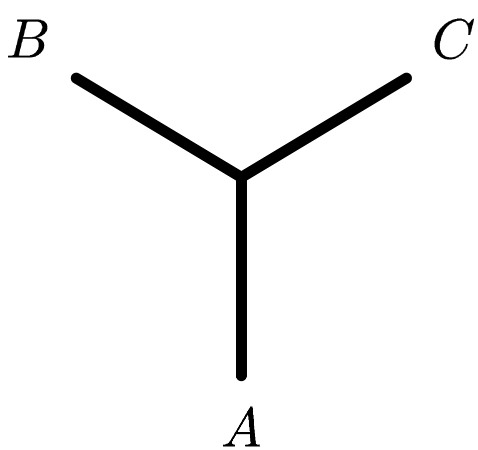
# Last class we saw that..

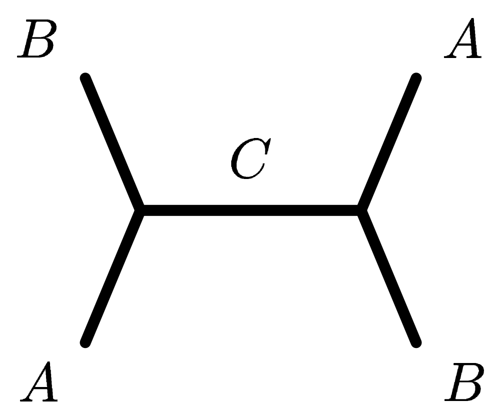
- ... the statistical mean lifetime of a particle is the inverse of the decay rate.
- ... the notion of a geometrical cross section can be generalized so as to incorporate a variety of classical scattering results and to carry over to the quantum realm of particle physics.
- ... Fermi's Golden Rule provides the prescription for combining dynamical information about the amplitude and kinematical information about the phase space in order to obtain observable quantities like decay rates and scattering cross sections.

Thanks to Ian Blockland and Randy Sobie for these slides

# Feynman diagrams

- The graphical representation of an interaction has a 1-to-1 correspondence with the mathematical expression describing the amplitude
- The time axis in these diagrams is vertically upward (as in the Griffiths text)
- We'll use arrows to indicate particle/anti-particle later; in today's lecture all particles are their own antiparticles



