Thanks to Ian Blockland and Randy Sobie for these slides

Recall from the previous lecture...

- Feynman diagrams provide a convenient picture of particle interactions.
- The Feynman rules allow us to translate Feynman diagrams into mathematical expressions for the amplitudes.
- ABC Theory is a toy theory which makes it easier to learn how to use Feynman rules.
- Sometimes higher-order Feynman diagrams are needed to improve the precision of a calculation.

Relativistic Wave Equations

- Klein-Gordon Equation
- Dirac Equation

Deriving Wave Equations

- Recall that when we substitute

$$\mathbf{p} \to -i\nabla \qquad E \to i\frac{\partial}{\partial t}$$

into the classical expression for energy conservation,

$$\frac{\mathbf{p}^2}{2m} + V = E$$

we obtain the Schrödinger equation for a particle of mass m

$$-\frac{1}{2m}\nabla^2\Psi + V\Psi = i\frac{\partial \Psi}{\partial t}$$

Relativistic Wave Equations

- Let's see what happens when we use the relativistic energy-momentum relation:

$$\begin{aligned} E^2 - \mathbf{p}^2 &= m^2 \\ p^\mu p_\mu - m^2 &= 0 \end{aligned}$$

- With the covariant substitution $p_\mu \to i\partial_\mu$,

$$\begin{aligned} -\partial^\mu \partial_\mu \phi - m^2 \phi &= 0 \\ -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi &= m^2 \phi \\ (\Box + m^2)\phi &= 0 \end{aligned}$$

This is the Klein-Gordon equation.

The Klein-Gordon Equation

- Discovered by Schrödinger, discarded, rediscovered by Klein and Gordon.
- Consider a *plane-wave* solution to the Klein-Gordon equation:

$$\phi(\mathbf{x}, t) = e^{-iEt + i\mathbf{p}\cdot\mathbf{x}} = e^{-ip\cdot x}$$

$$(\Box + m^2)\phi = 0 \qquad \Rightarrow \qquad -E^2 + \mathbf{p}^2 + m^2 = 0$$

- For a given $\mathbf{p}$, there are two possible solutions for E:

$$E = \pm\sqrt{\mathbf{p}^2 + m^2}$$

What's a negative energy supposed to mean?

An Even Bigger Problem

- For the Schrödinger equation we have

$$\begin{aligned} i\frac{\partial}{\partial t}(\psi^*\psi) &= -\frac{1}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) \\ \frac{\partial P}{\partial t} &= -\frac{1}{2mi}\nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*) = -\nabla\cdot\mathbf{j} \end{aligned}$$

- It can be shown (Noether's Theorem) that the Klein-Gordon equation satisfies the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0$$

for the probability ρ and probability current $\mathbf{j}$. But

$$\rho = i\left[\phi^*\frac{\partial\phi}{\partial t} - \left(\frac{\partial\phi^*}{\partial t}\right)\phi\right]$$

is *not* constant, nor is it even necessarily positive.

The Source of the Problems

- The second-order time derivative in the $\Box$ of the KG equation is responsible for both the negative-energy plane wave solutions and the misbehaving probability density.
- Dirac tried to fix this problem by looking for a relativistic equation that, like the Schrödinger equation $H\Psi = i\,\partial\Psi/\partial t$, only contained first-order time derivatives.

Dirac's Approach

- Suppose a particle is at rest (i.e., $\mathbf{p} = \mathbf{0}$). Then the (quadratic) energy-momentum relation

$$E^2 - \mathbf{p}^2 = m^2$$

can be factored into a pair of (linear) equations:

$$\begin{aligned} (p^0)^2 - m^2 &= 0 \\ (p^0 - m)(p^0 + m) &= 0 \\ \Rightarrow \; (p^0 - m) = 0 \quad \text{or} & \quad (p^0 + m) = 0 \end{aligned}$$

- Either linear equation leads to a configuration-space equation which is first-order in time and satisfies the relativistic energy-momentum relation.

- It is not a trivial matter, though, to extend this factorization for moving particles. Writing

$$(p^\mu p_\mu - m^2) = (\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m)$$

we need to determine the coefficients β^κ and γ^λ.

