Physics 214 **INTRODUCTION TO LABORATORY ELECTRONICS** **Lecture 7**

Topics: AC circuit theory: complex numbers, phases and RC, RL series impedance

Text: Bugg, Sections 6.1-6.6

COMPLEX NUMBERS: used to describe sinusoidal waveforms. Contain information about both the amplitude and phase. Make for easy handling of impedances of capacitors and inductors where amplitudes and phases are important.

Uses so-called ***imaginary numbers.*** These are numbers which are multiples of $j=\sqrt{-1}$. (j^2 = -1). E.g. 5j or -0.223j or πj . The square of these numbers are: -25, -0.050^2, -(π^2). ***Complex numbers*** have both a ***real*** part and an ***imaginary*** part. e.g. for 3+5j, Re(3-5j)=3, Im(3-5j) = -5.

Complex Plane: much like a number line in two dimensions. The x-axis is the number line for the Real part, the y-axis is the number line for the Imaginary part. Any complex number is represented by a point on the complex plane.

Complex Arithmetic:

$z = a + jb$ and $Z = A + jB$

then

$$\mathrm{z} + \mathrm{Z} = (a + A) + j(b + B)$$

$$z - Z = (a - A) + j(b - B)$$

$$z \times Z = (a + jb) \times (A + jB) = (aA - bB) + j(bA + aB)$$

$$\frac{z}{Z} = \frac{a + jb}{A + jB} = \frac{a + jb}{A + jB} \times \frac{A - jB}{A - jB} = \frac{(aA + bB) + j(bA - aB)}{A^2 + B^2}$$

$$\text{So: } \mathrm{Re}(z / Z) = \frac{(aA + bB)}{A^2 + B^2} \text{ and } \mathrm{Im}(z / Z) = \frac{(bA - aB)}{A^2 + B^2}$$

For example:

$$\frac{8 - 6j}{5 + 9j} = \frac{8 - 6j}{5 + 9j} \times \frac{5 - 9j}{5 - 9j} = \frac{(40 - 54) + j(-30 - 72)}{25 + 81}$$

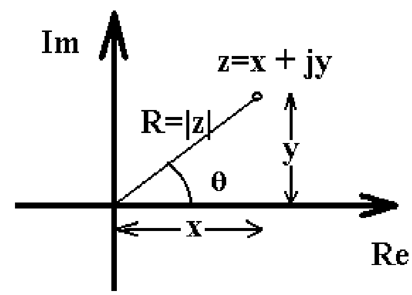
$$= \frac{-14 - j(102)}{106} = -\frac{14}{106} - j\frac{102}{106} = -0.132 - 0.962j$$

$$\mathrm{Re}\left(\frac{8 - 6j}{5 + 9j}\right) = -0.132 \quad \mathrm{Im}\left(\frac{8 - 6j}{5 + 9j}\right) = -0.962$$

Here we used the ***complex conjugate,*** $(5+9j)^*=(5-9j)$, of the denominator: if $z = a + jb$ then $z^* = a - jb$

Note: $|z|^2 = zz^* = a^2 + b^2$

The connection between complex numbers and electronics becomes more clear when we express complex numbers in polar coordinates: R and θ.
If z=x+jy then, from Pythagoras,
$R^2=x^2+y^2$, R is the ***modulus*** of z, usually denoted as |z|.
From trigonometry tanθ=y/x, θ is the ***phase*** of z.
Re(z)=x=Rcosθ and Im(z)=y=Rsinθ.
z=R(cosθ+ jsinθ).
The modulus can be associated with, for example, the peak voltage of a waveform and θ as the relative phase of the waveform.



But the power of complex numbers really comes from expressing them in exponential form using Euler's Formula:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Here's where the connection to sinusoidal waveforms is made: the phase of the waveform appears in the exponential and we make use of $e^a e^b = e^{(a+b)}$.

$$v(t) = V\cos(\omega t + \theta) = \mathrm{Re}(Ve^{j(\omega t + \theta)}) = V\,\mathrm{Re}(e^{j(\omega t + \theta)})$$

So use $v(t) = Ve^{j(\omega t + \theta)}$ to describe sinusoidal waveforms
then take the Real part of the expression after all calculations are done.

example:
express a sinusoidal wave with amplitude 20V and phase shift of 60^o in exponential form and Cartesian form, the frequency is 100Hz 60^o = 1.05 radians.
$v = 20V\, e^{j(2\pi\, 100\, t\ +\ 1.05)} = 20V\, e^{j(2\pi\, 100\, t)}\ (\cos 60^o\ + j \sin 60^o)$
$= 20\,(0.5 + 0.866\, j)\, e^{j(2\pi\, 100\, t)}\, V = (10 + 17.3\, j)V\, e^{j(2\pi\, 100\, t)}$
Here the frequency information has been factored out. If we want the full form:
v (t) = 20 V (cos (2π 100 t + 1.05) + j sin(π 100 t + 1.05)) then
at the end we'd take Re(v)=20V cos (2π 100 t + 1.05)

With the exponential form, multiplication and division are much easier:

Let $z_1 = R_1 e^{j\theta_1}$ and $z_2 = R_2 e^{j\theta_2}$

$$z_1 z_2 = (R_1 e^{j\theta_1})(R_2 e^{j\theta_2}) = R_1 R_2 e^{j(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{R_1 e^{j\theta_1}}{R_2 e^{j\theta_2}} = \frac{R_1}{R_2} e^{j(\theta_1 - \theta_2)}$$

OK, as an example, let's use this to add two sinusoidal wave forms with different amplitudes and phases but with the same frequency.

