Symmetries

- Conservation Laws
- Basic Group Theory
- Angular Momentum
- Flavor Symmetries

What is a Symmetry?

- Let's translate the wavefunction $\psi(x)$ by an amount a:

$$\psi(x) \longrightarrow \psi(x+a)$$

- Now expand $\psi(x+a)$ as a Taylor series about $\psi(x)$

$$\begin{aligned} \psi(x+a) &= \psi(x) + a\,\frac{\partial \psi}{\partial x}\bigg|_x + \frac{a^2}{2!}\,\frac{\partial^2 \psi}{\partial x^2}\bigg|_x + \ldots \\ &= \left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\,\frac{\partial^n}{\partial x^n}\right)\psi(x) \\ &= U(a)\psi(x) \qquad \text{where} \qquad U(a) = \exp\left[a\,\frac{\partial}{\partial x}\right] \end{aligned}$$

Why $U(a)$ is Unitary

- If our physical system is indeed invariant under translations, then

$$\begin{aligned}\langle\psi(x)|\psi(x)\rangle &= \langle\psi(x+a)|\psi(x+a)\rangle \\ &= \langle U(a)\psi(x)|U(a)\psi(x)\rangle \\ &= \left\langle\psi(x)|U^\dagger(a)U(a)\psi(x)\right\rangle\end{aligned}$$

- Clearly then, $U^\dagger U = 1$, or in other words $U^\dagger = U^{-1}$.

Connection to Hermitian Operators

- Recall that in Quantum Mechanics, every physical observable is represented by a Hermitian operator ($H^\dagger = H$).
- Every Hermitian operator is a *generator* of a unitary operator via $U = e^{iH}$.
- Factoring out a couple of constants so that $U(a) = \exp\left[iHa/\hbar\right]$ and comparing to our previous result of $U(a) = \exp\left[a\,\partial/\partial x\right]$, we find that

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

is a Hermitian operator which generates spatial translations.

Noether's Theorem

- Every symmetry is associated with a conservation law.

Symmetry	Conservation Law
Space translation	Linear momentum
Time translation	Energy
Rotation	Angular momentum
Gauge transformation	Electric charge

Approximate Symmetries

- Even approximate symmetries are useful to us, so long as we don't demand 100% accuracy.
- We'll soon be looking at an example of this, relating to the nearly identical masses of the proton and neutron.

Unknown Symmetries

- Sometimes we have conservation laws which do not correspond to a previously known symmetry.
- Perhaps this can inspire a theorist to come up with a model which includes this symmetry...
- Or perhaps there is no such symmetry (i.e., it is an approximate symmetry), and the conservation law will fail in the future with a more detailed experiment. For example, lepton and baryon number conservation are not associated with fundamental symmetries.

Basic Group Theory

- Group Theory is the mathematical description of symmetries, namely operations that leave a system invariant
- A group G is a set of elements with a binary composition law (i.e., a "multiplication") such that:
  1. Closure: $\forall\ a, b \in G : ab = c \in G$
  2. Identity: $\exists\ e \in G \mid \forall\ a \in G : ae = ea = a$
  3. Inverse: $\forall\ a \in G\ \exists\ a^{-1} \in G \mid aa^{-1} = a^{-1}a = e$
  4. Associativity: $\forall\ a, b, c \in G : (ab)c = a(bc)$
- G is an Abelian group if the group multiplication is commutative, i.e. $ab = ba\ \forall\ a, b \in G$. Otherwise, the group is non-Abelian.

Example - equilateral triangles

- The symmetry operations that leave an equilateral triangle invariant are
  - Rotation by $\pm\pi/3$ in the plane of the triangle
  - Rotation by π around any axis that bisects a vertex of the triangle
- Other symmetry operations can be built from combinations of these 5

Group Representations

- A representation of a group is a mapping of the group elements to a set of matrices, with matrix multiplication providing the appropriate composition law.
- The groups we see in particle physics are Lie Groups, in which the elements are continuously connected.
- In particular, we will be using $SO(N)$, $U(N)$, and $SU(N)$.
  $S \Rightarrow$ Special $\Rightarrow$ determinant 1
  $O \Rightarrow$ Orthogonal $\Rightarrow M^T M = 1$
  $U \Rightarrow$ Unitary $\Rightarrow M^\dagger M = 1$
  so, e.g., $SU(2)$ can be represented by a set of 2×2 unitary complex matrices of unit determinant.

