Outline

Thanks to Ian Blockland and Randy Sobie for these slides

- Lifetimes of Decaying Particles
- Scattering Cross Sections
- Fermi's Golden Rule

Observables

We want to relate experimental measurements to theoretical predictions

- Decay widths and lifetimes
  $\Gamma = h/\tau$ (units of energy)

- Scattering cross-sections
  σ is the total cross section
  $\frac{d\sigma}{d\Omega}$ is the angular distribution
  $\frac{d\sigma}{dE}$ is the energy distribution

Lifetime of an Unstable Particle

- The decay rate, Γ, represents the probability per unit time of the particle decaying:

$$\begin{aligned} dN &= -\Gamma N \, dt \\ \Rightarrow \quad N(t) &= N(0)\, e^{-\Gamma t} \end{aligned}$$

- The decay rate determines the (mean) lifetime of the particle:

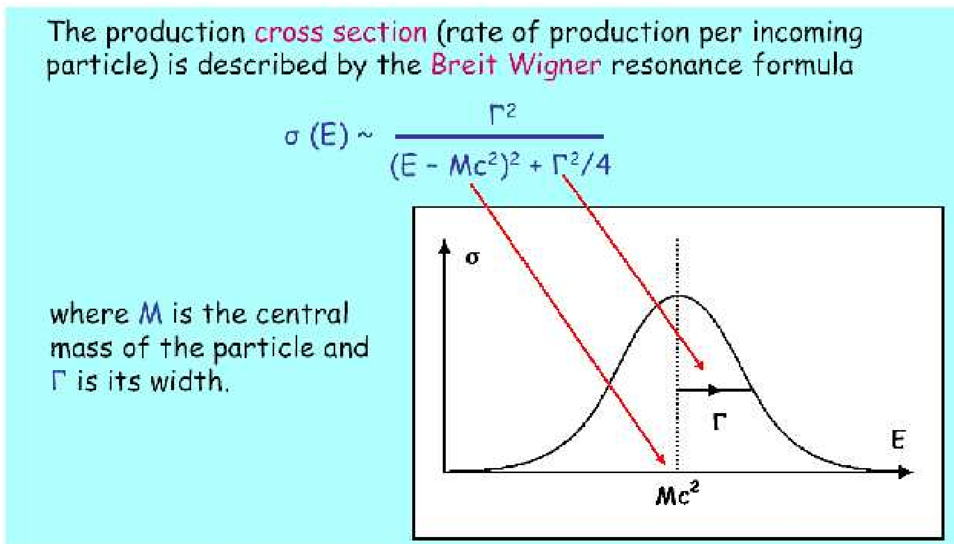
$$\tau = \frac{1}{\Gamma}$$

Breit-Wigner Resonance

Wavefunction for a particle with mass M and width Γ is $\Psi(t) = \Psi(0)e^{-i(M-i\Gamma/2)t}$

Fourier transform $\qquad \chi(E) \propto \frac{1}{(M-E)-i\Gamma/2}$

Breit-Wigner formula : $\qquad |\chi(E)|^2 \propto \frac{1}{(M-E)^2+\Gamma^2/4}$



Luminosity

- We relate cross sections to observed detection rates, per unit time, by
$$dN = \mathcal{L}\, d\sigma$$
- Since N is the number of events observed per unit time, $\mathcal{L}$ has the dimensions of an inverse cross-section per unit time. The PEP2 collider at SLAC has: $\mathcal{L} \simeq 10^{34}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$
- Only *peak luminosity* is typically quoted in per-unit-time form. Usually we integrate $\mathcal{L}$ over the running time of an experiment in order to determine what sorts of cross sections we are sensitive to. This *integrated luminosity* is measured in, say, pb^{-1}.

