Today: Electrodynamics of Quarks and Hadrons

- Elastic $e\ p$ scattering
- Inelastic $e\ p$ scattering

Slides from Sobie and Blokland

Elastic Electron-Proton Scattering

- This is our best probe of the internal structure of the proton.
- If the proton were structureless, we could simply recycle our result for electron-muon scattering:

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^4}{4(p_1 - p_3)^4} \left[4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2 - p_1 \cdot p_3) g^{\mu\nu}\right)\right] \times \left[4\left(p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + (M^2 - p_2 \cdot p_4) g_{\mu\nu}\right)\right]$$

$$= \frac{g_e^4}{q^4} L^{\mu\nu}_{\text{electron}} L_{\mu\nu\ \text{muon}}$$

with $q = p_1 - p_3$ and

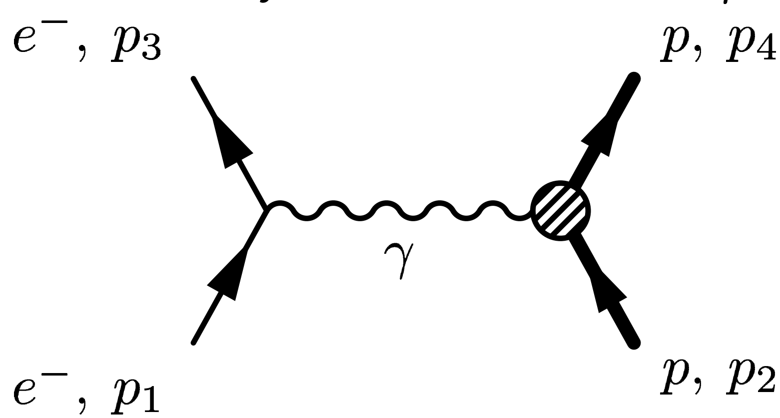
$$L^{\mu\nu}_{\text{electron}} = 2\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2 - p_1 \cdot p_3) g^{\mu\nu}\right)$$

But the Proton Isn't Structureless...

- Instead of just replacing $L_{\mu\nu\text{ muon}}$ with $L_{\mu\nu\text{ proton}}$, which assumes that the proton is a true point particle, we can generically account for proton structure via

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^4}{q^4} \, L^{\mu\nu}_{\text{electron}} \, K_{\mu\nu\text{ proton}}$$

- Notice that the implied proton structure does not affect the electron-photon coupling or the photon propagator. All of the complications are neatly stashed within $K_{\mu\nu}$. Pictorially,



So How Do We Calculate $K^{\mu\nu}_{\text{proton}}$?

- Even without assuming anything about the substructure of a proton, we know that $K^{\mu\nu}_{\text{proton}}$ is a second-rank tensor.
- In addition to $g^{\mu\nu}$, we can construct tensors from the four-vectors p_2, p_4, and q. Since $q = p_4 - p_2$, only 2 of these four-vectors are independent, from which we choose q and $p_2 = p$. Thus, our choices are

$$g^{\mu\nu} \qquad p^\mu p^\nu \qquad q^\mu q^\nu \qquad (p^\mu q^\nu + p^\nu q^\mu) \qquad (p^\mu q^\nu - p^\nu q^\mu)$$

  For electromagnetic interactions, $L^{\mu\nu}_{\text{electron}}$ is symmetric in μ and ν, therefore we need not include $(p^\mu q^\nu - p^\nu q^\mu)$. This term would be required for weak interactions (e.g., elastic *neutrino*-proton scattering).

Form Factors

- Using the four symmetric second-rank tensors available to us, we write

$$K^{\mu\nu}_{\rm proton} = -K_1 g^{\mu\nu} + \frac{K_2}{M^2}p^\mu p^\nu + \frac{K_4}{M^2}q^\mu q^\nu + \frac{K_5}{M^2}(p^\mu q^\nu + p^\nu q^\mu)$$

  where K_1, K_2, K_4, and K_5 are unknown functions which we refer to as *form factors*.

