Weak Interactions

- Feynman Rules for the W Bosons
- Muon Decay
- Fermi's Effective Theory of the Weak Interaction

Slides from Sobie and Blokland

Intermediate Vector Bosons

- Like QED and QCD, the weak interaction is mediated by spin-1 (vector) particle exchange.
- Unlike the photon and gluons, the weak mediators are *massive*:

$$M_W = 80.425(38)\ \text{GeV} \qquad M_Z = 91.1876(21)\ \text{GeV}$$

  This means that the longitudinal polarization mode is available, for a total of 3 independent polarizations.

Propagator for the W and Z Bosons

- We need to generalize the photon propagator

$$-\frac{ig_{\mu\nu}}{q^2}$$

in order to account for the mass, M, of the intermediate vector bosons.

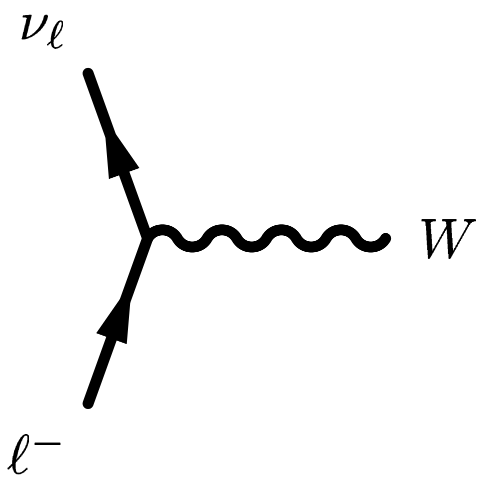
- This problem is more subtle than it looks. We will use the so-called *unitary gauge* propagator:

$$\frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{M^2}\right)}{q^2 - M^2}$$

Charged Current Vertex

- The W bosons mediate charged current (CC) weak interactions. They couple to leptons via

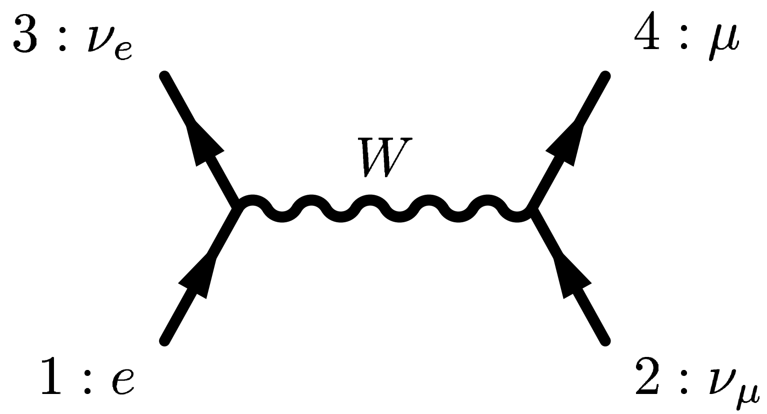
$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)$$



We will look at weak interactions involving quarks soon.

- Notice that this interaction mixes vector (γ^μ) and axial vector $(\gamma^\mu\gamma^5)$ terms. We call this a $V-A$ interaction and it leads to parity violation.

Example: $\nu_\mu + e^- \to \mu^- + \nu_e$



$$\mathcal{M} = i\left[\bar{u}_3\left(\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)\right)u_1\right]\left(\frac{-i\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{M_W^2}\right)}{q^2-M_W^2}\right)$$

$$\times\left[\bar{u}_4\left(\frac{-ig_w}{2\sqrt{2}}\gamma^\nu(1-\gamma^5)\right)u_2\right]$$

$$\xrightarrow{q^2 \ll M_W^2} \frac{g_w^2}{8M_W^2}\left[\bar{u}_3\gamma^\mu(1-\gamma^5)u_1\right]\left[\bar{u}_4\gamma_\mu(1-\gamma^5)u_2\right]$$

II: $\nu_\mu + e^- \to \mu^- + \nu_e$

$$\mathcal{M} = \frac{g_w^2}{8M_W^2}\left[\bar{u}_3\gamma^\mu(1-\gamma^5)u_1\right]\left[\bar{u}_4\gamma_\mu(1-\gamma^5)u_2\right]$$

$$\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{2}\left(\frac{g_w^2}{8M_W^2}\right)^2 \mathrm{Tr}\left[\gamma^\mu(1-\gamma^5)(\not{p}_1+m_e)\gamma^\nu(1-\gamma^5)\not{p}_3\right]$$
$$\times\, \mathrm{Tr}\left[\gamma_\mu(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5)(\not{p}_4+m_\mu)\right]$$

- Note the leading factor of $\frac{1}{2}$ (electrons have 2 spins and the neutrino has 1 spin).
- To evaluate the traces, it helps to bring the $(1-\gamma^5)$ factors together first:

$$\begin{aligned}(1-\gamma^5)\not{p}_2\gamma_\nu(1-\gamma^5) &= (1-\gamma^5)\not{p}_2(1+\gamma^5)\gamma_\nu \\ &= (1-\gamma^5)(1-\gamma^5)\not{p}_2\gamma_\nu \\ &= 2(1-\gamma^5)\not{p}_2\gamma_\nu\end{aligned}$$

We do the same thing for the first trace, wrapping around so as to avoid the m_e term.