Groups in The Standard Model

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$$

- The three groups in the expression above basically correspond to the three forces in the Standard Model; $SU(3)_C$ for the strong (color) force, $SU(2)_L$ for the weak force, and $U(1)$ for the electromagnetic force.
- Grand Unified Theories (GUTs) try to stuff these three groups into a larger group such as $SU(5)$ or $SO(10)$.
- String theory deals with even larger groups such as $SO(32)$ or $E_8 \times E_8$.

Angular Momentum

- In Quantum Mechanics, we cannot know everything about the angular momentum **J** of a particle at a given time.
- The best we can do is the simultaneous knowledge of J^2 and J_z, with eigenvalues:

$$\begin{aligned} J^2\psi &= \left[j(j+1)\hbar^2\right]\psi \\ J_z\psi &= (m_j\hbar)\,\psi \end{aligned}$$

- This formalism applies equally well to orbital angular momentum (**L**) as to intrinsic angular momentum, i.e. spin (**S**).

Spin-1/2

- Many of the particles we encounter have spin-1/2 (e.g. leptons, quarks, many baryons).
- States are then described as a 2-component spinor:

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\chi_+ + \beta\chi_-$$

- Define $\mathbf{S} = (\hbar/2)\sigma$, where the *Pauli matrices*, σ, are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that

$$\begin{aligned} \sigma_i\sigma_j &= \delta_{ij} + i\epsilon_{ijk}\sigma_k \\ [\sigma_i, \sigma_j] &= 2i\epsilon_{ijk}\sigma_k \end{aligned}$$

Fermions vs. Bosons

- J_z eigenvalues must be spaced in multiples of $\hbar$. Since $\left|m_j^{MAX}\right| = \left|m_j^{MIN}\right|$, this means that $j = m_j^{MAX}$ is either an integer or a half-integer.
- Particles with integer spin are bosons and obey Bose-Einstein statistics (i.e., symmetric w.r.t. exchange of identical particles).
- Particles with half-integer spin are fermions and obey Fermi-Dirac statistics (i.e., antisymmetric w.r.t. exchange of identical particles).
  The Pauli Exclusion Principle (i.e., 2 identical fermions can never share the same state) follows directly from this.

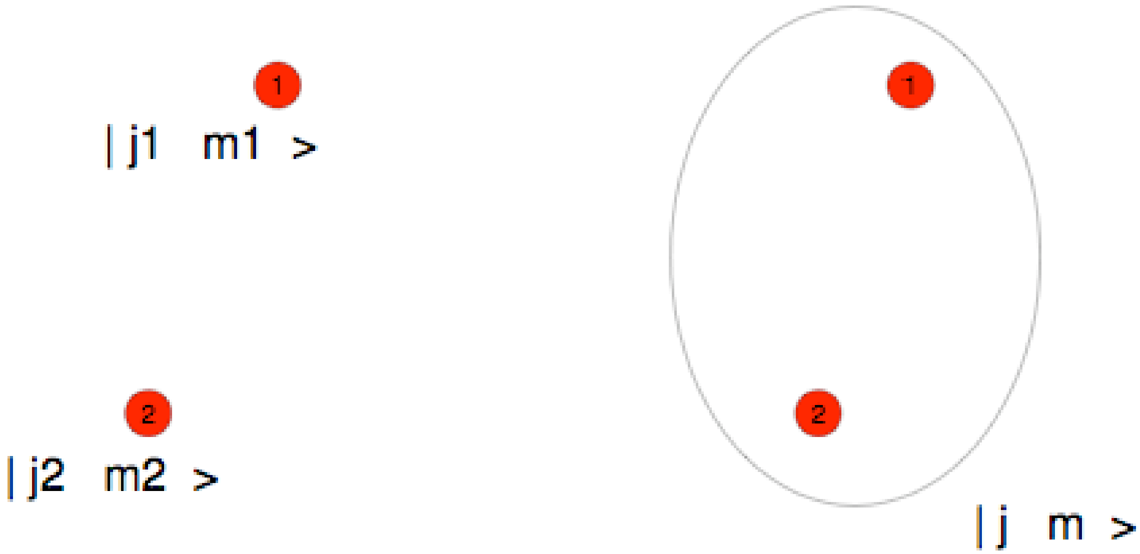
The Spin-Statistics Theorem

- Ribbon demo: Interchanging two particles is equivalent to a 2π relative rotation.
- The unitary transformation which effects rotations is

$$U(\theta) = \exp\left[\frac{i}{\hbar}\mathbf{J}\cdot\theta\right]$$

- For particles with integer spin,
  $U(2\pi) = \exp(2n\pi i) = 1$
  $\Rightarrow$ bosons.
- For particles with half-integer spin,
  $U(2\pi) = \exp[2(n+1/2)\pi i] = -1$
  $\Rightarrow$ fermions.

