Kinematics

- Lorentz Transformations
- Four-Vectors
- Energy, Momentum, and Mass
- Collisions
- Examples

Lorentz Transformations

- Relate coordinates in: $S \xrightarrow{v} S'$
- Derived from the postulates of relativity
- For motion along the x-axis:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

- Length Contraction: moving objects are shortened: $L = L'/\gamma$
- Time Dilation: moving clocks run slow: $T = \gamma T'$

Application: Cosmic Ray Muons

- With $\tau_\mu = 2.2\ \mu s$, a muon produced in the upper atmosphere could nominally travel (at $v \sim c$) 660 m before decaying
- The muon lifetime is enhanced through *time dilation* by a factor of γ. Supposing $\gamma \sim 10$, this allows a typical muon to travel 6.6 km before decaying.
- Recall $\gamma = E/m$ and $\beta = \sqrt{1 - 1/\gamma^2}$
- Decay length in laboratory frame is $\gamma c \tau$

Four-Vectors and Tensors

- Write $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ so that

$$(x')^\mu = \Lambda^\mu_\nu x^\nu \qquad \text{with} \qquad \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- A *four-vector* is a four-component object which behaves like x^μ under Lorentz transformations.
- x^μ is the contravariant and x_μ is the covariant four-vector
- The invariant $I = x^\mu x_\mu$

Relativistic Invariant

- It can be shown that

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$$

for any two frames related by a Lorentz transformation.

- If we define the *metric* tensor $g_{\mu\nu} = diag(1, -1, -1, -1)$, we have $x_\mu = g_{\mu\nu}x^\nu$ and

$$x^2 = x \cdot x = x_\mu x^\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

is a *Lorentz scalar*.

Energy, Momentum, and Mass

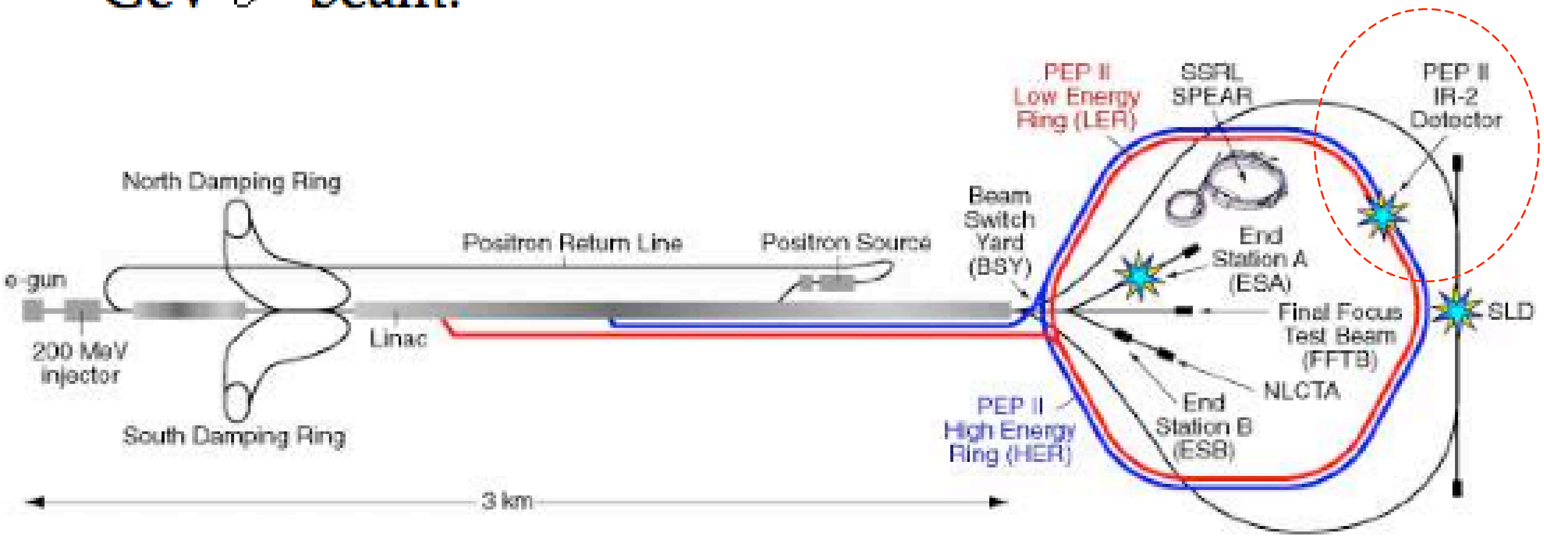
- Relativistic energy is $E = \gamma m$
- Relativistic momentum is $\mathbf{p} = \gamma m \mathbf{v}$
- Define the *four-momentum* by $p^\mu = (E, \mathbf{p})$.
  Then, $p^2 = m^2$ is a relativistic invariant.
- For massless particles, $E = |\mathbf{p}| = h\nu$
- Classically, we always conserve 3-momentum ($\mathbf{p}$), always conserve mass, sometimes conserve kinetic energy, and always conserve total energy even if we don't keep track of it all.
- Relativistically, we always conserve 3-momentum ($\mathbf{p}$), sometimes conserve mass, sometimes conserve kinetic energy, and always conserve total energy. More succinctly, four-momentum is conserved.