III: $\nu_\mu + e^- \to \mu^- + \nu_e$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2\left(\frac{g_w^2}{8M_W^2}\right)^2 \text{Tr}\left[\gamma^\mu(1-\gamma^5)(\not p_1 + m_e)\gamma^\nu \not p_3\right] \times \text{Tr}\left[\gamma_\mu(1-\gamma^5)\not p_2 \gamma_\nu(\not p_4 + m_\mu)\right]$$

- The m-dependent terms do not contribute to these traces, so we have

$$\begin{aligned}\left\langle |\mathcal{M}|^2 \right\rangle &= \frac{g_w^4}{2M_W^4}\left[p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1 \cdot p_3)g^{\mu\nu} - i\epsilon^{\mu\nu\lambda\sigma}p_{1\lambda}p_{3\sigma}\right] \\ &\quad \times \left[p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - (p_2 \cdot p_4)g_{\mu\nu} - i\epsilon_{\mu\nu\kappa\tau}p_2^\kappa p_4^\tau\right] \\ &= \frac{2g_w^4}{M_W^4}(p_1 \cdot p_2)(p_3 \cdot p_4)\end{aligned}$$

For the last step, remember that:

$$\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\kappa\tau} = -2\left(\delta_\kappa^\lambda \delta_\tau^\sigma - \delta_\tau^\lambda \delta_\kappa^\sigma\right)$$

IV: $\nu_\mu + e^- \to \mu^- + \nu_e$

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{2g_w^4}{M_W^4}(p_1 \cdot p_2)(p_3 \cdot p_4)$$

- In the CM frame and neglecting the mass of the electron,

$$\begin{aligned}
(p_1 \cdot p_2) &= \left[(p_1 + p_2)^2 - p_1^2 - p_2^2\right]/2 \\
&= \left[(2E)^2 - 0 - 0\right]/2 \\
&= 2E^2 \\
(p_3 \cdot p_4) &= \left[(p_3 + p_4)^2 - p_3^2 - p_4^2\right]/2 \\
&= \left[(p_1 + p_2)^2 - 0 - m_\mu^2\right]/2 \\
&= \left[4E^2 - m_\mu^2\right]/2 \\
&= 2E^2\left[1 - \left(\frac{m_\mu}{2E}\right)^2\right]
\end{aligned}$$

$$\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle = \frac{8g_w^4 E^4}{M_W^4}\left[1 - \left(\frac{m_\mu}{2E}\right)^2\right]$$

V: $\nu_\mu + e^- \to \mu^- + \nu_e$

$$\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle = \frac{8 g_w^4 E^4}{M_W^4} \left[1 - \left(\frac{m_\mu}{2E} \right)^2 \right]$$

- We can quickly convert this to a cross section using Fermi's Golden Rule:

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi} \right)^2 \frac{S \left\langle |\mathcal{M}|^2 \right\rangle}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}$$

It takes a bit of effort to show that

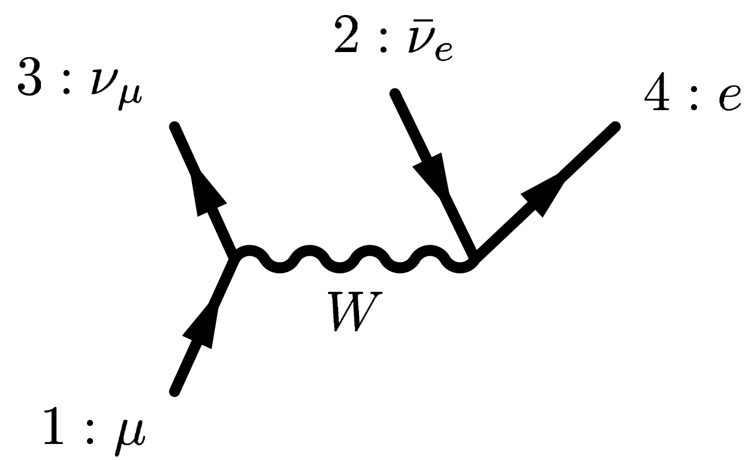
$$\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} = \left[1 - \left(\frac{m_\mu}{2E} \right)^2 \right]$$

but from this point on, the calculation is trivial:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} \left(\frac{g_w^2 E}{4\pi M_W^2} \right)^2 \left[1 - \left(\frac{m_\mu}{2E} \right)^2 \right]^2$$

$$\sigma = \frac{1}{8\pi} \left(\frac{g_w^2 E}{M_W^2} \right)^2 \left[1 - \left(\frac{m_\mu}{2E} \right)^2 \right]^2$$

