Calculations in QED

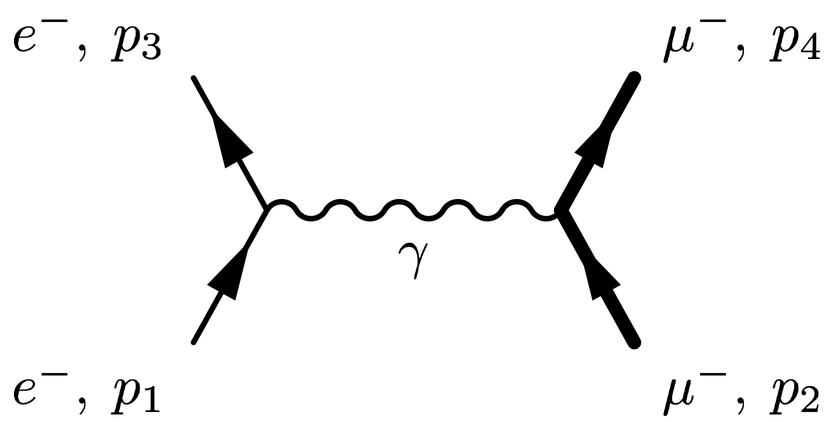
- Electron-Muon Scattering
  (Mott Scattering)
- Pair annihilation
- Higher-Order Diagrams in QED

Slides from Sobie and Blokland

Recap

- The Feynman rules for QED provide the recipe for translating Feynman diagrams into mathematical expressions for the amplitude.
- If we are interested in the spin-averaged amplitude $\left\langle |\mathcal{M}|^2 \right\rangle$ then we need not ever use explicit fermion spinors and photon polarization vectors.
- Instead, Casimir's Trick allows us to calculate spin-averaged amplitudes in terms of traces of γ-matrices.
- With practice, γ-matrix traces can be taken quite quickly.

Example: Electron-Muon Scattering



- Only one diagram,

$$\begin{aligned}\mathcal{M} &= i\,[\bar{u}_3(ig_e\gamma^\mu)u_1]\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)[\bar{u}_4(ig_e\gamma^\nu)u_2] \\ &= -\,\frac{g_e^2}{(p_1-p_3)^2}\,[\bar{u}_3\gamma^\mu u_1]\,[\bar{u}_4\gamma_\mu u_2]\end{aligned}$$

$$
\begin{aligned}
\mathcal{M} &= -\frac{g_e^2}{(p_1 - p_3)^2} \, [\bar{u}_3 \gamma^\mu u_1] \, [\bar{u}_4 \gamma_\mu u_2] \\
\left\langle |\mathcal{M}|^2 \right\rangle &= \frac{g_e^4}{4(p_1 - p_3)^4} \, \mathrm{Tr} \left[\gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_3 + m) \right] \\
&\qquad \times \mathrm{Tr} \left[\gamma_\mu (\not{p}_2 + M) \gamma_\nu (\not{p}_4 + M) \right] \\
&= \frac{g_e^4}{4(p_1 - p_3)^4} \left[4 \left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2 - p_1 \cdot p_3) g^{\mu\nu} \right) \right] \\
&\qquad \times \left[4 \left(p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + (M^2 - p_2 \cdot p_4) g_{\mu\nu} \right) \right] \\
&= \frac{4 g_e^4}{(p_1 - p_3)^4} \{ 2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) \\
&\qquad + 2m^2 (p_2 \cdot p_4) + 2M^2 (p_1 \cdot p_3) \\
&\qquad - 4(p_1 \cdot p_3)(p_2 \cdot p_4) \\
&\qquad + 4(m^2 - p_1 \cdot p_3)(M^2 - p_2 \cdot p_4) \big\}
\end{aligned}
$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{8g_e^4}{(p_1 - p_3)^4} \{(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - m^2(p_2 \cdot p_4) - M^2(p_1 \cdot p_3) + 2m^2 M^2\}$$

- So far, this is a very general result that can be applied to electron scattering off of any charged particle, except for another electron or positron (why?).
- What we will do now is impose a succession of approximations which will gradually convert this general expression to a more specialized result.