Luminosity Example

At PEP2 $L = 10^{34}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1}$

1 barn = 10^{-24} cm^2 or 1 nb = 10^{-33} cm^2

Hence L = 10 nb^{-1} s^{-1}

The $e^+e^- \to B\overline{B}$ cross section at the operating energy is about 1 nb, so the rate for producing $B\overline{B}$ pairs in BaBar is

$$N_{B\overline{B}} = \sigma_{B\overline{B}} L = 10\ \mathrm{s}^{-1}$$

Branching Ratios

- It is the decay rate that we will be calculating from Feynman diagrams. If a particle can decay via multiple routes, we have

$$\Gamma_{tot} = \sum_i \Gamma_i \qquad \tau = \frac{1}{\Gamma_{tot}}$$

- We define the branching ratio for a particular decay mode as

$$B_i = \frac{\Gamma_i}{\Gamma_{tot}}$$

- This is all just terminology and basic probability...

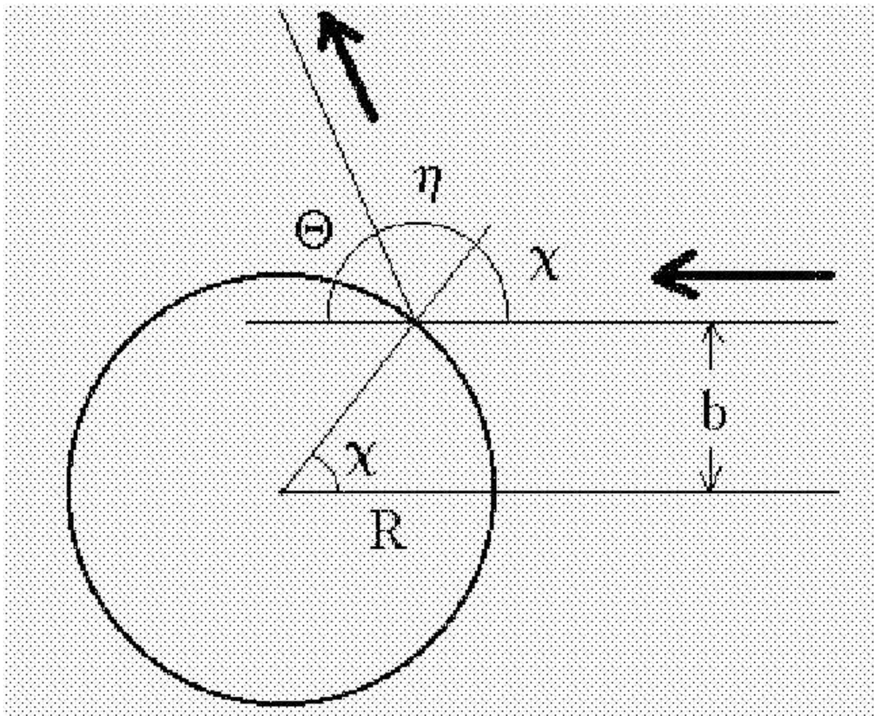
Scattering Cross Sections

- We need to generalize the intuitive notion of the geometrical cross section of a target.
- First, the interaction between the projectile and the target may be a *long-range* one, such that it is not a case of “hit or miss”, but rather, “how much deflection”? This might depend on the particle energies.
- Second, the cross section will no longer be the sole property of the target but rather a joint characteristic of both the projectile and the target.
- Third, we need to be able to account for the inelastic processes in which the final-state particles are different from those of the initial state.

Scattering Formalism

- We can interpret classical scattering experiments as prescribing a unique relationship between the impact parameter, b, and the scattering angle, θ.
- Expressing things in terms of $\theta(b)$ already allows us to treat short-range and long-range forces on the same footing. Let's look at an example of each in classical physics.
- Short-range interaction example: Hard-Sphere Scattering
- Long-range interaction example: Rutherford Scattering

Hard-Sphere Scattering



$$\eta = \chi \qquad \chi + \eta + \theta = \pi \quad \Rightarrow \quad \chi = \pi/2 - \theta/2$$

$$b = R\sin\chi = R\sin(\pi/2 - \theta/2) = R\cos(\theta/2)$$

Note that $\theta = 0$ for all $b \geq R$.