Two particle systems I



Conservation of angular momentum requires:

$$\mathbf{J} = \mathbf{J_1} + \mathbf{J_2}$$

$$m = m_1 + m_2$$

We must also consider the relative angular momentum of the two particles with respect to each other.

Two particle systems II

However, one must also consider the relative angular momentum in addition to the intrinsic angular momentum.

Generally intrinisc angular momentum ($\mathbf{S}$) is

$$\mathbf{S} = \mathbf{J_1} + \mathbf{J_2}$$

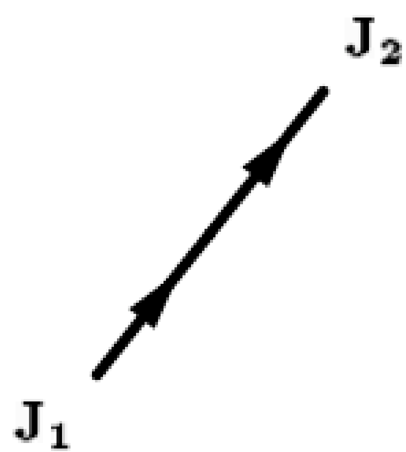
and the total angular momentum ($\mathbf{J}$) is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

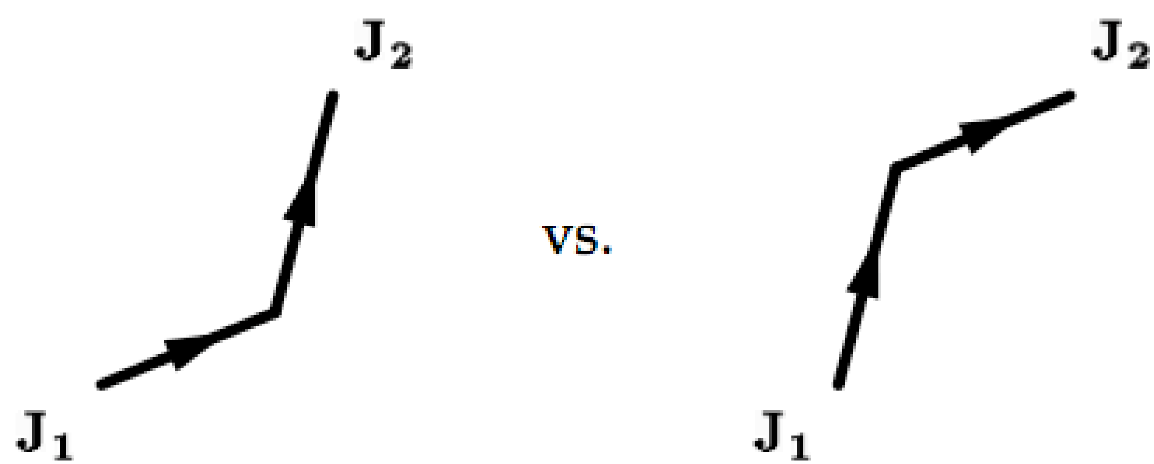
Angular Momentum Addition

- Consider $\mathbf{J} = \mathbf{J_1} + \mathbf{J_2}$.
- Since we do not know every component of $\mathbf{J_1}$ and $\mathbf{J_2}$, we cannot even fix the magnitude of $\mathbf{J}$. All we can say is that $m = m_1 + m_2$ and $|j_1 - j_2| \leq j \leq j_1 + j_2$

vs.



- Or, we could work in the other direction, whereby it is $\mathbf{J}$ that is known.
- Given j_1 and j_2, we wish to determine m_1 and m_2 subject to the constraints of $m_1 + m_2 = m$, $|m_1| \leq j_1$, and $|m_2| \leq j_2$.



Clebsch-Gordan Coefficients

- CG's represent the quantum mechanical overlap between the two different descriptions of a coupled system:

$$C^{\,j\ \ j_1\ \ j_2}_{\,m\ m_1\ m_2} = \langle\, j\ m \mid j_1\ m_1\ ,\ j_2\ m_2\,\rangle$$

- For specific choices of the eigenvalues, CG's can be slowly derived by using the raising and lowering operators of the angular momentum algebra
- The general formula is known and is useful for inclusion in a computer program...