Mott Scattering

- Our first approximation is to assume that $M \gg m, E, \mathbf{p}$ and that the scattering takes place in the lab frame where M is at rest. We will neglect any recoil of the target.
- The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{(8\pi M)^2}$$

Mott Scattering: Kinematics I

- The four-momenta are

$$p_1 = (E, \mathbf{p}_1) \quad p_2 = (M, \mathbf{0}) \quad p_3 \simeq (E, \mathbf{p}_3) \quad p_4 \simeq (M, \mathbf{0})$$

- The momentum transfer is then

$$\begin{aligned}(p_1 - p_3)^2 &= (0, \mathbf{p}_1 - \mathbf{p}_3)^2 \\ &= -\mathbf{p}_1^2 - \mathbf{p}_3^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_3 \\ &= -2\mathbf{p}^2(1 - \cos\theta) \\ &= -4\mathbf{p}^2 \sin^2 \frac{\theta}{2}\end{aligned}$$

Mott Scattering: Kinematics II

- We also need to evaluate the various $(p_i \cdot p_j)$ factors in the spin-averaged amplitude.

$$
\begin{aligned}
(p_1 \cdot p_3) &= \left[p_1^2 + p_3^2 - (p_1 - p_3)^2\right]/2 \\
&= m^2 + 2\mathbf{p}^2 \sin^2\frac{\theta}{2} \\
(p_2 \cdot p_4) &= M^2 \\
(p_1 \cdot p_2) &= ME \\
(p_3 \cdot p_4) &= ME \\
(p_1 \cdot p_4) &= ME \\
(p_2 \cdot p_3) &= ME
\end{aligned}
$$

Mott Scattering: Amplitude

- Using the kinematic results, the spin-averaged amplitude is

$$
\begin{aligned}
\left\langle |\mathcal{M}|^2 \right\rangle &= \frac{8 g_e^4}{(p_1 - p_3)^4} \{ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \\
&\qquad - m^2 (p_2 \cdot p_4) - M^2 (p_1 \cdot p_3) + 2 m^2 M^2 \} \\
&= \frac{g_e^4}{2 \mathbf{p}^4 \sin^4 \frac{\theta}{2}} \{ 2 M^2 E^2 - m^2 M^2 \\
&\qquad - M^2 (m^2 + 2 \mathbf{p}^2 \sin^2(\theta/2)) + 2 m^2 M^2 \} \\
&= \left(\frac{g_e^2 M}{\mathbf{p}^2 \sin^2(\theta/2)} \right)^2 \{ E^2 - \mathbf{p}^2 \sin^2(\theta/2) \} \\
&= \left(\frac{g_e^2 M}{\mathbf{p}^2 \sin^2(\theta/2)} \right)^2 \{ m^2 + \mathbf{p}^2 \cos^2(\theta/2) \}
\end{aligned}
$$

Mott Scattering: $\frac{d\sigma}{d\Omega}$

- Substituting the spin-averaged amplitude into the appropriate expression for the differential cross section, we have

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \left(\frac{1}{8\pi M}\right)^2 \left(\frac{g_e^2 M}{\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 \{m^2 + \mathbf{p}^2 \cos^2(\theta/2)\} \\ &= \left(\frac{\alpha}{2\mathbf{p}^2 \sin^2(\theta/2)}\right)^2 \{m^2 + \mathbf{p}^2 \cos^2(\theta/2)\}\end{aligned}$$

- This is the Mott formula. It describes Coulomb scattering off a nucleus, so long as the scattering particle is not too heavy or energetic (i.e. $m, E, \mathbf{p} \ll M$). It assumes that the target is a point particle.