- Expanding out the right-hand side, we have

$$(p^\mu p_\mu - m^2) = \beta^\kappa \gamma^\lambda p_\kappa p_\lambda + m(\gamma^\kappa - \beta^\kappa)p_\kappa - m^2$$

To eliminate the linear p term, we require $\beta^\kappa = \gamma^\kappa$. The quadratic term leads to

$$p^2 \quad = \quad \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$$

$$\begin{aligned} p^2 &= \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda \\ &= \gamma^\kappa \gamma^\lambda (p_\kappa p_\lambda + p_\lambda p_\kappa)/2 \\ &= (\gamma^\kappa \gamma^\lambda + \gamma^\lambda \gamma^\kappa) p_\kappa p_\lambda /2 \\ \Rightarrow \quad (\gamma^\kappa \gamma^\lambda + \gamma^\lambda \gamma^\kappa) &= 2g^{\kappa\lambda} \end{aligned}$$

- We typically write this last relationship as an *anticommutator*:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

- With $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$, we might consider $\gamma^0 = 1$ and $\gamma^i = i$, but then we would get $\{\gamma^0, \gamma^i\} \neq 0$. Our anticommutator equation cannot be solved by any set of complex numbers!

The γ Matrices

- Dirac's clever idea was to let γ represent a matrix. Specifically, the 4×4 matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

  satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

- This set of γ matrices is known as the Bjorken and Drell representation and it is commonly used at low energies. Other choices exist, most notable of which is the chiral representation which is useful at high energies. Of course, the physics is independent of the specific choice of γ matrices.

The Dirac Equation

- Having successfully factored the relativistic energy-momentum relation,

$$(p^\mu p_\mu - m^2) = (\gamma^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = 0$$

we can set either factor to zero.

- In momentum space, the Dirac equation is

$$\gamma^\mu p_\mu - m = 0$$

where m implicitly multiplies the $4-d$ unit matrix.

- With $p_\mu \to i\partial_\mu$, we get the configuration-space Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Slash Notation

- When we contract γ^μ with a four-vector q_μ, we can abbreviate this using the *Feynman slash* notation

$$\gamma^\mu q_\mu = \not{q}$$

- With the slash notation, the Dirac equation becomes

$$\begin{aligned} \not{p} - m &= 0 \\ (i\not{\partial} - m)\psi &= 0 \end{aligned}$$

Spinors

$$(i\partial\!\!\!/ - m)\psi = 0$$

- Since the γ matrices are 4×4, ψ must be a 4-component column matrix. We call this a bi-spinor, Dirac spinor, or just plain *spinor*. It is *not* a four-vector.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Solutions of the Dirac Equation

- Let's start by looking for solutions that are independent of position:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial z} = 0$$

This simplifies the Dirac equation to

$$i\gamma^0 \frac{\partial \psi}{\partial t} - m\psi = 0$$

We will split the spinor into a pair of 2-component pieces:

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \qquad \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

- This leads to the pair of equations

$$\frac{\partial \psi_A}{\partial t} = -im\psi_A \qquad \frac{\partial \psi_B}{\partial t} = +im\psi_B$$

whose solutions are

$$\psi_A(t) = \psi_A(0)e^{-imt} \qquad \psi_B(t) = \psi_B(0)e^{+imt}$$

- Recall that for the Schrödinger equation, the characteristic time dependence of the solutions is e^{-iEt}. Evidently, ψ_A is a solution with energy $E = +m$, as we should expect, but ψ_B seems to have a negative energy $E = -m$.
- Dirac had hoped that a first-order (in $\partial/\partial t$) equation would avoid these negative energy solutions.

Plan B

- Seeing as we seem to be stuck with the negative energy solutions, Dirac's next suggestion was that all possible negative energy states were already filled by a *Dirac sea* of particles. The Pauli Exclusion Principle would then leave only the positive energy states available.
- The excitation of a sea electron would leave a *hole* which would behave like a positive energy particle with a positive charge. Eventually, Dirac worked up the courage to predict the existence of the positron.
- Experimentalists had secretly been observing evidence for antimatter for years, but had always discarded these unphysical particles. Needless to say, the positron was quickly "discovered".

Electrons and Positrons

- With the interpretation of negative energy states as positive energy antiparticles, we can see how the Dirac equation has four independent solutions for a particle at rest:

$$(e^- \uparrow) \quad \psi^{(1)} = e^{-imt}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \psi^{(2)} = e^{-imt}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad (e^- \downarrow)$$

$$(e^+ \uparrow) \quad \psi^{(3)} = e^{+imt}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \psi^{(4)} = e^{+imt}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad (e^+ \downarrow)$$