Rutherford Scattering

- Coulomb repulsion of a heavy stationary target of charge q_2 and a light incident particle of charge q_1 and kinetic energy E.
- With a great deal of effort, classical mechanics can be used to relate the impact parameter to the scattering angle:

$$b = \frac{q_1 q_2}{2E} \cot(\theta/2)$$

In this case note that $\theta > 0$ for any finite value of b.

The Differential Cross Section...

- ... is written as $d\sigma/d\Omega$
- ... often depends on θ.
- Geometrically, it is easy to see that

$$d\sigma = |\, b \; db \; d\phi \,| \qquad\qquad d\Omega = |\, \sin\theta \; d\theta \; d\phi \,|$$

and so the (classical) differential cross section is

$$\frac{d\sigma}{d\Omega} = \left|\, \frac{b}{\sin\theta} \left(\frac{db}{d\theta}\right) \,\right|$$

Hard-Sphere Scattering

$$b = R\cos(\theta/2) \qquad \Rightarrow \quad \frac{db}{d\theta} = -\,\frac{R}{2}\,\sin(\theta/2)$$

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left| \frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right) \right| \\ &= \frac{R^2\,\sin(\theta/2)\cos(\theta/2)}{2\,\sin\theta} \\ &= R^2/4\end{aligned}$$

$$\sigma = \int (R^2/4)\, d\Omega = \pi R^2$$

Rutherford Scattering

$$b = \frac{q_1 q_2}{2E}\cot(\theta/2) \qquad \Rightarrow \quad \frac{db}{d\theta} = -\frac{q_1 q_2}{4E}\csc^2(\theta/2)$$

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left| \frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right) \right| \\
&= \frac{q_1^2 q_2^2 \cot(\theta/2)\csc^2(\theta/2)}{8E^2 \sin\theta} \\
&= \left(\frac{q_1 q_2}{4E\sin^2(\theta/2)}\right)^2
\end{aligned}$$

$$\sigma = 2\pi\left(\frac{q_1 q_2}{4E}\right)^2 \int_0^\pi \frac{\sin\theta}{\sin^4(\theta/2)}\, d\theta \to \infty$$

Fermi's Golden Rule

$$\text{Transition rate} \sim |\mathcal{M}|^2 \times (\text{Phase space})$$

- The amplitude $\mathcal{M}$ contains the dynamical information about the process. We use Feynman diagrams to calculate this.
- The phase space is a kinematical factor. The bigger the phase space available, the larger the transition rate.
- Alternate terminology:
  Amplitude ⇔ Matrix Element
  Phase Space ⇔ Density of Final States

Golden Rule for Decays

- For the decay $1 \longrightarrow 2 + 3 + 4 + \ldots + n$

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2m_1} \left[\left(\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \right) \left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \cdots \left(\frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \right]$$
$$\times (2\pi)^4 \delta^4 (p_1 - p_2 - p_3 - \ldots - p_n)$$

- S is a symmetry factor of $1/j!$ for every group of j identical particles in the final state.

- $p_i = (E_i, \mathbf{p}_i)$ is the four-momentum of the i-th particle. The volume elements of the final-state particles are (subtly) Lorentz invariant:

$$\int d^4 p_i \, \delta \left(p_i^2 - m_i^2 \right) = \int \frac{d^3\mathbf{p}_i}{2E_i}$$