Rutherford Scattering Limit

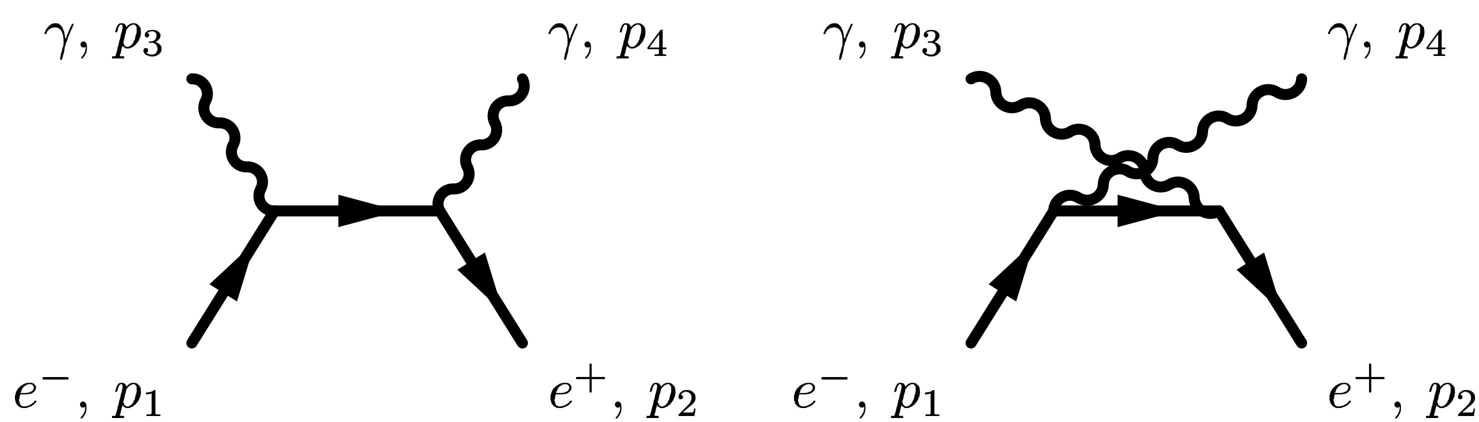
- If the incident particle is non-relativistic, we can simplify the Mott formula further:

$$\begin{aligned} \{m^2 + \mathbf{p}^2 \cos^2(\theta/2)\} &\rightarrow m^2 \\ \mathbf{p}^2 &\rightarrow 2mE \qquad (E \text{ is kinetic energy}) \\ \alpha &\rightarrow q_1 q_2 \qquad (\text{Gaussian units}) \end{aligned}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{q_1 q_2}{4E \sin^2(\theta/2)} \right)^2$$

- This is the Rutherford formula that we first saw in Chapter 6.

Example: Pair Annihilation



- No antisymmetrization $\Rightarrow\ \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$

$$\begin{aligned} \mathcal{M}_1 &= i\left[\bar{v}_2(ig_e\gamma^\mu)\left(\frac{i(\not{p}_1 - \not{p}_3 + m)}{(p_1-p_3)^2 - m^2}\right)(ig_e\gamma^\nu)u_1\right]\epsilon^*_{3\nu}\epsilon^*_{4\mu} \\ &= \frac{g_e^2}{(p_1-p_3)^2 - m^2}\left[\bar{v}_2\not{\epsilon}^*_4(\not{p}_1 - \not{p}_3 + m)\not{\epsilon}^*_3 u_1\right] \end{aligned}$$

$$
\begin{aligned}
\mathcal{M} \quad = \quad & \frac{g_e^2}{(p_1 - p_3)^2 - m^2} \left[\bar{v}_2 \not{\epsilon}_4^* (\not{p}_1 - \not{p}_3 + m) \not{\epsilon}_3^* u_1 \right] \\
& + \frac{g_e^2}{(p_1 - p_4)^2 - m^2} \left[\bar{v}_2 \not{\epsilon}_3^* (\not{p}_1 - \not{p}_4 + m) \not{\epsilon}_4^* u_1 \right]
\end{aligned}
$$

- From here, Griffiths proceeds immediately to positronium decay, wherein the incoming particles are actually bound together. He uses explicit forms for the spinors and polarization vectors and it's a mess. Go ahead and read through it if you like. We will obtain (the square of) his final result,

$$\mathcal{M}_{\text{singlet}} = -4 g_e^2$$

in an equally messy way which makes use of the tools we have developed.

Our Approach

- We will be calculating $\left\langle |\mathcal{M}|^2 \right\rangle$ using traces. This will obviate the need for explicit spinors.
- We will avoid imposing additional assumptions on the momenta until after we have calculated $\left\langle |\mathcal{M}|^2 \right\rangle$.
- We can simplify the denominator factors arising from the electron propagators:

$$
\begin{aligned}
(p_1 - p_3)^2 - m^2 &= p_1^2 + p_3^2 - 2(p_1 \cdot p_3) - m^2 \\
&= m^2 + 0 - 2(p_1 \cdot p_3) - m^2 \\
&= -2(p_1 \cdot p_3)
\end{aligned}
$$