CG General Formula

$$
\begin{aligned}
C^{j\ j_1\ j_2}_{m\ m_1\ m_2} \quad = \quad & \delta_{m_1+m_2,m}\sqrt{(j+m)!(j-m)!(2j+1)} \\
& \times \sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!} \\
& \times \sqrt{\frac{(j+j_1-j_2)!(j-j_1+j_2)!(j_1+j_2-j)!}{(j+j_1+j_2+1)!}} \\
& \times \sum_n \frac{(-1)^n}{n!} \frac{1}{(j_1+j_2-j-n)!(j_1-m_1-n)!} \\
& \times \frac{1}{(j_2+m_2-n)!(j-j_2+m_1+n)!(j-j_1-m_2+n)!}
\end{aligned}
$$

CG Table (from PDG)

http://pdg.lbl.gov/2005/reviews/clebrpp.pdf

$1 \times 1/2$

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

PDG web site

The "Particle Data Group" exists as a data repository for particle physics, and has summaries of all experiments and measurements. It also contains useful tables and mini-reviews. Check it out:
http://pdg.lbl.gov/
http://pdg.lbl.gov/2005/reviews/contents_sports.html

How to Read a CG Table

- First, find the appropriate table for the j_1 and j_2 of interest.
- If it's m_1 and m_2 that are known and we want to find j, read the appropriate row across. Don't forget that all entries have an implicit square root. For example:

$$|\,1\;0\,\rangle\;|\,1/2\;\;1/2\,\rangle = \sqrt{2/3}\;|\,3/2\;\;1/2\,\rangle - \sqrt{1/3}\;|\,1/2\;\;1/2\,\rangle$$

- If it's j that is known and we want to find m_1 and m_2, read the appropriate column down. For example:

$$\begin{aligned} |\,3/2\;\;1/2\,\rangle &= \sqrt{1/3}\;|\,1\;1\,\rangle\;|\,1/2\;\;-1/2\rangle \\ &\quad +\sqrt{2/3}\;|\,1\;0\,\rangle\;|\,1/2\;\;1/2\,\rangle \end{aligned}$$

Angular momentum of a $q\overline{q}$ state

- $q = \left|\frac{1}{2}\frac{1}{2}\right\rangle$ and $\overline{q} = \left|\frac{1}{2} - \frac{1}{2}\right\rangle$
- We can form 2 spin states $|10\rangle$ and $|00\rangle$

  (We are ignoring the angular momentum between the two quarks)
- For a $u\overline{u}$ system, this corresponds to the π^0 meson $(J = 0)$ and the ρ^0 meson $(J = 1)$

Direct Products and Direct Sums

- A spin-1 particle has 3 possible spin orientations ($m = -1, 0, 1$) and a spin-1/2 particle has 2 ($m = -1/2, +1/2$).
- When we couple these two particles together, we can get either spin-1/2 or spin-3/2 (with 2 or 4 possible spin orientations, respectively).
- We represent this information via the statement:

$$\mathbf{3} \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{4}$$

Isospin

- neutron M = 939 MeV
- proton M = 938 MeV
- n-n, n-p and p-p interactions are identical if EM interaction ignored
- Consider the (n,p) as components of the Nucleon (N) with I-spin 1/2

$$p = |\,1/2 \;\; 1/2\,\rangle \qquad n = |\,1/2 \;\; -1/2\,\rangle$$

Spinors

- When we write the wavefunction for an electron as

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\chi_+ + \beta\chi_-$$

  are χ_+ and χ_- merely two different aspects of a single particle or can we regard them as two separate entities?

- Perhaps we could try to use the spinor formalism for other closely related pairs of particles, such as Isospin (proton and neutron) w.r.t. the strong force.

Isospin Spinors

- Define a nucleon spinor $$N = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

  with $$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- The strong force is invariant under rotations in this *isospin space*.

Isospin Assignments

- For any hadrons made up of u and d quarks, construct isospin multiplets:

$$p = |\,1/2 \;\; 1/2\,\rangle \qquad n = |\,1/2 \;\; -1/2\,\rangle$$

$$\pi^+ = |\,1\,1\,\rangle \qquad \pi^0 = |\,1\,0\,\rangle \qquad \pi^- = |\,1\;-1\,\rangle$$

$$\Delta^{++} = |\,3/2 \;\; 3/2\,\rangle \qquad \Delta^{+} = |\,3/2 \;\; 1/2\,\rangle$$

$$\Delta^{0} = |\,3/2 \;\; -1/2\,\rangle \qquad \Delta^{-} = |\,3/2 \;\; -3/2\,\rangle$$

$$\Lambda = |\,0\,0\,\rangle$$

What About the Deuteron?

- The deuteron is composed of a proton and a neutron, which in the language of isospin couple together as:

$$|\,1/2\;\;1/2\,\rangle\;|\,1/2\;\;-1/2\,\rangle = \sqrt{1/2}\;|\,1\,0\,\rangle + \sqrt{1/2}\;|\,0\,0\,\rangle$$

- Is the deuteron the isotriplet $|\,1\;0\,\rangle$ state or the isosinglet $|\,0\;0\,\rangle$ state?