- The form factors can depend on q^2, the only scalar variable available to us, since $p^2 = M^2$ and $p \cdot q = -q^2/2$. (This last identity follows from squaring $p_4 = p_2 + q$ and recognizing that $p_4^2 = M^2$ for elastic scattering.)

Simplifying Things Further

- Using the *Ward identity*

$$q_\mu K^{\mu\nu}_{\rm proton} = 0$$

we find that there are only 2 independent form factors:

$$K^{\mu\nu}_{\rm proton} = K_1 \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{K_2}{M^2} \left(p^\mu + \frac{q^\mu}{2} \right) \left(p^\nu + \frac{q^\nu}{2} \right)$$

- The goal is then to measure these form factors experimentally and to try to calculate them theoretically.

Skipping a Bunch of Nasty Algebra...

- Working in the lab frame with the target proton at rest, we assume that the energy of the incident electron, E, is sufficiently large that m can be ignored. Both q^2 and the energy of the scattered electron, E', is fixed by the scattering angle:

$$E' = \frac{E}{1 + (2E/M)\sin^2(\theta/2)}$$

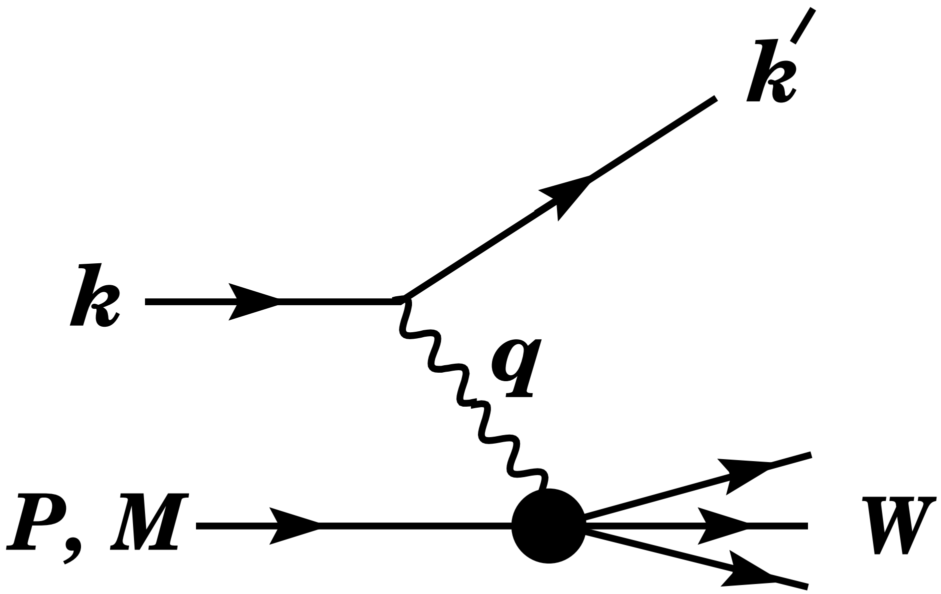
- We can obtain the differential cross section in terms of the form factors K_1 and K_2:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4ME\sin^2(\theta/2)}\right)^2 \frac{E'}{E}\left[2K_1\sin^2(\theta/2) + K_2\cos^2(\theta/2)\right]$$

This is the *Rosenbluth formula*.

Inelastic Electron-Proton Scattering

- If the incident electron is sufficiently energetic, it is quite unlikely that the proton will stay intact. Instead, we should be considering the more general *inelastic* process $e + p \to e + X$



Masking Our Ignorance

- As with the elastic case, we introduce a second-rank tensor $W_{\mu\nu}$ to describe the unknown details about the subprocess $\gamma + p \to X$. The electron vertex and the photon propagator are known, therefore
$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^4}{q^4} \, L^{\mu\nu}_{\text{electron}} W_{\mu\nu}(X)$$
- We can insert this spin-averaged amplitude into the Golden Rule for scattering in order to compute a differential cross section:
$$\begin{aligned} d\sigma &= \frac{S \left\langle |\mathcal{M}|^2 \right\rangle}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \\ &\times \left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \left(\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \right) \cdots \left(\frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n} \right) \\ &\times (2\pi)^4 \, \delta^4(p_1 + p_2 - p_3 - p_4 - \ldots - p_n) \end{aligned}$$

Inclusive Cross Sections

- It is not feasible to measure every single piece of hadronic shrapnel and to compute a cross section for each possible set of final state particles. Instead, we typically measure *only* the scattered electron.