Similarly, $$(p_1 - p_4)^2 - m^2 = -2(p_1 \cdot p_4)$$

Spin-Averaging

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_1 + \mathcal{M}_2 \\
\mathcal{M}_1 &= \frac{-g_e^2}{2(p_1 \cdot p_3)} \left[\bar{v}_2 \not{\epsilon}_4^* (\not{p}_1 - \not{p}_3 + m) \not{\epsilon}_3^* u_1 \right] \\
\mathcal{M}_2 &= \frac{-g_e^2}{2(p_1 \cdot p_4)} \left[\bar{v}_2 \not{\epsilon}_3^* (\not{p}_1 - \not{p}_4 + m) \not{\epsilon}_4^* u_1 \right]
\end{aligned}$$

- The spin-averaged amplitude will consist of three terms:

$$\begin{aligned}
\left\langle |\mathcal{M}|^2 \right\rangle &= \left\langle |\mathcal{M}_1 + \mathcal{M}_2|^2 \right\rangle \\
&= \left\langle |\mathcal{M}_1|^2 \right\rangle + \left\langle |\mathcal{M}_2|^2 \right\rangle + 2\mathrm{Re} \left\langle \mathcal{M}_1 \mathcal{M}_2^* \right\rangle
\end{aligned}$$

First Term

$$\mathcal{M}_1 = \frac{-g_e^2}{2(p_1 \cdot p_3)} \left[\bar{v}_2 \not{\epsilon}_4^* (\not{p}_1 - \not{p}_3 + m) \not{\epsilon}_3^* u_1\right]$$

$$\Rightarrow \left\langle |\mathcal{M}_1|^2 \right\rangle = \frac{g_e^4}{16(p_1 \cdot p_3)^2} \sum_{\text{pol.}} \epsilon_{4\mu}^* \epsilon_{3\nu}^* \epsilon_{3\rho} \epsilon_{4\sigma}$$

$$\times \mathrm{Tr}\left[\gamma^\mu (\not{p}_1 - \not{p}_3 + m) \gamma^\nu (\not{p}_1 + m) \gamma^\rho (\not{p}_1 - \not{p}_3 + m) \gamma^\sigma (\not{p}_2 - m)\right]$$

- To perform the sum over photon polarizations, we need the following completeness relation:

$$\sum_{\text{pol.}} \epsilon_\mu^* \epsilon_\nu = -g_{\mu\nu}$$

Note that this is just $(-i)$ times the numerator of the photon propagator. Similarly, $(-i)$ times the numerator of the electron propagator yields the spin sum $(\not{p} + m)$.

$$\left\langle |\mathcal{M}_1|^2 \right\rangle = \frac{g_e^4}{16(p_1 \cdot p_3)^2} (-g_{\mu\sigma})(-g_{\nu\rho})$$
$$\times \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 - \not{p}_3 + m)\gamma^\nu(\not{p}_1 + m)\gamma^\rho(\not{p}_1 - \not{p}_3 + m)\gamma^\sigma(\not{p}_2 - m)\right]$$

- At this stage, we have a few choices:
  1. Expand the brackets and evaluate 36 separate traces, some of which contain 8 γ-matrices. (Very stupid)
  2. Use the g-tensors to reduce the number of distinct indices in the trace to 2 and then apply various contraction identities of the form
  $$\gamma_\mu \Gamma \gamma^\mu = \Gamma'$$
  This leaves (still 36) traces which contain no more than 4 γ-matrices. (Slightly less stupid)
  3. Evaluate the traces on a computer. (Lazy but clever)

The Computer Says...

$$
\begin{aligned}
\left\langle |\mathcal{M}_1|^2 \right\rangle &= \frac{g_e^4}{16(p_1\cdot p_3)^2}(-g_{\mu\sigma})(-g_{\nu\rho}) \\
&\quad \times \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 - \not{p}_3 + m)\gamma^\nu(\not{p}_1 + m)\gamma^\rho(\not{p}_1 - \not{p}_3 + m)\gamma^\sigma(\not{p}_2 - m)\right] \\
&= -64m^4 + 16p_1^2(p_1\cdot p_2) - 32p_1^2(p_2\cdot p_3) \\
&\quad -16p_3^2(p_1\cdot p_2) - 48m^2(p_1\cdot p_2) + 32(p_1\cdot p_3)(p_2\cdot p_3) \\
&\quad +64m^2(p_1\cdot p_3) + 64m^2(p_2\cdot p_3) - 64m^2p_3^2
\end{aligned}
$$