- If the deuteron is $|\,1\;0\,\rangle$, then the other two members of the isotriplet should also exist with similar properties. BUT... neither the nn and pp states exist as stable nuclei, hence we conclude that the deuteron is an isosinglet.

What Can Isospin do for Us?

- It can help us predict relative cross sections without having to know anything about the absolute cross sections. For example, consider the reactions:

$$
\begin{array}{llcl}
\text{(a)} & p+p & \longrightarrow & d+\pi^+ \\
\text{(b)} & p+n & \longrightarrow & d+\pi^0 \\
\text{(c)} & n+n & \longrightarrow & d+\pi^-
\end{array}
$$

- Since the deuteron is an isosinglet, the isospin of the final states is just that of the pions. When we use a CG table to find the overlap with the total isospin of the initial states, we obtain the numbers 1, $\sqrt{1/2}$, and 1, respectively, leading to:

$$\sigma_a : \sigma_b : \sigma_c = 2 : 1 : 2$$

πp Cross Sections

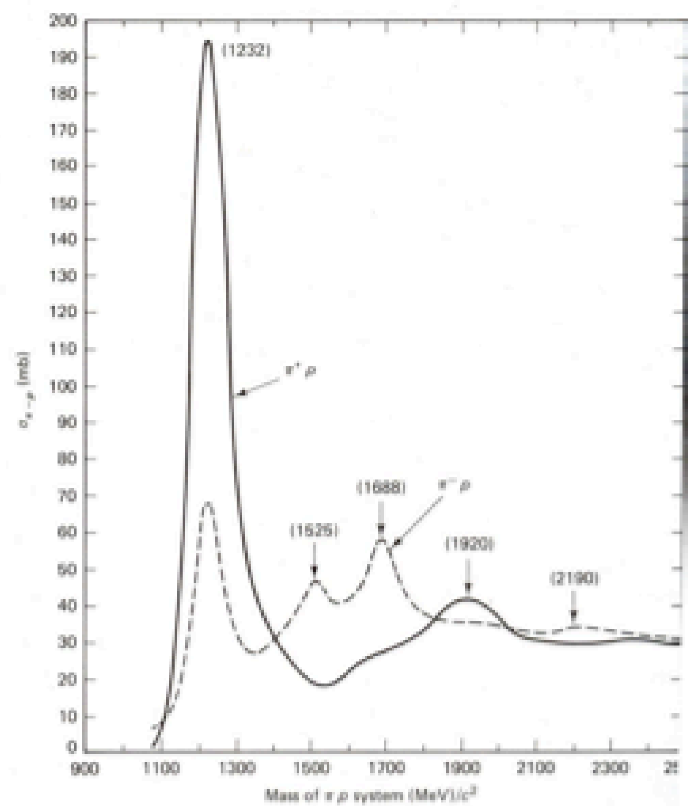


Figure 4.6 Total cross sections for π^+p (solid line) and π^-p (dashed line) sc
(*Source:* S. Gasiorowicz, *Elementary Particle Physics* (New York: Wiley, copy
1966, page 294. Reprinted by permission of John Wiley and Sons, Inc.)

Cross sections:

$\pi^+ p \to \pi^+ p$ and $\pi^- p \to \pi^- p$

If only I=3/2 component contributes then,

$\sigma(\pi^+ p) : \sigma(\pi^- p) = 3 : 1$

$\Delta(1236)$ is an $I = \frac{3}{2}$ resonance

Flavor Symmetries

- Isospin is an $SU(2)$ flavor symmetry whose origin lies in the near degeneracy of the masses of the u and d quarks.
- Insofar that the s quark is also somewhat lighter than most hadrons, an $SU(3)$ flavor symmetry will also provide a useful description of hadron physics. This $SU(3)_F$ is what got Murray Gell-Mann his nobel prize. (Be careful, though, not to confuse this with the color $SU(3)$ which is the basis for QCD in the Standard Model.)
- Since the c, b, and t are so much heavier than the other quarks, there is not much point in working with $SU(4)$, $SU(5)$, and $SU(6)$ flavor symmetries.

Summary

- Symmetries play a huge role in particle physics and are the basis for conservation laws.
- Group theory is the mathematical language of symmetry.
- The quantum mechanical description of angular momentum is a key ingredient of particle physics.
- Flavor symmetries, arising from the near-degeneracy of the light quark masses, provide us with free information about the strong interaction.