Plane-Wave Solutions

- Next we will look for solutions to the Dirac equation of the form

  $$\psi(x) = a e^{-ip\cdot x} u(p)$$

  $u(p)$ is a momentum-space solution of the Dirac equation, satisfying

  $$(\not{p} - m)u = 0$$

  Using

  $$\begin{aligned} \not{p} &= \gamma^0 p^0 - \gamma \cdot \mathbf{p} \\ &= E \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \mathbf{p} \cdot \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix} \\ &= \begin{pmatrix} E & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -E \end{pmatrix} \end{aligned}$$

$$(\not{p} - m)u = \begin{pmatrix} E - m & -\mathbf{p}\cdot\sigma \\ \mathbf{p}\cdot\sigma & -E - m \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$
$$= \begin{pmatrix} (E - m)u_A - (\mathbf{p}\cdot\sigma)u_B \\ (\mathbf{p}\cdot\sigma)u_A - (E + m)u_B \end{pmatrix}$$

- The Dirac equation $(\not{p} - m)u = 0$ then gives us a pair of coupled equations for u_A and u_B:

$$u_A = \frac{(\mathbf{p}\cdot\sigma)}{E - m}u_B \qquad u_B = \frac{(\mathbf{p}\cdot\sigma)}{E + m}u_A$$

- These equations can easily be solved by substituting one into the other and noting that

$$(\mathbf{p}\cdot\sigma)^2 = \mathbf{p}^2 \cdot 1$$

$$u_A = \frac{(\mathbf{p}\cdot\sigma)}{E-m}u_B \qquad u_B = \frac{(\mathbf{p}\cdot\sigma)}{E+m}u_A$$

- Substituting the second equation into the first, we have

$$u_A = \frac{\mathbf{p}^2}{E^2-m^2}u_A$$

which requires $E^2 - m^2 = \mathbf{p}^2$, just as we should expect. The same thing happens with u_B. Either way, we have two solutions for E:

$$E = \pm\sqrt{\mathbf{p}^2 + m^2}$$

- By picking specific forms for u_A or u_B (remember, one is fixed by the other), we can construct a set of four solutions to the Dirac equation for a moving particle.

- With the normalization $u^\dagger u = 2|E|$, we have two particle solutions

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

and two antiparticle solutions (satisfying $(\not{p} + m)v = 0$)

$$v^{(1)} = N \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v^{(2)} = -N \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

- In all cases, $N = \sqrt{E+m}$ and E is positive.

Spins of the Plane-Wave Solutions

- We can generalize the Pauli spin matrices to the 4×4 matrices required for Dirac spinors:

$$\mathbf{S} = \frac{\hbar}{2}\Sigma \qquad \Sigma \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

- If (and only if) the particles are traveling along the z-axis, the plane-wave solutions u and v will be eigenstates of S_z. $u^{(1)}$ and $v^{(1)}$ are spin up, while $u^{(2)}$ and $v^{(2)}$ are spin down.

Particles and Antiparticles

- In a typical experiment, we are dealing with particles of specific energies and momenta, therefore it is the u and v plane-wave solutions which are of interest to us.
- While the particle states are solutions to the original momentum-space Dirac equation

$$(\not{p} - m)u = 0$$

The antiparticle states, by virtue of reinterpreting the negative energy particle states as positive energy antiparticles, satisfy

$$(\not{p} + m)v = 0$$

Big and Small Components

- Using the Bjorken and Drell representation of the γ matrices, we have shown that the plane-wave solution for a spin-up electron is

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

- At low energies, the *upper* components are much larger than the *lower* components. As a result, it is possible to construct a 2-component representation of the particle which takes some of the relativistic effects of antiparticles into account. This is sometimes used in quantum chemistry and nuclear physics.

Problems With Relativistic Wave Equations

- Well, we see that both the Klein-Gordon and Dirac equations have negative energy solutions. In hindsight, we realize that these are not pathologies of the theory, but rather that they represent antiparticle solutions.
- The Dirac equation, meanwhile, leads to a sensible definition for the probability density, since the antiparticle content is made explicit. With the Klein-Gordon equation, antiparticles also exist, and their negative number density leads to the possibility of negative densities (i.e. because of pair creation, relativistic wave equations are not strictly one-particle wave equations).

Summary

- The relativistic energy-momentum relation leads to the Klein-Gordon equation, but a naive examination of this equation leads to problems.
- Dirac tried to get around these problems with a first-order equation. The KG problems remained and their solution was to postulate the existence of antiparticles.
- We now realize that the Klein-Gordon equation describes spin-0 particles and the Dirac equation describes spin-$\frac{1}{2}$ particles.
- The particle and antiparticle plane-wave solutions of the Dirac equation will be used frequently in our formulation of QED.