Example: $\pi \to \gamma + \gamma$

- From the general formula, set $S = 1/2$ and gather the factors of 2 and (2π):
$$d\Gamma = \frac{|\mathcal{M}|^2}{(8\pi)^2 m_1} \frac{d^3\mathbf{p}_2}{E_2} \frac{d^3\mathbf{p}_3}{E_3} \, \delta^4(p_1 - p_2 - p_3)$$
- Factor the δ-function into an energy part and a momentum part:
$$\delta^4(p_1 - p_2 - p_3) = \delta(m - E_2 - E_3) \, \delta^3(\mathbf{0} - \mathbf{p}_2 - \mathbf{p}_3)$$
- The momentum part of the δ-function, upon integration over $\mathbf{p}_3$, sets $\mathbf{p}_3 = -\mathbf{p}_2$. Writing E_2 and E_3 in terms of $\mathbf{p}_2$ and $\mathbf{p}_3$, we obtain
$$d\Gamma = \frac{|\mathcal{M}|^2}{(8\pi)^2 m} \frac{d^3\mathbf{p}_2}{|\mathbf{p}_2|^2} \, \delta(m - 2|\mathbf{p}_2|)$$

- The δ-function identity

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|} \qquad \forall\, i \mid f(x_i) = 0$$

allows us to write the remaining δ-function as

$$\delta(m - 2|\mathbf{p}_2|) = \frac{1}{2}\,\delta\left(|\mathbf{p}_2| - \frac{m}{2}\right)$$

- Next we write the $d^3\mathbf{p}_2$ integration element in spherical coordinates:

$$d^3\mathbf{p}_2 = |\mathbf{p}_2|^2\; d\mathbf{p}_2\;\sin\theta\; d\theta\; d\phi = p^2\; dp\; d\Omega$$

- For a 2-body decay without spin, $|\mathcal{M}|^2$ can only depend on p, therefore we can integrate over $d\Omega \to 4\pi$. In more complicated cases, $|\mathcal{M}|^2$ will also depend on θ, in which case we can only perform the ϕ-integral in advance.

$$d\Gamma = \frac{|\mathcal{M}|^2}{32\pi m} \, \frac{p^2 \, dp}{p^2} \, \delta\left(p - \frac{m}{2}\right)$$

- $d\Gamma$ becomes Γ once we have integrated over every bit of phase space, and even without knowing the specific form of $\mathcal{M}$, we can complete the integration in this case. Restoring the factor of S,

  $$\Gamma = \frac{S}{16\pi m} \, |\mathcal{M}|^2$$

  where $\mathcal{M}$ is to be evaluated using $\mathbf{p}_3 = -\mathbf{p}_2$ and $|\mathbf{p}_2| = m/2$.

- Remember that Γ has dimensions of energy, therefore $\mathcal{M}$ will also have dimensions of energy for 2-body decays.

Example: $\rho \to \pi + \pi$

- Although the basic procedure remains the same when the final-state particles have mass, the algebra becomes more complicated.
- Starting from the Golden Rule, collecting the constants, and setting $\mathbf{p}_3 = -\mathbf{p}_2$, we have

$$
\begin{aligned}
d\Gamma &= \frac{S|\mathcal{M}|^2}{2(4\pi)^2 m_1} \frac{d^3\mathbf{p}_2}{\sqrt{\mathbf{p}_2^2 + m_2^2}\sqrt{\mathbf{p}_2^2 + m_3^2}} \\
&\quad \times \delta\left(m_1 - \sqrt{\mathbf{p}_2^2 + m_2^2} - \sqrt{\mathbf{p}_2^2 + m_3^2}\right)
\end{aligned}
$$

- Going to spherical coordinates ($p = |\mathbf{p}_2|$) and integrating over the angles,

$$\Gamma = \frac{S}{8\pi m_1} \int \frac{|\mathcal{M}|^2 \, \delta(m_1 - \sqrt{p^2 + m_2^2} - \sqrt{p^2 + m_3^2})}{\sqrt{p^2 + m_2^2}\sqrt{p^2 + m_3^2}} \, p^2 \, dp$$

- In order to make use of the δ-function, we can either solve its argument for p and apply the chain rule δ-function identity (this is not advised) or we can hope to find a change of integration variables within which the δ-function is more manageable.