Muon Decay $\mu^- \rightarrow e^- + \nu_\mu + \overline{\nu}_e$



- Again, working in the limit of $q^2 \ll M_W^2$, the amplitude is

$$\mathcal{M} \quad = \quad \frac{g_w^2}{8M_W^2} \left[\bar{u}_3 \gamma^\mu (1-\gamma^5) u_1\right] \left[\bar{u}_4 \gamma_\mu (1-\gamma^5) v_2\right]$$

- This is identical to the amplitude in the previous example, except for $u_2 \rightarrow v_2$, but since either spinor gives us $\not{p}_2$ in the trace (since $m_\nu = 0$), we can recycle the result

$$\left\langle |\mathcal{M}|^2 \right\rangle \quad = \quad \frac{2g_w^4}{M_W^4} (p_1 \cdot p_2)(p_3 \cdot p_4)$$

Preliminary Kinematics

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{2g_w^4}{M_W^4}(p_1 \cdot p_2)(p_3 \cdot p_4)$$

- Since the kinematics of μ decay are different from those of $e\,\nu_\mu$ scattering, we will need to start our work here.
- In the muon rest frame, $(p_1 \cdot p_2) = m_\mu E_2$, and

$$\begin{aligned}(p_3 \cdot p_4) &= \left[(p_3 + p_4)^2 - p_3^2 - p_4^2\right]/2 \\ &= \left[(p_1 - p_2)^2 - 0 - 0\right]/2 \\ &= \left[p_1^2 + p_2^2 - 2p_1 \cdot p_2\right]/2 \\ &= m_\mu(m_\mu - 2E_2)/2\end{aligned}$$

$$\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_w^4}{M_W^4}\, m_\mu^2 E_2(m_\mu - 2E_2)$$

Fermi's Golden Rule

- Since $\left\langle |\mathcal{M}|^2 \right\rangle$ (via E_2) depends nontrivially on θ, we will have to work out the decay rate from scratch with Fermi's Golden Rule:

$$d\Gamma_\mu = \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{2m_\mu} \left(\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \right) \left(\frac{d^3\mathbf{p}_3}{(2\pi)^3 2E_3} \right) \left(\frac{d^3\mathbf{p}_4}{(2\pi)^3 2E_4} \right) \times (2\pi)^4 \, \delta^4(p_1 - p_2 - p_3 - p_4)$$

- As always, we gather the factors of 2 and π, and we use the spatial parts of the δ-function to do one of the integrals. We'll do $\mathbf{p}_3$ first, since $\left\langle |\mathcal{M}|^2 \right\rangle$ depends on E_2 and we'll want our penultimate result to depend on the electron energy E_4

$$d\Gamma_\mu = \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{16(2\pi)^5 m_\mu} \frac{d^3\mathbf{p}_2 \, d^3\mathbf{p}_4}{E_2 E_3 E_4} \, \delta(m_\mu - E_2 - E_3 - E_4)$$

The Next Integral

- Next, we will integrate over $\mathbf{p}_2$. Working in spherical coordinates,

$$d^3\mathbf{p}_2 = |\mathbf{p}_2|^2 \, d|\mathbf{p}_2| \, d\Omega = E_2^2 \, dE_2 \ \sin\theta \, d\theta \, d\phi$$

  where θ is the angle between particles 2 and 4 ($\bar{\nu}_e$ and e)

$$\Rightarrow d\Gamma_\mu \quad = \quad \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{(4\pi)^4 m_\mu} \frac{E_2 \, dE_2 \ \sin\theta \, d\theta \, d^3\mathbf{p}_4}{E_3 E_4} \delta(m_\mu - E_2 - E_3 - E_4)$$

  where E_3 represents $\sqrt{E_2^2 + E_4^2 + 2E_2 E_4 \cos\theta}$.

- We could either use the chain rule to evaluate the θ-integral, or we can search for a miraculous change of variables to save the day.

The Miraculous Change of Variables

- It's staring right at us:

$$\begin{aligned} E_3 &= \sqrt{E_2^2 + E_4^2 + 2E_2E_4\cos\theta} \\ \Rightarrow dE_3 &= -\frac{E_2E_4 \sin\theta\, d\theta}{E_3} \end{aligned}$$

- Let's rewrite E_3 as x. We have to be a little bit careful here, because

$$\int_0^\pi \frac{\sin\theta\, d\theta}{E_3}\,\delta(m_\mu - E_2 - E_3 - E_4) = \frac{1}{E_2E_4}\int_{x-}^{x+} \delta(m_\mu - E_2 - E_4 - x)\, dx$$

Conservation of Energy: Algebraic

- From $x = \sqrt{E_2^2 + E_4^2 + 2E_2E_4 \cos\theta}$, we have

$$\begin{aligned} x_\pm &= \sqrt{E_2^2 + E_4^2 \pm 2E_2E_4} \\ &= |E_2 \pm E_4| \end{aligned}$$