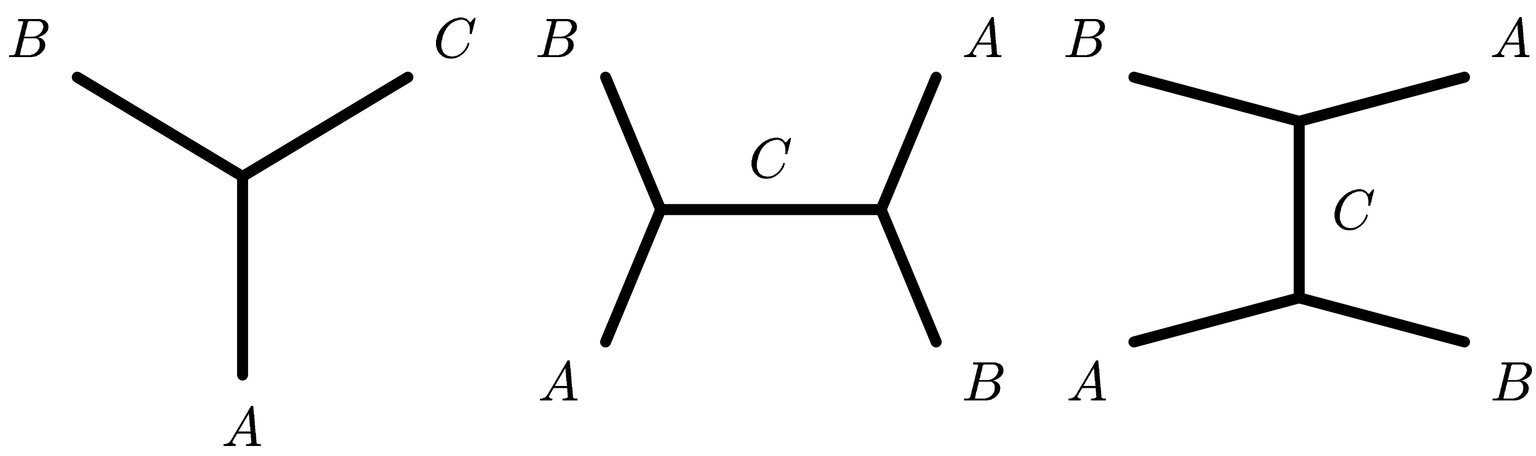
## *ABC* Theory

This follows closely the development in Griffiths' text "Introduction to Elementary Particles"

- Feynman Diagrams
- Feynman Rules
- Calculating Decay Rates
- Calculating Cross Sections
- Higher-Order Diagrams

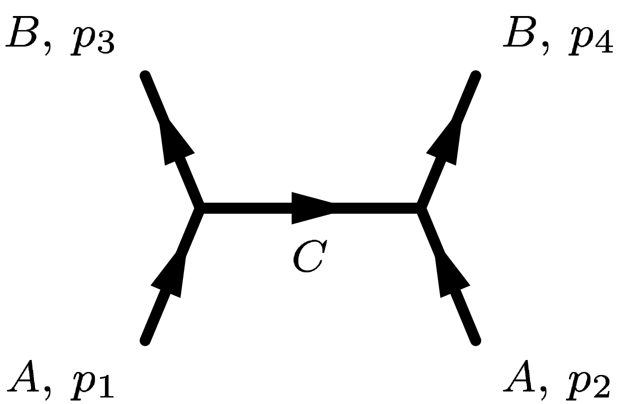
# ABC Theory

- 3 spinless particles, $A$, $B$, and $C$, each of which is its own antiparticle.
- One only primitive vertex, with coupling $g$ (dimensions of momentum).
- If $m_A > m_B + m_C$ then $A$ can decay.



# The Feynman Rules

- The Feynman rules provide the recipe for constructing an amplitude $\mathcal{M}$ from a Feynman diagram.
- **Step 1:** Draw the Feynman diagram(s) with the minimum number of vertices. There may be more than one.
- **Step 2:** Label the four-momentum of each line (with arrows), enforcing four-momentum conservation at every vertex.




$p_1, p_2, \ldots$ external momenta

$q_1, q_2, \ldots$ internal momenta

arrows indicate positive direction

- **Step 3:** Each vertex contributes a factor of $(-ig)$.
  Each internal line, with mass $m$ and four-momentum $q$, contributes a propagator of $\frac{i}{q^2-m^2}$
- **Step 4:** Conserve 4-momentum at each vertex

$$(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)$$

where $k_i$ are the momenta coming into the vertex.

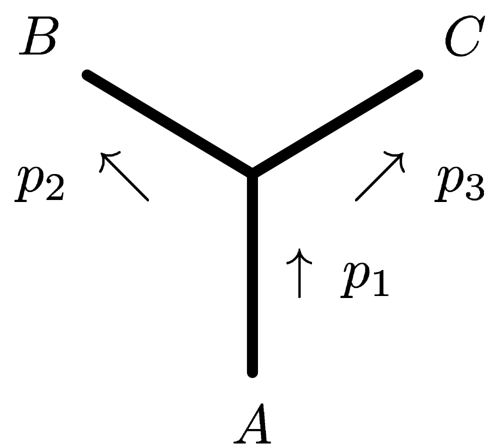
- **Step 5** Form the amplitude

$$\mathcal{M} = i\,(\text{vertex factors})(\text{propagators})(\text{momentum conservation})$$

- **Step 6:** Integrate over the internal momenta $\frac{1}{(2\pi)^4}d^4q_j$
- **Step 7:** Drop the extra $(2\pi)^4\,\delta^{(4)}$-function and $\mathcal{M}$ remains.

