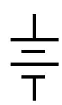
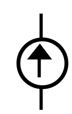
P214 Midterm Practice Problem SOLUTIONS

(note that these are a little bit harder than the problems on the actual midterm will be, but are very good practice for them)

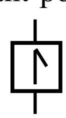
1. Show that, if you have a 15 volt battery, it is *not* possible to exceed the power rating of a ¼ watt resistor of resistance greater than 1 kΩ (no matter how you connect it).

If you directly connect up a 1 kΩ resistor to a 15 volt battery (with, at most, only wires with negligible resistance to make the connections), the power dissipated in the resistor will be $\boldsymbol{P = V^2/R}$ **= 225/1000 = just a bit under ¼**. *Any* alternative way of connecting the resistor to the battery (using simple elements such as other resistors, capacitors, inductors, wires, etc) would only cause the power dissipated in the resistor to be *less* than that value, and increasing the resistance of the resistor above 1 kΩ would also only decrease the dissipated power. Thus there is no way to exceed the power rating of a ¼ watt resistor of resistance greater than 1 kΩ with a 15 volt battery.

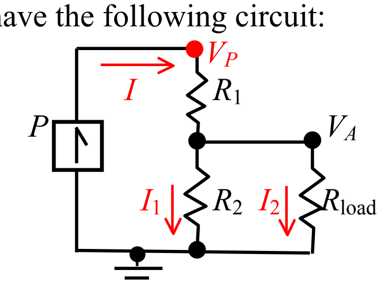
2. (A quite hard [especially parts b) – d)] but good problem!) Let's say that, instead of a voltage source (like a battery) or a constant-current source, we have a constant-*power* source. A constant-power source will do whatever it can to make sure that the voltage across its terminals, *multiplied by* the current it provides, is kept at a constant value (say, for example, 10 watts, or alternatively 100 milliwatts, etc). We know that the symbols for a voltage source and for a current source are as follows:

Voltage source:  Current source: 

Let us let the symbol for a constant-power source be:

Power source: 

And let's say we have the following circuit:



a) Find the voltage V_A, as a function of P, R_1, R_2, and R_{load}.

I've drawn in additional variables I, I_1, I_2 (representing currents) and V_P (the voltage coming out of the power source) in the drawing above. We know 5 relationships between all these variables:

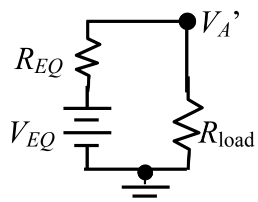
1) $I = I_1 + I_2$ (from Kirchhoff's node law)
2) $V_P - V_A = IR_1$ (from Ohm's law, on resistor R_1)
3) $V_A = I_1 R_2$ (from Ohm's law, on resistor R_2)

4) $V_A = I_2 R_{\text{load}}$ (from Ohm's law, on resistor R_{load})

5) $P = IV_P$ (the definition of our constant power source)

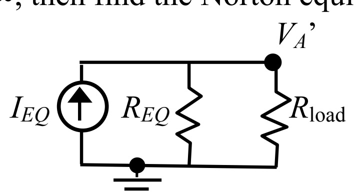
We need to solve those 5 equations in 5 variables, so that we're left with V_A in terms of P, R_1, R_2, and R_{load}. That's not as hard as it might seem. Plugging 3) and 4) into equation 1), we get $I = V_A/R_2 + V_A/R_{\text{load}}$. Call that equation 1'). Then plugging in equation 5) for I into 1') gives us $P/V_P = V_A/R_2 + V_A/R_{\text{load}}$. Call that equation 5'). From equation 2), we know that $V_P = V_A + IR_1$, and plugging equation 1') for I into that gives us that $V_P = V_A + (V_A/R_2 + V_A/R_{\text{load}})R_1 = V_A(1 + R_1/R_2 + R_1/R_{\text{load}})$. Now plugging that into equation 5') gives us $P = V_A^2(1 + R_1/R_2 + R_1/R_{\text{load}})(1/R_2 + 1/R_{\text{load}})$, and thus $\boldsymbol{V_A = [P/\{(1 + R_1/R_2 + R_1/R_{\text{load}})(1/R_2 + 1/R_{\text{load}})\}]^{1/2}}$.

b) If we decide to "formally" define the Thevenin equivalent voltage V_{EQ}, and Thevenin equivalent resistance R_{EQ}, as: 1) $\boldsymbol{V_{EQ}} \equiv V_A' = V_A$ in both of the circuits below and above respectively when taking the limit that $R_{\text{load}} \rightarrow \infty$, and 2) $\boldsymbol{R_{EQ}} \equiv R_{\text{load}}((V_{EQ}/V_A) - 1) = R_{\text{load}}((V_{EQ}/V_A') - 1)$ in both of the circuits above and below respectively when taking the limit that $R_{\text{load}} << R_{EQ}$, then find the Thevenin equivalents V_{EQ} and R_{EQ} as functions of P, R_1, and R_2.