As a result, the x-integral is

$$\frac{1}{E_2E_4}\int_{x-}^{x+} \delta(m_\mu - E_2 - E_4 - x)\, dx = \frac{1}{E_2E_4}$$

provided that

$$\begin{array}{ccccc} |E_2 - E_4| & < & (m_\mu - E_2 - E_4) & < & E_2 + E_4 \\ |E_2 - E_4| + E_2 + E_4 & < & m_\mu & < & 2(E_2 + E_4) \\ \max\{E_2, E_4\} & < & \frac{1}{2}m_\mu & < & (E_2 + E_4) \end{array}$$

Conservation of Energy: Intuitive

$$\max\{E_2, E_4\} \quad < \quad \tfrac{1}{2}m_\mu \quad < \quad (E_2 + E_4)$$

$$\Rightarrow \left\{ \begin{array}{r} E_2 < \frac{1}{2}m_\mu \\ E_4 < \frac{1}{2}m_\mu \\ (E_2 + E_4) > \frac{1}{2}m_\mu \end{array} \right\}$$

- Since all three final-state particles are assumed to be massless, energy and (3-) momentum are the same.
- The sum of the momenta for the 3 final-state particles must be zero, therefore no single particle can have *more than half* of the available energy and no two particles can have *less than half* of the available energy.

Solving the E_2 Integral

- Returning to the decay rate,

$$
\begin{aligned}
d\Gamma_\mu &= \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{(4\pi)^4 m_\mu} \frac{E_2 \, dE_2 \; \sin\theta \; d\theta \; d^3\mathbf{p}_4}{E_3 E_4} \, \delta(m_\mu - E_2 - E_3 - E_4) \\
&= \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{(4\pi)^4 m_\mu} \frac{d^3\mathbf{p}_4}{E_4^2} \, dE_2
\end{aligned}
$$

where, if E_4 ranges from 0 to $\frac{1}{2}m_\mu$, then E_2 runs from $\frac{1}{2}m_\mu - E_4$ to $\frac{1}{2}m_\mu$ (i.e., work from the outside in). Now we can substitute for $\left\langle |\mathcal{M}|^2 \right\rangle$ and evaluate the E_2 integral.

Electron energy spectrum

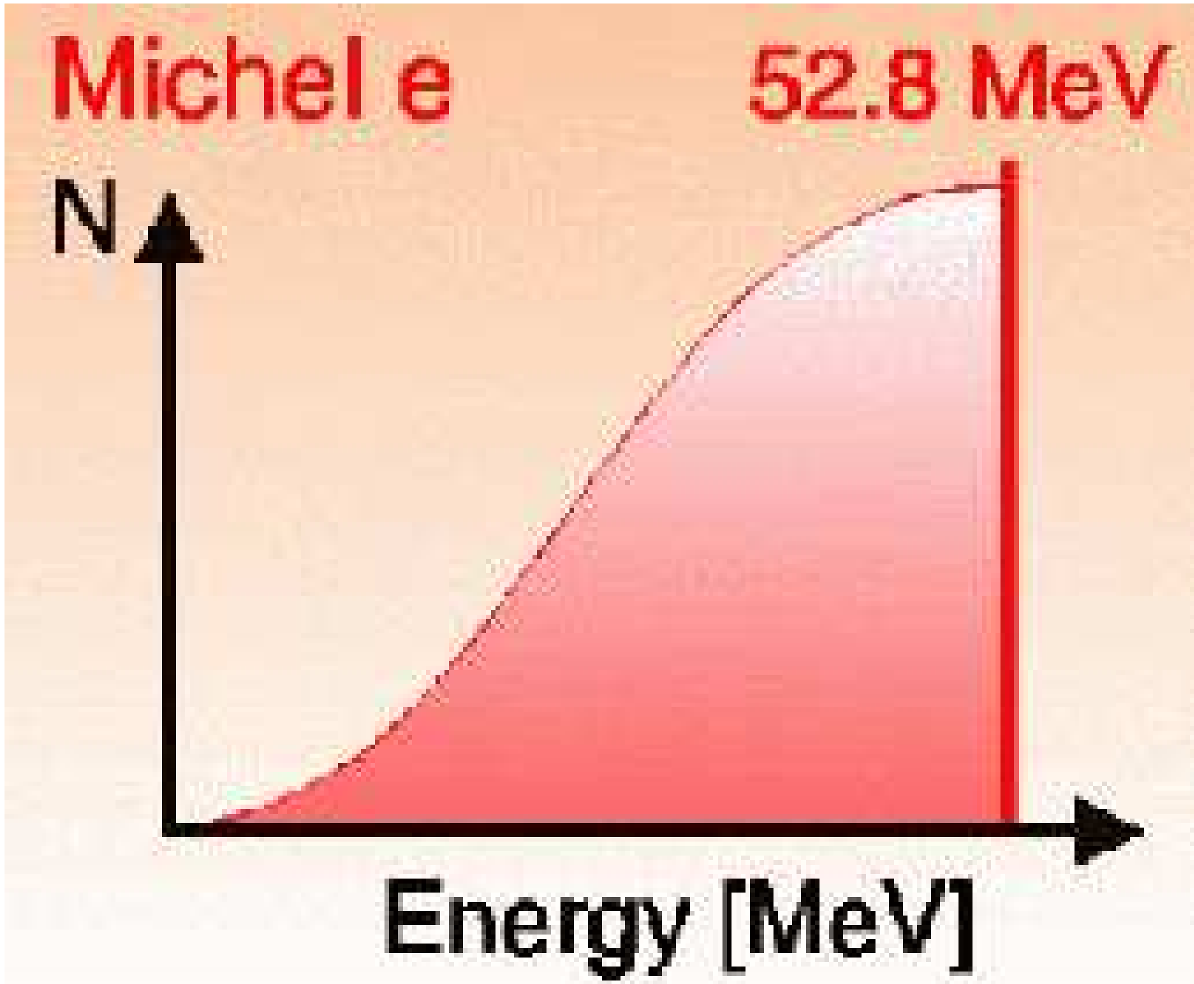
$$
\begin{aligned}
d\Gamma_\mu &= \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{(4\pi)^4 m_\mu} \frac{d^3\mathbf{p}_4}{E_4^2}\, dE_2 \\
&= \left(\frac{g_w}{4\pi M_W}\right)^4 \frac{m_\mu\, d^3\mathbf{p}_4}{E_4^2} \int_{m_\mu/2-E_4}^{m_\mu/2} E_2(m_\mu - 2E_2)\, dE_2 \\
&= \left(\frac{g_w}{4\pi M_W}\right)^4 \frac{m_\mu\, d^3\mathbf{p}_4}{E_4^2} \left(\frac{m_\mu E_4^2}{2} - \frac{2E_4^3}{3}\right)
\end{aligned}
$$