## **Example:** $A \to B + C$

- To lowest order ($\mathcal{O}(g)$), we have just one diagram:




- There is just one vertex and no propagators, therefore

$$\mathcal{M} = i(-ig) = g$$

# Lifetime of $A$

- From Fermi's Golden Rule, the decay rate is given by

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi m_1^2} \, |\mathcal{M}|^2$$

in the rest frame of $A$

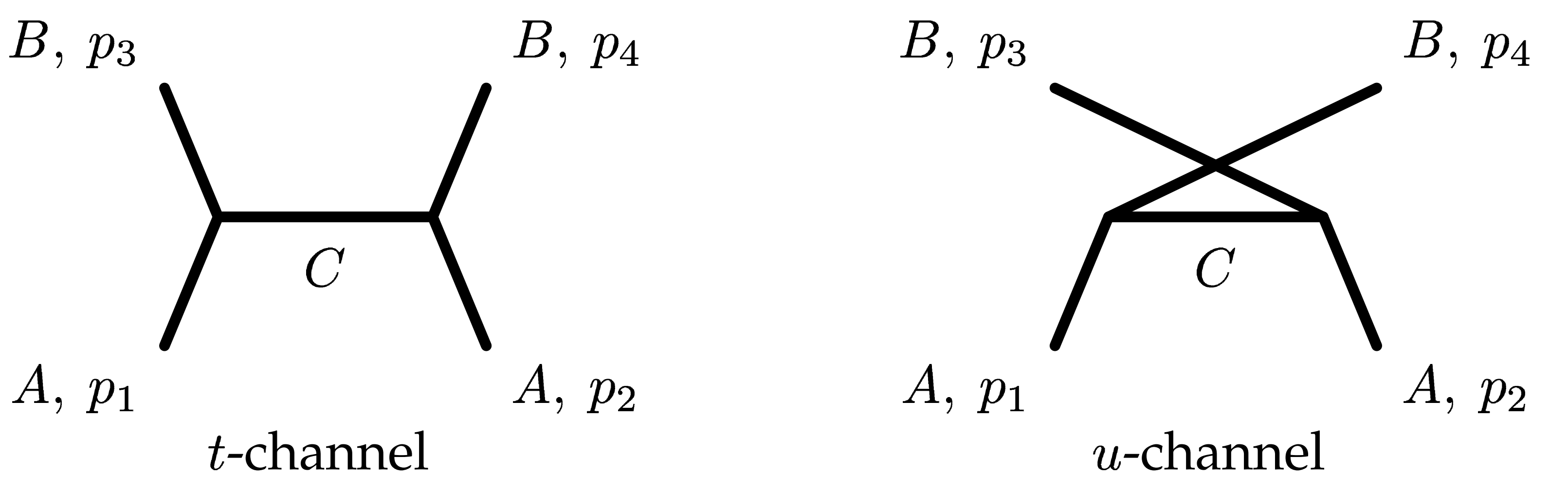
- With $S = 1$, $m_1 = m_A$, $\mathcal{M} = g$, and $|\mathbf{p}|$ representing the magnitude of the spatial momentum of either $B$ or $C$, we find that

$$\tau_A = \frac{8\pi m_A^2}{g^2|\mathbf{p}|}$$

- Note that $g$ has dimensions of mass.

