Transformation properties of spinors

- Lorentz Transformations of Spinors
- Bilinear Covariants
- The Photon

Slides from Sobie and Blokland

Lorentz Transformations of Spinors

- Spinors are not four-vectors, therefore they do not transform via Λ. How do they transform?

$$\psi \quad \rightarrow \quad S\psi$$

where for motion along the x-axis,

$$\begin{aligned} S &= \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} \\ a_\pm &= \pm\sqrt{(\gamma \pm 1)/2} \\ \gamma &= (1 - v^2)^{-1/2} \end{aligned}$$

Making a Scalar With a Spinor

- Consider

$$\psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

Under a Lorentz transformation,

$$\begin{aligned} \psi^\dagger \psi &\rightarrow (S\psi)^\dagger (S\psi) \\ &\rightarrow \psi^\dagger (S^\dagger S)\psi \end{aligned}$$

Since $S^\dagger S \neq 1$ (check for yourself using the explicit representation of S on the previous page), $\psi^\dagger \psi$ is not a Lorentz scalar.

The Adjoint Spinor

- Just as four-vector contractions need a few well-placed minus signs (i.e., $g^{\mu\nu}$) in order to make a scalar, we can add a couple of minus signs to a spinor by defining the *adjoint spinor*:

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = (\psi_1^* \quad \psi_2^* \quad -\psi_3^* \quad -\psi_4^*)$$

- Since $S^\dagger \gamma^0 S = \gamma^0$ (again, check this yourself),

$$\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$

is a Lorentz scalar.

γ^5: The Black Sheep of the Family

- Define an additional γ-matrix by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

(You really don't want to know what happened to γ^4.)

- In the Bjorken and Drell representation,

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Note: $(\gamma^5)^2 = 1$ and γ^5 anticommutes with every other γ:

$$\{\gamma^\mu, \gamma^5\} = 0 \qquad \Rightarrow \qquad \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$$

Another Scalar?

- We have already seen how $\bar{\psi}\psi$ is a Lorentz scalar.
- Since $S^\dagger\gamma^0\gamma^5 S = \gamma^0\gamma^5$ (check this too),

$$\bar{\psi}\gamma^5\psi$$

  is also a Lorentz scalar.
- This gives us 2 Lorentz scalars: $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$. What's the difference?

Parity

- Under a parity transformation

$$\psi \to \gamma^0 \psi$$

- Since

$$\begin{aligned} \bar\psi\psi &\to (P\psi)^\dagger\gamma^0(P\psi) \\ &\to \psi^\dagger(\gamma^0)^\dagger\gamma^0\gamma^0\psi \\ &\to \psi^\dagger(\gamma^0)^\dagger\psi \\ &\to \bar\psi\psi \end{aligned}$$

$$\begin{aligned} \bar\psi\gamma^5\psi &\to (P\psi)^\dagger\gamma^0\gamma^5(P\psi) \\ &\to \psi^\dagger(\gamma^0)^\dagger\gamma^0\gamma^5\gamma^0\psi \\ &\to -\psi^\dagger(\gamma^0)^\dagger\gamma^5\psi \\ &\to -\bar\psi\gamma^5\psi \end{aligned}$$

$\bar\psi\psi$ is a true scalar and $\bar\psi\gamma^5\psi$ is a pseudoscalar.

Bilinear Covariants

- There are 16 possible products of the form $\psi_i^* \psi_j$. These 16 products can be grouped together into *bilinear covariants*:

$\bar{\psi}\psi$	Scalar	1 component
$\bar{\psi}\gamma^5\psi$	Pseudoscalar	1 component
$\bar{\psi}\gamma^\mu\psi$	Vector	4 components
$\bar{\psi}\gamma^\mu\gamma^5\psi$	Pseudovector	4 components
$\bar{\psi}\sigma^{\mu\nu}\psi$	Antisymmetric tensor	6 components

Note that: $\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$

Why This is Useful

- We have a simple basis set $\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$ for any 4×4 matrix, therefore we can always simplify more complicated combinations of γ matrices.
- The tensorial and parity character of each bilinear is evident. This makes it easy to see why the QED interaction Lagrangian

$$-eA_\mu \bar{\psi}\gamma^\mu\psi$$

leads to a parity-conserving electromagnetic force mediated by a vector (i.e., spin-1) boson.

- To describe the parity-violating weak interaction, we could (and do) mix vector $(\bar{\psi}\gamma^\mu\psi)$ and axial $(\bar{\psi}\gamma^\mu\gamma^5\psi)$ interactions.

EM and photons

- Maxwell's equation

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 4\pi J^\nu$$

where $\Box = \partial_\mu \partial^\mu$, $A^\nu = (\phi, \mathbf{A})$ and $J^\nu = (\rho, \mathbf{J})$

- $(\phi, \mathbf{A})$ are not uniquely determined and so we are allowed to make a gauge transformation $A^\mu \to A^\mu + \partial^\mu \lambda$
- We can demand the Lorentz condition $\partial_\mu A^\mu = 0$
- The Lorentz condition simplifies the Maxwell equations to

$$\Box A^\mu = 4\pi J^\mu$$

Another Constraint

- Even with the Lorentz condition, we can make further gauge transformations of the form $A^\mu \to A^\mu + \partial^\mu \lambda$ without disturbing $\Box A^\mu = 4\pi J^\mu$ so long as $\Box\lambda = 0$.
- As a result, we can impose an additional constraint. We typically choose to set $A^0 = 0$ and thereby work in the *Coulomb gauge*:

$$\nabla \cdot \mathbf{A} = 0$$

Free Photons

- For a photon in free space ($J^\mu = 0$), the potential is given by $\Box A^\mu = 0$.
- The plane-wave solution is

$$A^\mu(x) = ae^{-ip\cdot x}\epsilon^\mu(p)$$

where ϵ^μ is the *polarization vector* and $p_\mu p^\mu = 0$.

- Although ϵ^μ has 4 components, not all are independent. The Lorentz condition requires that $p^\mu \epsilon_\mu = 0$ Furthermore, the Coulomb gauge implies that $\epsilon^0 = 0$ and $\epsilon \cdot \mathbf{p} = 0$
- Since ϵ is perpendicular to $\mathbf{p}$, the photon is *transversely polarized* and there are only 2 independent polarization states.