- Writing $d^3\mathbf{p}_4$ in spherical coordinates and integrating over the angles, we have

$$
\begin{aligned}
d\Gamma_\mu &= 4\pi \left(\frac{g_w}{4\pi M_W}\right)^4 m_\mu\, dE_4 \left(\frac{m_\mu E_4^2}{2} - \frac{2E_4^3}{3}\right) \\
\frac{d\Gamma_\mu}{dE} &= \left(\frac{g_w}{M_W}\right)^4 \frac{m_\mu^2 E^2}{2(4\pi)^3} \left(1 - \frac{4E}{3m_\mu}\right)
\end{aligned}
$$

This describes the electron-energy spectrum of muon decay.

$$\frac{d\Gamma_\mu}{dE} \sim E^2 \left(1 - \frac{4E}{3m_\mu}\right) \qquad (E \leq m_\mu/2)$$



Electron energy spectrum data

$$\frac{d\Gamma_\mu}{dE} \sim E^2 \left(1 - \frac{4E}{3m_\mu}\right) \qquad\qquad (E \leq m_\mu/2)$$

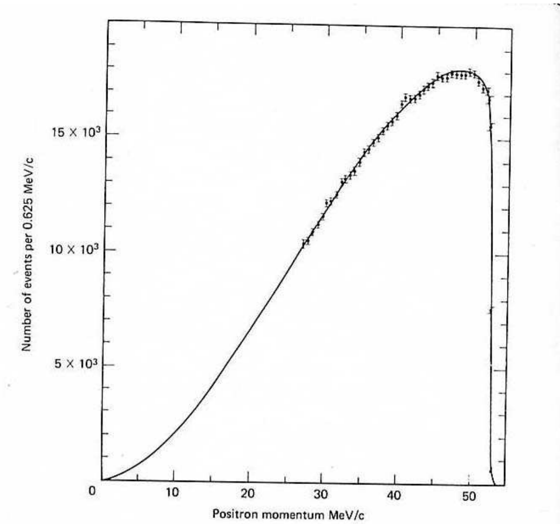


Figure 10.1 Experimental spectrum of positrons in $\mu^+ \to e^+ + \nu_e + \nu_\mu$. The solid line is the theoretically predicted spectrum based on equation (10.35), corrected for electromagnetic effects. (*Source:* M. Bardon et al., *Phys. Rev. Lett.* **14,** 449 (1965).)

The Muon Decay Rate

- Integrating over the electron energy, we (finally) obtain the muon decay rate:

$$
\begin{aligned}
\Gamma_\mu &= \left(\frac{g_w}{M_W}\right)^4 \frac{m_\mu^2}{2(4\pi)^3} \int_0^{m_\mu/2} E^2 \left(1 - \frac{4E}{3m_\mu}\right) \, dE \\
&= \left(\frac{g_w}{M_W}\right)^4 \frac{m_\mu^2}{2(4\pi)^3} \left(\frac{m_\mu^3}{48}\right) \\
&= \frac{1}{6144\pi^3} \left(\frac{g_w}{M_W}\right)^4 m_\mu^5
\end{aligned}
$$

- This is exactly as we predicted! (Up to the minor cosmetic factor of $1/6144\pi^3$.)