In this example, we use fact that the complex conjugate of a complex number is obtained by changing the sign of the coefficients of all the *j* terms, even those in the exponential.

e.g. if $z=10e^{4j}$ then $z^*=10e^{-4j}$ and $|z|^2=10e^{4j}\,10e^{-4j}=100\,e^{(4j-4j)}=100\,e^0 = 100$.

$$v_1(t)=V_1e^{(j\,\omega\mathrm{t}+\varphi)}=V_1\,e^{j\,\omega\,\mathrm{t}}\,e^{j\,\varphi} \quad \text{and} \quad v_2(t)=V_2e^{j(\omega\,\mathrm{t})}$$

$$\begin{aligned}
v_s=v_1+v_2 &= e^{j\,\omega\,\mathrm{t}}\,(V_2e^0+V_1e^{j\,\varphi})\\
&= e^{j\,\omega\,\mathrm{t}}\,(V_2+V_1\cos\varphi+jV_1\sin\varphi)\\
&= |v_s|e^{j(\omega\,\mathrm{t}\,+\,\theta)}
\end{aligned}$$

$$\begin{aligned}
|v_s|^2=v_sv_s^* &= e^{\,j\,\omega\,\mathrm{t}}\,(V_2+V_1\cos\varphi+jV_1\sin\varphi)\times\\
&= e^{j\,\omega\,\mathrm{t}}e^{-j\,\omega\,\mathrm{t}}[\;(V_2+V_1\cos\varphi)^2+(V_1\sin\varphi)^2\;]\\
&= [\;(V_2+V_1\cos\varphi)^2+(V_1\sin\varphi)^2\;]
\end{aligned}$$

$$\tan\theta=\mathrm{Im}(v_s)/\mathrm{Re}(v_s)=\frac{V_1\sin\varphi}{V_2+V_1\cos\varphi}$$

Let's relate this back to the electronics applications, impedances in particular are more completely described by not only their magnitudes, but also their phases:

For resistors, voltage and current have same phase:

$$Z_R=\frac{v_R}{i_R}=\frac{|v_R|e^{j\,\omega\,\mathrm{t}}}{|i_R|e^{j\,\omega\,\mathrm{t}}}=\frac{|v_R|}{|i_R|}=R$$

For capacitors, current leads the voltage by $\pi/2$ (ICE).

$$Z_C=\frac{v_C}{i_C}=\frac{|v_C|e^{j\,\omega\,\mathrm{t}}}{|i_C|e^{j\,\omega\,\mathrm{t}\,+\,\pi/2}}=\frac{|v_C|}{|i_C|}e^{-j\,\pi/2}=X_C(\cos(-\pi/2)+j\sin(\text{-}\pi/2))=-jX_C=\frac{-j}{\omega\,C}$$

For inductors, current lags the voltage by $\pi/2$ (ELI).

$$Z_L=\frac{v_L}{i_L}=\frac{|v_L|e^{j\,\omega\,\mathrm{t}}}{|i_L|e^{j\,\omega\,\mathrm{t}\,\text{-}\,\pi/2}}=\frac{|v_L|}{|i_L|}e^{+j\,\pi/2}=X_L(\cos(\pi/2)+j\sin(\pi/2))=jX_L=j\,\omega\,\mathrm{L}$$

In the Lab 7, you'll be measuring the full information (phase and magnitude) about the impedances of RC and RL series circuits.

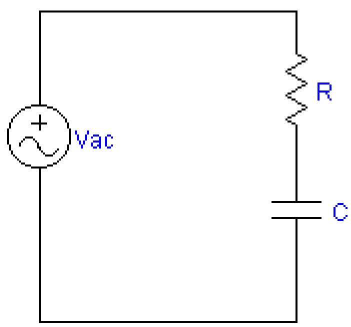
With complex impedances, the calculations are quite simple:

Series RC Impedance:

$$Z_{RC} = Z_R + Z_C = R - jX_C = |Z_{RC}| e^{j\,\theta_{\mathrm{RC}}}$$

$$|Z_{RC}| = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + 1/\omega^2\ \mathrm{C}^2}$$

$$\theta_{RC} = \tan^{-1}(-X_C / R) = -\tan^{-1}(1/\omega\,\mathrm{CR})$$



Series RL Impedance:

$$Z_{RL} = Z_R + Z_L = R + jX_L = |Z_{RL}| e^{j\,\theta_{\mathrm{RL}}}$$

$$|Z_{RL}| = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + \omega^2\ \mathrm{L}^2}$$

$$\theta_{RL} = \tan^{-1}(X_L / R) = \tan^{-1}(\omega\,\mathrm{L}\ / \mathrm{R})$$