- By integrating over *all* accessible final states X with *all* possible momenta, we obtain the *inclusive cross section*:

$$
\begin{aligned}
d\sigma &= \frac{g_e^4 \, L^{\mu\nu}}{4q^4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3}\right) 4\pi M W_{\mu\nu} \\
W_{\mu\nu} &\equiv \frac{1}{4\pi M}\sum_X \int W_{\mu\nu}(X) \left(\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4}\right) \cdots \left(\frac{d^3\mathbf{p}_n}{(2\pi)^3 2E_n}\right) \\
&\quad \times (2\pi)^4 \delta^4(q + p - p_4 - \ldots - p_n)
\end{aligned}
$$

$$d\sigma = \frac{g_e^4 \, L^{\mu\nu}}{4q^4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) 4\pi M W_{\mu\nu}$$

- With an initial electron energy of E, whose mass we will neglect,

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = ME$$

- The outgoing electron has energy E' and

$$\begin{aligned} \frac{d^3\mathbf{p}_3}{E_3} &= \frac{|\mathbf{p}_3|^2 d|\mathbf{p}_3| d\Omega}{E'} \\ &= E' \, dE' \, d\Omega \end{aligned}$$

- Substituting these results, the differential cross section simplifies to

$$\frac{d\sigma}{dE' \, d\Omega} = \frac{\alpha^2}{q^4} \, \frac{E'}{E} \, L^{\mu\nu} W_{\mu\nu}$$

$$\frac{d\sigma}{dE'\, d\Omega} = \frac{\alpha^2}{q^4}\, \frac{E'}{E}\, L^{\mu\nu} W_{\mu\nu}$$

- Note that, unlike elastic scattering, E' is *not* kinematically fixed by E and θ because the outgoing hadrons can have a range of masses. Equivalently, the total momentum,

$$p_{\rm total} = p_4 + p_5 + \ldots + p_n$$

is not constrained by the condition $p_{\rm total}^2 = M^2$

- This leaves us with 2 independent variables, for a given incident energy:

Experimentalist: $E'\,,\ \theta$

Theorist: $q^2\,,\ x \qquad \left(x \equiv -\frac{q^2}{2q \cdot p} \right)$

Many other choices exist (ν, y, Q^2, etc.).

Structure Functions

- From here, we proceed as with the elastic scattering case and write the most general tensor $W_{\mu\nu}$ that depends on q, p, and satisfies the Ward identity $q_\mu W^{\mu\nu}$.
- This leads to an expression for the differential cross section in terms of the two *structure functions* $W_1(q^2, x)$ and $W_2(q^2, x)$:

$$\frac{d\sigma}{dE'\, d\Omega} = \left(\frac{\alpha}{2E\sin^2(\theta/2)}\right)^2 \left[2W_1 \sin^2(\theta/2) + W_2 \cos^2(\theta/2)\right]$$

  The structure functions are the inelastic generalization of the elastic form factors $K_1(q^2)$ and $K_2(q^2)$.

Summary

- Electron-proton scattering experiments provide us with a great deal of information about the structure of the proton.
- For elastic electron-proton scattering, we can write the cross section (the Rosenbluth formula) in terms of 2 form factors: $K_1(q^2)$ and $K_2(q^2)$.
- For inelastic electron-proton scattering, we write an inclusive cross section in terms of 2 structure functions: $W_1(q^2, x)$ and $W_2(q^2, x)$.