# **Example:** $A + A \to B + B$

- To lowest order ($\mathcal{O}(g^2)$), we have two diagrams:




mom. transf. $= t = (p_1 - p_3)^2$

mom. transf. $= u = (p_1 - p_4)^2$

$$\mathcal{M}_t = i(-ig)^2 \frac{i}{(p_1 - p_3)^2 - m_C^2} = \frac{g^2}{t - m_C^2}$$

# Cross Section for $A + A \to B + B$

$$\mathcal{M} = \frac{g^2}{t - m_C^2} + \frac{g^2}{u - m_C^2}$$

- Notice that $\mathcal{M}$ is Lorentz invariant. This is *always* true.
- To convert $\mathcal{M}$ to a cross section, we use Fermi's Golden Rule. In the CM frame,

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{S\,|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

$S = 1/2$ because we have two identical particles $(B + B)$ in the final state. Also, $E_1 = E_2 = E$.

$$\frac{d\sigma}{d\Omega} = \frac{1}{2(16\pi E)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \left| \frac{g^2}{t - m_C^2} + \frac{g^2}{u - m_C^2} \right|^2$$

- To take this calculation further, let's assume that $m_A = m_B = m$ and $m_C = 0$. Then $|\mathbf{p}_f| = |\mathbf{p}_i|$ and

$$\begin{aligned} t = (p_1 - p_3)^2 &= -2\mathbf{p}^2(1 - \cos\theta) \\ u = (p_1 - p_4)^2 &= -2\mathbf{p}^2(1 + \cos\theta) \end{aligned}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{2} \left( \frac{g^2}{16\pi E \mathbf{p}^2 \sin^2\theta} \right)^2$$

- Note that $\sigma \to \infty$, just as for Rutherford scattering.

# Higher-Order Diagrams

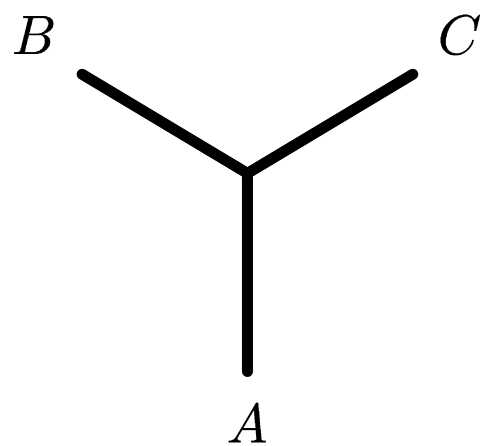
- By considering more complicated Feynman diagrams, we can generate additional contributions to the amplitude:

$$\begin{aligned}\mathcal{M}_{A\to B+C} &= g\mathcal{A}_1 + g^3\mathcal{A}_3 + g^5\mathcal{A}_5 + \dots \\ \mathcal{M}_{A+A\to B+B} &= g^2\mathcal{A}_2 + g^4\mathcal{A}_4 + g^6\mathcal{A}_6 + \dots\end{aligned}$$

  If $g \ll 1$ (or, more precisely, $(g/m_A) \ll 1$ in $ABC$ Theory), we can see how each successive term in the *perturbation series* provides smaller and smaller corrections to the amplitude.

# Corrections to $\tau_A$

- We have already calculated the lifetime of $A$ due to the simple vertex diagram

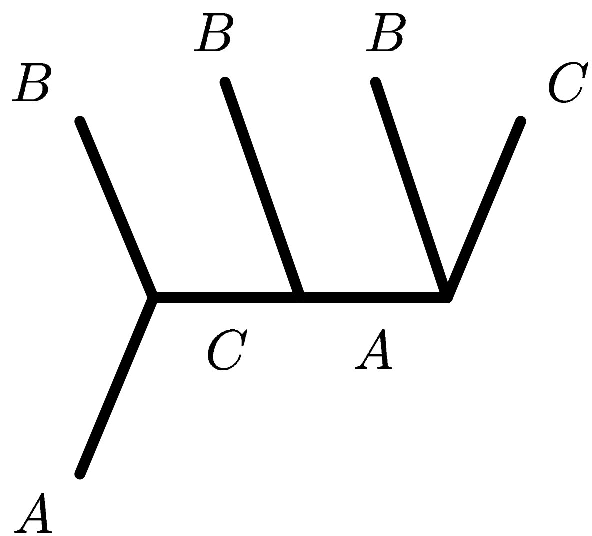



- Since $\mathcal{M} \sim g,\ \Gamma \sim g^2$. The leading corrections to $\Gamma$ will be $\mathcal{O}(g^4)$ and they will arise from the interference of the $\mathcal{O}(g)$ diagram with a $\mathcal{O}(g^3)$ diagram in the coherent sum:

$$|\mathcal{M}|^2 = \left|g\mathcal{A}_1 + g^3\mathcal{A}_3 + \ldots\right|^2$$

# Other Decay Modes

- Note that we are only interested in the $\mathcal{O}(g^3)$ diagrams in which $A \to B + C$. If $A$ is sufficiently heavy, other decay modes such as $A \to 3B + C$ and $A \to B + 3C$ are possible.



# Incoherent Sums

- Since $A \to 3B + C$ is a distinct decay mode, we calculate its $\Gamma$ separately from that of $A \to B + C$. As a result, even though $\mathcal{M} \sim g^3,\ \Gamma \sim g^6$, and so we need not consider these diagrams for a $\Gamma \sim g^4$ calculation.

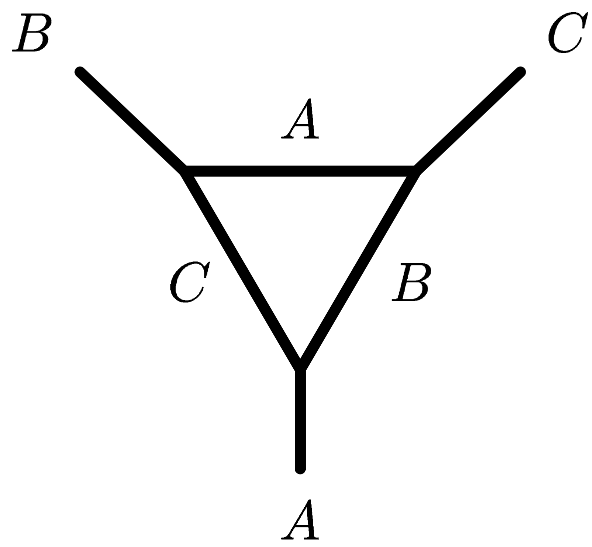
- Terminology: The decay of $A$ involves both *coherent* and *incoherent* sums.

$$\begin{aligned}\Gamma(A \to anything) &= \Gamma_{A\to BC} + \Gamma_{A\to BBBC} + \Gamma_{A\to BCCC} + \ldots \\ &= C_1 \left|\sum \mathcal{M}_{A\to BC}\right|^2 + C_2 \left|\sum \mathcal{M}_{A\to BBBC}\right|^2 \\ &\quad + C_3 \left|\sum \mathcal{M}_{A\to BCCC}\right|^2 + \ldots\end{aligned}$$

($C_1$, $C_2$, and $C_3$ arise from Fermi's Golden Rule.)

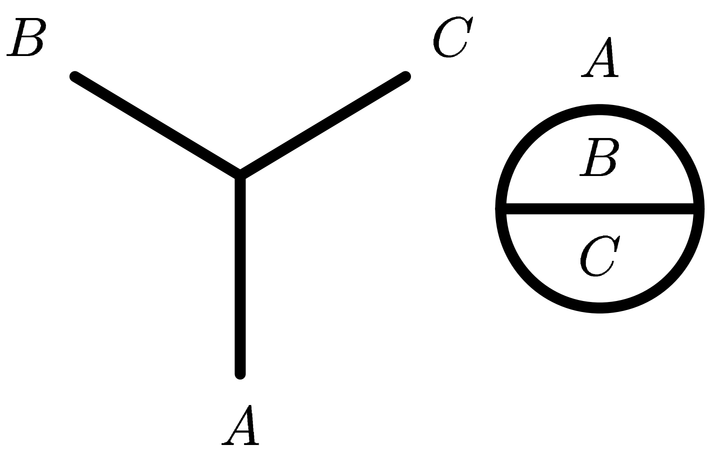
# Third-Order $A \to B + C$ Diagrams

- There is one legal third-order diagram to consider:

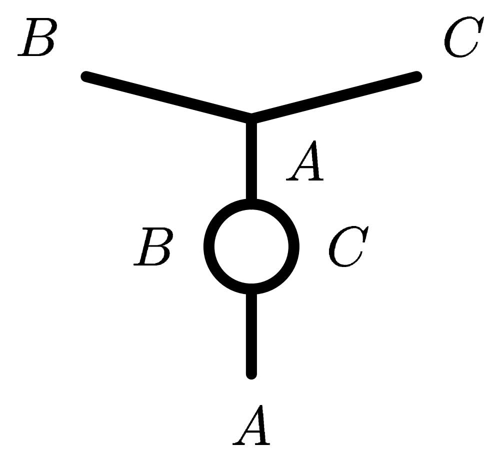


# Illegal Loop Diagrams

- There are several other $\mathcal{O}(g^3)$ diagrams that can be drawn for $A \rightarrow B + C$, however these are not to be calculated using the Feynman Rules.



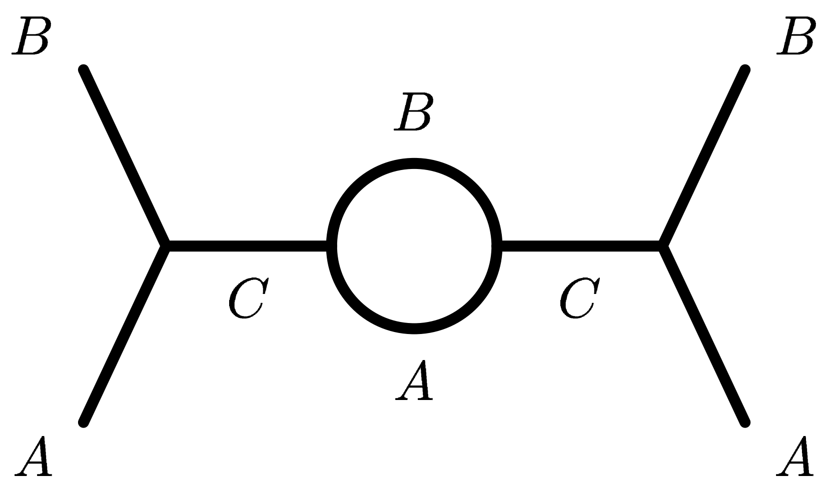

Disconnected




Reducible

## Corrections to $A + A \to B + B$

- The interference of the *one-loop* diagrams ($\mathcal{O}(g^4)$) with the tree-level diagram ($\mathcal{O}(g^2)$) provides $\mathcal{O}(g^6)$ corrections to the cross section.




- If you go through this calculation (see Griffiths) you'll find that the amplitude associated with this diagram is *divergent*.

# Renormalization

- All divergences in the final physical observables ($\Gamma$ or $\sigma$) seem to be affiliated with the coupling constants and the masses.
- In other words, we can define *renormalized* couplings and masses which absorb the divergences. We then assume that it is these renormalized parameters which we have been measuring all along. This would mean that the *bare* parameters aren't physical.
- Renormalization is a feature of all quantum field theories, including those found outside of particle physics.

# Summary

- Feynman diagrams provide a convenient and intuitive representation of particle interactions.
- The Feynman rules allow us to translate Feynman diagrams into mathematical expressions for the amplitudes.
- $ABC$ Theory is a toy theory which makes it easier to learn how to use Feynman rules without having to worry about some of the complexities which accompany more realistic theories.
- Sometimes higher-order Feynman diagrams are needed to improve the precision of a calculation. This is a minefield, but maps are available.