- The prescient change of variables is to

$$\begin{aligned} E &= \left(\sqrt{p^2+m_2^2}+\sqrt{p^2+m_3^2}\right) \\ \Rightarrow \; dE &= \frac{Ep\,dp}{\sqrt{p^2+m_2^2}\sqrt{p^2+m_3^2}} \end{aligned}$$

- This simplifies things dramatically, leaving us with

$$\Gamma = \frac{S}{8\pi m_1}\int |\mathcal{M}|^2 \,\frac{p}{E}\,\delta(m_1 - E)\,dE$$

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2}\,|\mathcal{M}|^2$$

- $|\mathbf{p}|$ is fixed by energy conservation. With $|\mathbf{p}| = m_1/2$, we recover our previous result.

3-Body Decays

- In the previous two examples, we have shown how the phase space for the decay rate of a 2-body decay can be integrated completely without any information about $\mathcal{M}$.
- For 3-body decays (and beyond), this is no longer possible, as the amplitude will typically depend non-trivially upon several of the phase space integration variables, so that we have to do the integration by hand for each specific $\mathcal{M}$.

Golden Rule for Scattering

- For the process $1 + 2 \longrightarrow 3 + 4 + \ldots + n$

$$
\begin{aligned}
d\sigma \quad = \quad & |\mathcal{M}|^2 \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \\
& \times \left[\left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \left(\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \right) \cdots \left(\frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \right] \\
& \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - \ldots - p_n)
\end{aligned}
$$

- The factor S contains a $1/j!$ for each group of j identical particles

Example: 2 to 2 Scattering in the CM Frame

- In the CM frame,

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} = (E_1 + E_2)|\mathbf{p}_1|$$

  Substituting this result into the Golden Rule and collecting the constants, we have

$$d\sigma \quad = \quad \frac{S\,|\mathcal{M}|^2}{(8\pi)^2 (E_1 + E_2)|\mathbf{p}_1|}\,\frac{d^3\mathbf{p}_3\, d^3\mathbf{p}_4}{E_3 E_4}\,\delta^4(p_1 + p_2 - p_3 - p_4)$$

- As usual, we use the momentum part of the δ-function to set $\mathbf{p}_4 = -\mathbf{p}_3$ and we express the energies E_3 and E_4 in terms of the corresponding momenta.

$$d\sigma = \frac{S\,|\mathcal{M}|^2}{(8\pi)^2(E_1+E_2)|\mathbf{p}_2|} \times \frac{\delta(E_1+E_2-\sqrt{\mathbf{p}_3^2+m_3^2}-\sqrt{\mathbf{p}_3^2+m_4^2})}{\sqrt{\mathbf{p}_3^2+m_3^2}\sqrt{\mathbf{p}_3^2+m_4^2}}\; d^3\mathbf{p}_3$$

- Next, we go to spherical coordinates (with $p = |\mathbf{p}_3|$). Unlike the 2-body decays, $\mathcal{M}$ can depend on the scattering angle θ, therefore only the azimuthal integral can be performed for the general case. We choose not to do this, instead moving the entire $d\Omega$ factor to the left-hand side to form a conventional differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{S}{(8\pi)^2\,(E_1+E_2)|\mathbf{p}_1|}\int |\mathcal{M}|^2 \times \frac{\delta(E_1+E_2-\sqrt{p^2+m_3^2}-\sqrt{p^2+m_4^2})}{\sqrt{p^2+m_3^2}\sqrt{p^2+m_4^2}}\; p^2\; dp$$

- With $(E_1 + E_2)$ taking the place of m_1 in the δ-function, this is precisely the integral we encountered in the previous example of a general 2-body decay. Applying the result we derived there, we obtain the final expression

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{S\,|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

- We're going to be using the above result quite a bit in the chapters ahead.

Summary

- The statistical mean lifetime of a particle is the inverse of the decay rate.
- The notion of a geometrical cross section can be generalized so as to incorporate a variety of classical scattering results and to carry over to the quantum realm of particle physics.
- Fermi's Golden Rule provides the prescription for combining dynamical information about the amplitude and kinematical information about the phase space in order to obtain observable quantities like decay rates and scattering cross sections.