- Simplify this further with

$$p_1^2 = p_2^2 = m^2 \qquad p_3^2 = p_4^2 = 0$$

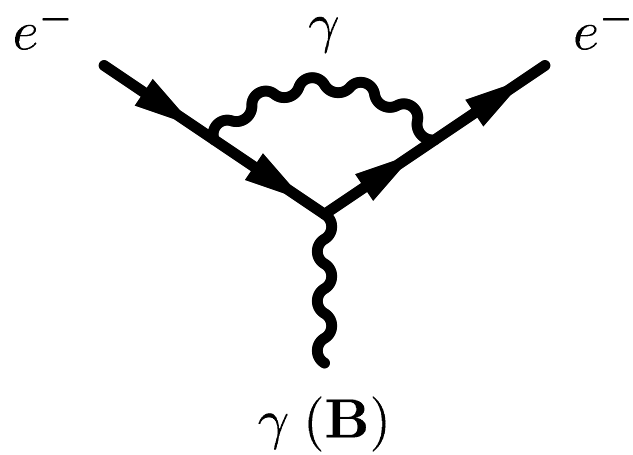
$$(p_2\cdot p_3) = (p_1\cdot p_4) \qquad (p_3\cdot p_4) = (p_1\cdot p_2) + m^2$$

The Other Terms

- In the same fashion, we can obtain the other two traces. At this stage, our result depends on m, $(p_1 \cdot p_2)$, $(p_1 \cdot p_3)$, and $(p_1 \cdot p_4)$ (but it's a little bit too long to show here).
- Everything we have done so far has been completely general; it applies just as well to annihilation events in a high-energy $e^+ \, e^-$ collider as it does to the low-energy $e^+ \, e^-$ bound state: positronium.
- The decay width of para-positronium can be derived from the spin-averaged amplitude determined here, but it would take too much class time to go through it.

Higher-Order Diagrams in QED

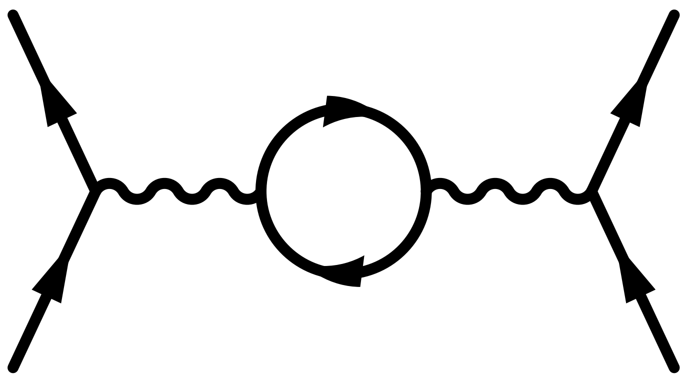
- The most famous higher-order process in QED is the anomalous magnetic moment of the electron (or muon), arising from the diagram



In 1948, Schwinger showed that this modifies the electron g-factor from 2 to $(2 + \alpha/\pi)$. It is currently known to α^4, corresponding to an uncertainty in g_e of about 10^{-12}.

Vacuum Polarization

- Recall from Chapter 5, that the Lamb Shift arises from *vacuum polarization* effects in QED:



Running of α

- Intuitively, we expect the electromagnetic force to strengthen at high energies (short distances), as two particles will see each other's unscreened charges more than at low energies. Quantitatively, the leading-order effect due to virtual $e^+ e^-$ pairs leads to

$$\alpha(|q^2|) = \frac{\alpha(0)}{1 - \left(\frac{\alpha(0)}{3\pi}\right) \ln\left(\frac{|q^2|}{m^2}\right)}$$

  Other types of virtual pairs modify this expression as various thresholds are passed.

- Experimentally, it was observed at LEP that

$$\alpha(M_W^2) \simeq \frac{1}{128}$$

Summary

- The Feynman rules for QED lead to a straightforward, albeit sometimes tedious, algorithm for calculating $\mathcal{M}$, as we saw in the case of $e\,\mu \to e\,\mu$ and $e^+\,e^- \to \gamma\,\gamma$.
- Once we calculate $\mathcal{M}$, we can then impose additional assumptions in order to get a specific physical result. We obtained the Mott and Rutherford formulas, as well as the spin-averaged pair annihilation matrix element.
- Higher-order QED diagrams reveal an even richer theory in which the vacuum has observable interactions with the particles we study.