Conserved vs. Invariant quantities

- A conserved quantity remains the same, *in a particular frame*, before and after an event.
- An invariant quantity is the same *in all inertial reference frames*.
  - Energy is conserved, but not invariant.
  - Mass is invariant, but not conserved.

Examples

- BaBar experiment : Here, a 9 GeV e^- beam collides with a 3.1 GeV e^+ beam.



- What are the speeds of the colliding particles?
- What are the energies of the particles in the center of momentum (CM) frame?

Speeds

- Use $E = \gamma m$ and $m = 0.511$ MeV to determine that $\gamma_- = 17600$ for the electrons and $\gamma_+ = 6070$ for the positrons.
- Then, with $\gamma = 1/\sqrt{1-\beta^2}$, we solve for $\beta = v/c$

$$\Rightarrow \beta = \sqrt{1 - \frac{1}{\gamma^2}} \simeq 1 - \frac{1}{2\gamma^2}$$

- Using γ_- and γ_+, we have

$$\begin{aligned} v_{e^-} &= (1 - 10^{-9})c \\ v_{e^+} &= (1 - 10^{-8})c \end{aligned}$$

- In the CM frame, $p_{e^-} = (E_{CM}, \mathbf{p}_{CM})$ and $p_{e^+} = (E_{CM}, -\mathbf{p}_{CM})$ so that the (invariant) square of the total four-momentum is:

$$
\begin{aligned}
(p_{e^-} + p_{e^+})^2 &= (2E_{CM}, \mathbf{0})^2 \\
&= 4E_{CM}^2
\end{aligned}
$$

- In the lab frame, $p_{e^-} = (E_-, \mathbf{p}_-)$ and $p_{e^+} = (E_+, \mathbf{p}_+)$ so that

$$
\begin{aligned}
(p_{e^-} + p_{e^-})^2 &= p_{e^-}^2 + p_{e^+}^2 + 2p_{e^-} \cdot p_{e^+} \\
&= m^2 + m^2 + 2(E_-E_+ - \mathbf{p}_- \cdot \mathbf{p}_+) \\
&\simeq 2(E_-E_+ + |\mathbf{p}_-||\mathbf{p}_+|) \\
&\simeq 4E_-E_+
\end{aligned}
$$

- Equating the CM and lab expressions for the invariant, we have

$$E_{CM} = \sqrt{E_-E_+} = \sqrt{(9\ \mathrm{GeV})(3.1\ \mathrm{GeV})} = 5.3\ \mathrm{GeV}$$

Fixed Targets vs. Colliding Beams

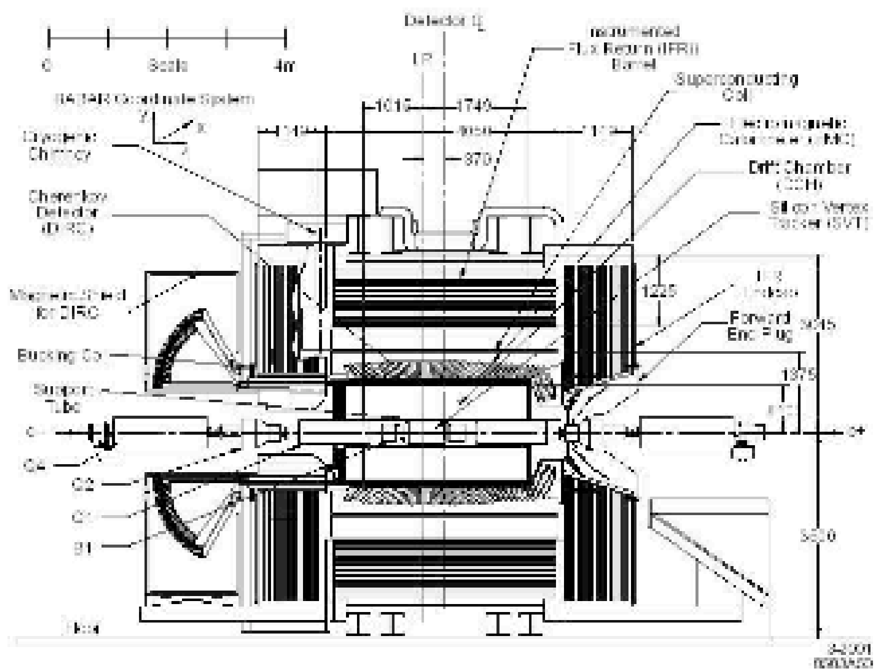
- In BaBar, $(9 + 3.1) = 12.1$ GeV of beam energy leads to $(2 \times 5.3) = 10.6$ GeV of CM energy that can be used to make new particles (in this case, the $\Upsilon(4S)$).
- How much beam energy would it take to produce this CM energy if the target were fixed?
- Use the total four-momentum as an invariant. The individual four-momenta will be $(m, 0)$ and $(E, \mathbf{p})$, and therefore

$$\begin{aligned} (p_{e^-} + p_{e^-})^2 &= m^2 + m^2 + 2\left[(m, 0) \cdot (E, \mathbf{p})\right] \\ &\simeq 2Em \end{aligned}$$

With $2Em = 4E_{CM}^2$ we find that $E = 10^5$ GeV!

