4m_K^2} - m_\ell^2$, thus

$|\vec{p}_f| = \frac{\sqrt{m_K^4 + (m_\ell^2 - m_\nu^2)^2 - 2m_K^2(m_\ell^2 + m_\nu^2)}}{2m_K}$. Thus

$$\Gamma = \frac{\sqrt{m_K^4 + (m_\ell^2 - m_\nu^2)^2 - 2m_K^2(m_\ell^2 + m_\nu^2)}}{256\pi m_K^3}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 \left[m_K^2(m_\ell^2 + m_\nu^2) - (m_\ell^2 - m_\nu^2)^2\right].$$

b) Repeat the above calculation with a purely vector current for the charged weak interaction (i.e. replace $\gamma^\mu(1-\gamma^5)/\sqrt{2}$ by γ^μ in the Feynman rules).

With a purely vector current, we have $\mathcal{M} = \frac{g_W^2}{4M_W^2}\left[\bar{u}(3)\gamma_\mu v(2)\right] f_K p^\mu \quad \Rightarrow$

$\left\langle |\mathcal{M}|^2 \right\rangle = \left[\frac{g_W^2 f_K}{4M_W^2}\right]^2 p_\mu p_\nu Tr\left[\gamma^\mu(\not{p}_2 + m_\nu)\gamma^\nu(\not{p}_3 + m_\ell)\right] =$

$\left[\frac{g_W^2 f_K}{4M_W^2}\right]^2 p_\mu p_\nu \left\{Tr[\gamma^\mu \not{p}_2 \gamma^\nu \not{p}_3] + Tr[\gamma^\mu \not{p}_2 \gamma^\nu m_\ell] + Tr[\gamma^\mu m_\nu \gamma^\nu \not{p}_3] + Tr[\gamma^\mu \gamma^\nu m_\nu m_\ell]\right\}$. The middle two terms contain an odd number of gamma matrices and thus are zero.

Thus we have $\left\langle |\mathcal{M}|^2 \right\rangle = \left[\frac{g_W^2 f_K}{4M_W^2}\right]^2 \left[4 p_{1\mu} p_{1\nu} p_{2\lambda} p_{3\sigma}(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\lambda}g^{\nu\sigma}) + 4m_\nu m_\ell g^{\mu\nu}\right]$

$= \frac{1}{4}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 \left[2(p_1 \cdot p_2)(p_1 \cdot p_3) + m_K^2(-p_2 \cdot p_3 + m_\nu m_\ell)\right]$, which is twice the spin-averaged square amplitude from part (a) above. Thus, analogously to part (a) above, $\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{8}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 \left[m_K^2(m_\ell^2 + m_\nu^2) - (m_\ell^2 - m_\nu^2)^2\right]$ and

$$\Gamma = \frac{\sqrt{m_K^4 + (m_\ell^2 - m_\nu^2)^2 - 2m_K^2(m_\ell^2 + m_\nu^2)}}{128\pi m_K^3}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 \left[m_K^2(m_\ell^2 + m_\nu^2) - (m_\ell^2 - m_\nu^2)^2\right].$$

(c) Compare your answers to parts (a) and (b) in the limit $m_\nu \to 0$. Try to give a physical meaning to this.

In the limit $m_\nu \to 0$, we have $\Gamma_{(a)} = \frac{1}{256\pi m_K^3}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 m_\ell^2(m_K^2 - m_\ell^2)^2$ and $\Gamma_{(b)} = \frac{1}{128\pi m_K^3}\left(\frac{g_w^2 f_K}{M_W^2}\right)^2 m_\ell^2(m_K^2 - m_\ell^2)^2$, so (as is in fact the case for any m_ν), $\Gamma_{(b)} = 2\Gamma_{(a)}$. This factor of 2 is just because the weak interaction just couples left-hand to left-hand particles, whereas the electromagnetic interaction couples both left-hand to left-hand and right-hand to right-hand particles.