The Fermi Coupling Constant

- In the limit of $q^2 \ll M_W^2$, our results always depend on the *ratio* of g_w and M_W, and not the two constants separately.
- Define the *Fermi coupling constant*, G_F, by

$$G_F = \frac{\sqrt{2} g_w^2}{8 M_W^2}$$

This allows us to write the muon lifetime as

$$\tau_\mu = \frac{192\pi^3}{G_F^2 m_\mu^5}$$

Using τ_μ and m_μ, we actually determine G_F from this equation:

$$G_F = 1.16637(1) \times 10^{-5}\ \mathrm{GeV}^{-2}$$

How Weak is the Weak Interaction?

- With the muon lifetime measurement giving us

$$G_F = \frac{\sqrt{2} g_w^2}{8 M_W^2} = 1.166 \times 10^{-5} \text{ GeV}^{-2}$$

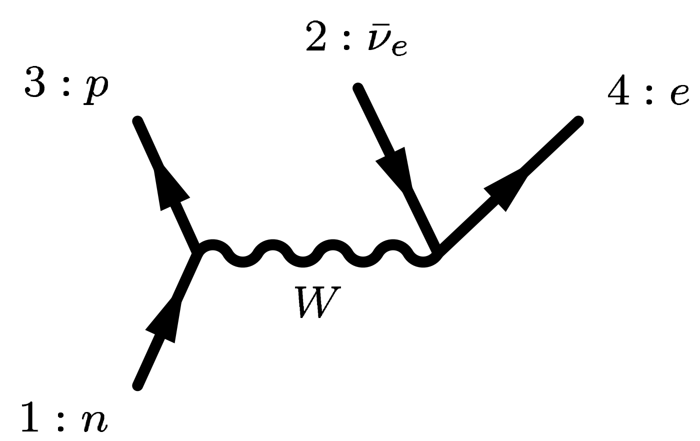
we can use the W mass measurement $M_W = 80.4$ GeV to determine g_w. The result,

$$g_w = 0.65 \qquad\qquad \Rightarrow \alpha_w = \frac{g_w^2}{4\pi} = \frac{1}{29}$$

indicates that the weak interaction is *inherently stronger* than the electromagnetic interaction! It is only the suppression factor E^2/M_W^2 which makes the weak force seem so feeble.

Neutron Decay

- Insofar that neutron substructure doesn't come into play, we could model neutron decay as a weak interaction process much like muon decay:



- In muon decay, all 3 final-state particles (ν_μ, $\bar{\nu}_e$, and e) are essentially massless.

Kinematics of Neutron Decay

- In neutron decay, the proton mass is obviously quite large. In addition, the mass of the electron (0.5 MeV) is a significant fraction of the neutron-proton mass difference (1.3 MeV), so we cannot ignore m_e.
- As a result, the phase-space calculation for neutron decay is more difficult than that of muon decay. Consult Griffiths if you would like to see the details.
- Using a pure $V - A$ vertex factor, we obtain a neutron lifetime of

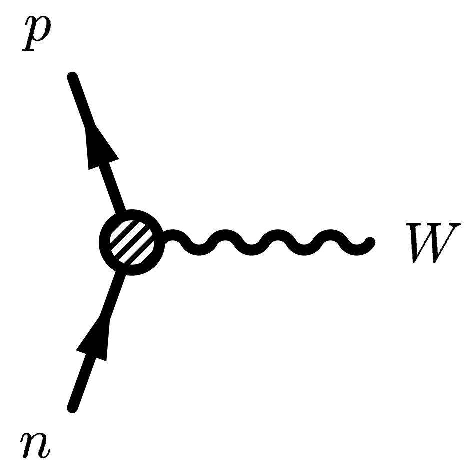
$$\tau_n = 1316 \text{ s}$$

- The experimentally measured value is 886 s (about 15 min).

Effects of Substructure

- We should generalize the $n\,p\,W$ vertex to

$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(c_V - c_A\gamma^5)$$

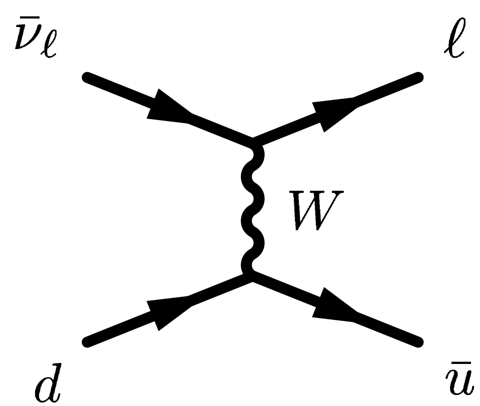


- Experiments indicate that

c_V	$=$	$1.000(3)$	CVC: Conserved Vector Current
c_A	$=$	$1.26(2)$	PCAC: Partially Conserved Axial Current