Fixed Targets vs. Colliding Beams III

- Why is the energy of the electrons and positrons at SLAC different?
- What does this mean for design of the BaBar detector?



Two-Body Decays

- Consider the decay $\pi \to \mu + \nu$
- In the CM frame, the final-state energies are unique, since the two particles must emerge back to back (to conserve momentum).
- How can we calculate these energies? Use an invariant, of course.

- Four-vectors: $p_\pi = (m_\pi, \mathbf{0})$, $p_\mu = (E_\mu, \mathbf{p})$, and $p_\nu = (E_\nu, -\mathbf{p})$.
- Conservation of four-momentum: $p_\pi = p_\mu + p_\nu$
- $p_\mu = p_\pi - p_\nu$ leads to the invariant

$$
\begin{aligned}
p_\mu^2 &= (p_\pi - p_\nu)^2 \\
m_\mu^2 &= p_\pi^2 + p_\nu^2 - 2p_\pi \cdot p_\nu \\
m_\mu^2 &= m_\pi^2 - 2m_\pi E_\nu \\
\Rightarrow E_\nu &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi}
\end{aligned}
$$

- Similarly, $p_\nu = p_\pi - p_\mu$ leads to the invariant

$$
\begin{aligned}
p_\nu^2 &= (p_\pi - p_\mu)^2 \\
0 &= p_\pi^2 + p_\mu^2 - 2p_\pi \cdot p_\mu \\
0 &= m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu \\
\Rightarrow E_\mu &= \frac{m_\pi^2 + m_\mu^2}{2m_\pi}
\end{aligned}
$$

- Notice that

$$
\begin{aligned}
E_\nu + E_\mu &= \frac{m_\pi^2 - m_\mu^2}{2m_\pi} + \frac{m_\pi^2 + m_\mu^2}{2m_\pi} \\
&= m_\pi
\end{aligned}
$$

as we require for energy conservation.

Three-Body Decays

- Consider decays such as $n \to p + e + \overline{\nu}_e$ and $\mu^- \to e^- + \overline{\nu}_e + \nu_\mu$
- In the CM frame, the final-state energies are *not* unique.
- The observation that there was a range of electron energies in the two decays above played a large role in the postulate of the existence of neutrinos.

Mandelstam Invariants

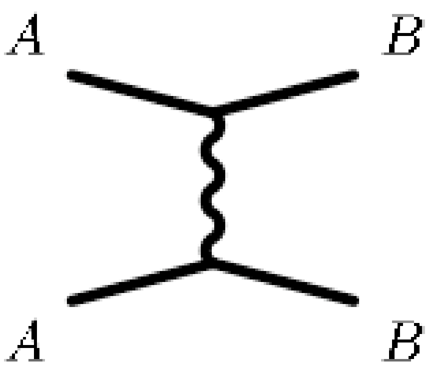
- For a scattering process like $A + B \longrightarrow C + D$, define the Mandelstam invariants by red

$$
\begin{aligned}
s &= (p_A + p_B)^2 \\
t &= (p_A - p_C)^2 \\
u &= (p_A - p_D)^2
\end{aligned}
$$

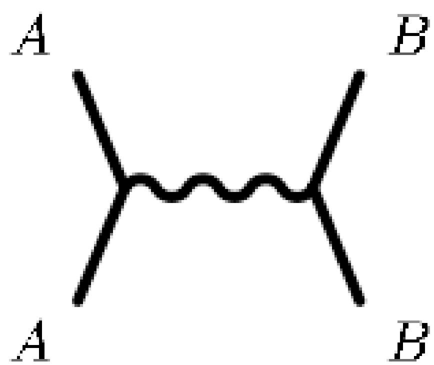
- We typically define a scattering angle θ in terms of the direction of C with respect to A.

Channels

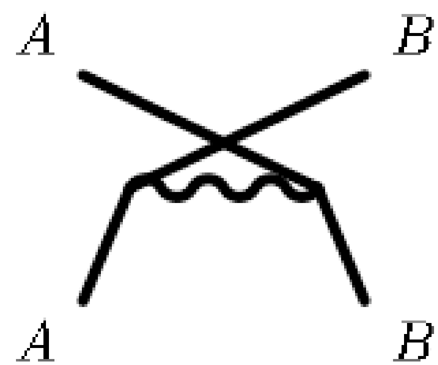
- For $A + B \longrightarrow A + B$ scattering in some unspecified theory, the Mandelstam invariants s, t, and u are related to 3 distinct topological *channels* with which Feynman diagrams might be drawn to represent the interaction:



s-channel



t-channel



u-channel

Summary

- Special Relativity is an essential foundation of particle physics.
- The CM frame vs laboratory frame
- 2-body decays are much simpler than 3-body decays.
- Whenever possible, work with invariants formed from the contraction of four-vectors.
- The Mandelstam invariants are so common and useful that we give them their own symbols: s, t, and u.