Pion Decay

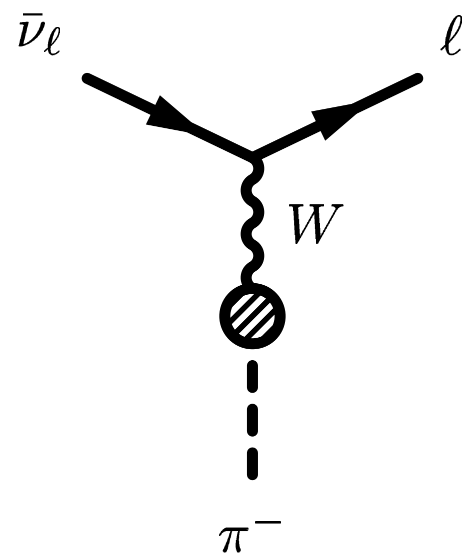
- While the π^0 $(u\bar{u} - d\bar{d})$ decays to $\gamma + \gamma$ via an electromagnetic interaction, the charged pions ($u\bar{d}$ and $d\bar{u}$) decay to a lepton-neutrino pair through the weak interaction.
- In some respects, π^- decay can be regarded as a scattering process:



- Although we know how the W couples to quarks (and we will look at this soon), we would eventually need to know $|\psi(0)|^2$ to calculate τ_π.

An Alternative Approach

- If we're going to have some unknown factor appearing in our results, we should make the rest of the calculation as simple as possible. Let's model π^- decay by:



where the $\pi\, W$ interaction at the blob is described by the vertex factor

$$\frac{-i g_w}{2\sqrt{2}}\, f_\pi p_\pi^\mu$$

The Pion Decay Amplitude I

- With our ansatz for the $\pi\, W$ interaction, the amplitude is

$$\mathcal{M} = \frac{g_w^2 f_\pi}{8M_W^2}\, p_{1\mu} \left[\bar{u}_3 \gamma^\mu (1-\gamma^5) v_2\right]$$

- Since the pion is a spin-0 particle, f_π can only depend on the pion momentum, p_1. The only scalar we can make from p_1 is $p_1^2 = m_\pi^2$, so f_π is, in fact, constant!
- We call f_π the *pion decay constant*, and experiments suggest that

$$f_\pi \simeq 93 \text{ MeV}$$

The Pion Decay Amplitude II

$$\mathcal{M} = \frac{g_w^2 f_\pi}{8M_W^2} p_{1\mu} \left[\bar{u}_3 \gamma^\mu (1-\gamma^5) v_2\right]$$

$$\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle = \left(\frac{g_w^2 f_\pi}{8M_W^2}\right)^2 p_{1\mu} p_{1\nu}$$
$$\times \operatorname{Tr}\left[\gamma^\mu (1-\gamma^5) \not{p}_2 \gamma^\nu (1-\gamma^5)(\not{p}_3 + m_\ell)\right]$$
$$= \left(\frac{g_w^2 f_\pi}{8M_W^2}\right)^2 p_{1\mu} p_{1\nu} \operatorname{Tr}\left[\gamma^\mu (1-\gamma^5)^2 \not{p}_2 \gamma^\nu (\not{p}_3 + m_\ell)\right]$$

- We start by using $(1-\gamma^5)^2 = 2(1-\gamma^5)$. The m_ℓ terms will not contribute, as they all involve traces of an odd # of γ-matrices. Finally, the ϵ-tensor produced by the trace involving γ^5 will vanish when contracted with $p_{1\mu} p_{1\nu}$.

$$\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle = \left(\frac{g_w^2 f_\pi}{8M_W^2}\right)^2 p_{1\mu} p_{1\nu}\, 8 \left[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - (p_2 \cdot p_3) g^{\mu\nu}\right]$$

The Pion Decay Amplitude III

$$\left\langle |\mathcal{M}|^2 \right\rangle = \left(\frac{g_w^2 f_\pi}{8M_W^2}\right)^2 p_{1\mu} p_{1\nu}\, 8\left[p_2^\mu p_3^\nu + p_2^\nu p_3^\mu - (p_2 \cdot p_3) g^{\mu\nu}\right]$$
$$= \frac{1}{8}\left(\frac{g_w^2 f_\pi}{M_W^2}\right)^2 \left[2(p_1 \cdot p_2)(p_1 \cdot p_3) - m_\pi^2 (p_2 \cdot p_3)\right]$$

- We can evaluate the various dot products by using $p_1 = p_2 + p_3$:

$$\begin{aligned} p_1^2 &= (p_2 + p_3)^2 \\ m_\pi^2 &= m_\ell^2 + 2(p_2 \cdot p_3) \\ \Rightarrow (p_2 \cdot p_3) &= (m_\pi^2 - m_\ell^2)/2 \\ \text{Similarly, } (p_1 \cdot p_2) &= (m_\pi^2 - m_\ell^2)/2 \\ (p_1 \cdot p_3) &= (m_\pi^2 + m_\ell^2)/2 \end{aligned}$$

$$\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{16}\left(\frac{g_w^2 f_\pi}{M_W^2}\right)^2 m_\ell^2 (m_\pi^2 - m_\ell^2)$$

The Pion Decay Rate I

- From Fermi's Golden Rule,

$$\Gamma = \frac{|\mathbf{p}_f|\,|\mathcal{M}|^2}{8\pi m_\pi^2}$$

- With $|\mathbf{p}_f|$ equalling the neutrino energy,

$$\begin{aligned} |\mathbf{p}_f| &= E_2 \\ &= (p_1 \cdot p_2)/m_\pi \\ &= (m_\pi^2 - m_\ell^2)/2m_\pi \end{aligned}$$

$$\Rightarrow \Gamma = \frac{f_\pi^2}{\pi m_\pi^3} \left(\frac{g_w}{4M_W}\right)^4 m_\ell^2 (m_\pi^2 - m_\ell^2)^2$$

The Pion Decay Rate II

$$\Gamma = \frac{f_\pi^2}{m_\pi^3}\left(\frac{g_w}{4M_W}\right)^4 m_\ell^2(m_\pi^2 - m_\ell^2)^2$$

- The otherwise unknown f_π can be extracted from this expression.
- Better yet, let's compare the π^- decay rates to electrons and muons so as to cancel f_π:

$$\frac{\Gamma(\pi^- \to e^- + \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \simeq 10^{-4}$$

  Surprisingly, the muon mode is heavily favored in spite of the smaller phase space available.
- The suppression of the electron mode can be understood in terms of angular momentum.

What About Quarks?

- For leptons, the W couples within a particular generation:

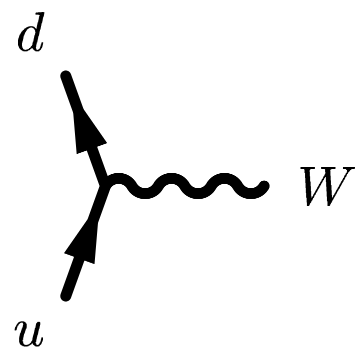
$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \qquad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \qquad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$$

- Things are more complicated for quarks, as the W couplings can mix generations:

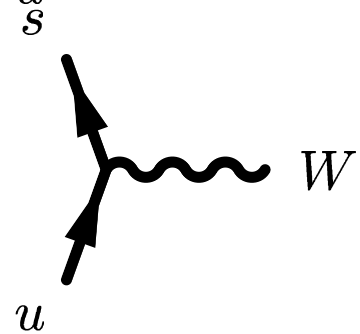
$$\begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{pmatrix} c \\ s \end{pmatrix} \qquad \begin{pmatrix} t \\ b \end{pmatrix}$$

Cabibbo angle

$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)cos\theta_C$$



$$\frac{-ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5)sin\theta_C$$



$$\Gamma(\pi^- \to l^-\overline{\nu}_\mu) = \frac{f_\pi^2}{\pi m_\pi^3}\left(\frac{g_w}{4M_W}\right)^4 m_l^2(m_\pi^2 - m_l^2)cos^2\theta_c$$

$$\Gamma(K^- \to l^-\overline{\nu}_\mu) = \frac{f_K^2}{\pi m_K^3}\left(\frac{g_w}{4M_W}\right)^4 m_l^2(m_K^2 - m_l^2)sin^2\theta_c$$

Kaon and pion decays

The ratio of the widths

$$\frac{\Gamma(K^- \to l^-\overline{\nu}_\mu)}{\Gamma(\pi^- \to l^-\overline{\nu}_\mu)} = \frac{f_K^2}{f_\pi^2}\,\frac{m_\pi^3}{m_K^3}\left(\frac{m_K^2 - m_l^2}{m_\pi^2 - m_l^2}\right) tan^2\theta_c$$

The lifetimes of the pion and kaon are 2.60×10^{-8} and 1.24×10^{-8} seconds, respectively.

The branching ratios for $\pi^- \to \mu^-\overline{\nu}_\mu$ is 100% and $K^- \to \mu^-\overline{\nu}_\mu$ is 64%.

We need other measurements to get $f_\pi = 132$ MeV and $f_K = 160$ MeV.

Result is $\theta_c = 13.1$ degrees and $cos\theta_c \; = \; 0.974$

Hence $d' = 0.97d \; + \; 0.23s$

Summary of charged-weak interaction

- The $W^{\pm}$ bosons mediate charged weak interactions and the Feynman rules incorporate the mass of the W and the parity-violating nature of the weak interaction.
- With these Feynman rules, we are able to calculate the lifetime of the muon.
- At low energies, the relevant weak interaction parameter is the Fermi coupling constant G_F.
- The weak interaction is not weak because the coupling constant is small, but rather because of the large mass of the virtual W which must be exchanged.
- The W boson is responsible for both neutron decay and the decay of the charged pion.