To find V_{EQ}, just take the limit $R_{\text{load}} \rightarrow \infty$ in the formula we calculated for V_A above in part a). One obtains $V_{EQ} = [P/\{(1 + R_1/R_2)(1/R_2)\}]^{1/2} = \boldsymbol{R_2(P/(R_1+R_2))^{1/2}}$. To find R_{EQ}, note that $R_{\text{load}}((V_{EQ}/V_A) - 1) = R_{\text{load}}[R_2^2\{(1 + R_1/R_2 + R_1/R_{\text{load}})(1/R_2 + 1/R_{\text{load}})\}/(R_1+R_2)]^{1/2} - R_{\text{load}}$. In the limit that R_{load} is *small*, this just equals $R_{\text{load}}[R_2^2\{(R_1/R_{\text{load}})(1/R_{\text{load}})\}/(R_1+R_2)]^{1/2} = [R_2^2 R_1/(R_1+R_2)]^{1/2} = \boldsymbol{R_2[R_1/(R_1+R_2)]^{1/2}}$.

c) If we decide to "formally" define the Norton equivalent current I_{EQ}, and Norton equivalent resistance R_{EQ}, as: 1) $\boldsymbol{I_{EQ}} \equiv V_A'/R_{\text{load}} = V_A/R_{\text{load}}$ in both of the circuits below and above respectively when taking the limit that $R_{\text{load}} << R_{EQ}$, and 2) $\boldsymbol{R_{EQ}} \equiv V_A/I_{EQ} = V_A'/I_{EQ}$ in both of the circuits above and below respectively when taking the limit that $R_{\text{load}} \rightarrow \infty$, then find the Norton equivalents I_{EQ} and R_{EQ} as functions of P, R_1, and R_2.



To find I_{EQ}, do similarly to what we did for the Thevenin R_{EQ} in part b) above. $I_{EQ} = V_A/R_{\text{load}} = [P/\{(1 + R_1/R_2 + R_1/R_{\text{load}})(1/R_2 + 1/R_{\text{load}})\}]^{1/2} / R_{\text{load}}$. In the limit that R_{load} is small, this equals $[P/\{(R_1/R_{\text{load}})(1/R_{\text{load}})\}]^{1/2} / R_{\text{load}} = \boldsymbol{(P/R_1)^{1/2}}$. To find the Norton R_{EQ}, we have that $R_{EQ} = V_A/I_{EQ} = [R_1/\{(1 + R_1/R_2 + R_1/R_{\text{load}})(1/R_2 + 1/R_{\text{load}})\}]^{1/2}$. In the limit that $R_{\text{load}} \rightarrow \infty$, this becomes $[R_1/\{(1 + R_1/R_2)(1/R_2)\}]^{1/2} = \boldsymbol{R_2[R_1/(R_1+R_2)]^{1/2}}$.

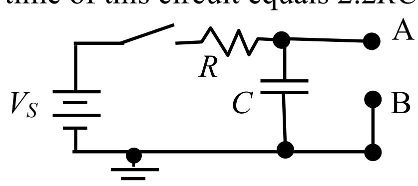
d) Does the Thevenin equivalent resistance R_{EQ} equal the Norton equivalent resistance R_{EQ} in your calculations above? Does $V_{EQ} = I_{EQ}R_{EQ}$, where R_{EQ} is the Thevenin equivalent resistance above?

Yes, and **yes**!

e) Is the power P' dissipated in the Thevenin equivalent circuit equal to P? If not, why? In the limit of small R_{load}, what ratio between R_1 and R_2 is needed to make $P' = P$?

No: $P' = V_{EQ}^2/(R_{EQ}+R_{load}) = (PR_2^2/(R_1+R_2))/[\ R_2[R_1/(R_1+R_2)]^{1/2} + R_{load}]$ which is certainly not identical to P. This is because V_{EQ} and R_{EQ} were calculated in very different limiting conditions: V_{EQ} was calculated in the limit that $R_{load} \to \infty$, and R_{EQ} was calculated in the limit that $R_{load} \ll R_{EQ}$. In the limit of small R_{load}, $P' = (PR_2^2/(R_1+R_2))/[R_2[R_1/(R_1+R_2)]^{1/2}] = PR_2/[R_1(R_1+R_2)]^{1/2}$. If $P' = P$, then $R_2/[R_1(R_1+R_2)]^{1/2} = 1$, which implies $R_2^2 = R_1^2 + R_1R_2$, i.e. $R_2^2 - R_1R_2 - R_1^2 = 0$, implying $R_2 = (R_1 \pm \mathrm{sqrt}(R_1^2+4R_1^2))/2$. Both R_1 and R_2 must be positive, so the + option is the only physical option, and thus $R_2 = R_1(1+\mathrm{sqrt}(5))/2$, i.e. $R_2{:}R_1 =$ **(1+sqrt(5))/2**, which is sometimes known as the golden ratio.

3. At time $t = 0$, someone closes the switch of the circuit below. If we define the rise time of the circuit to be the time the circuit takes to go from 10% to 90% of its final value, show that the rise time of this circuit equals $2.2RC$.



We want to determine the voltage at A, which we can call V_A, as a function of time. We know that the charge Q on the capacitor will be $Q = CV_A$. We additionally know from Ohm's law that $V_S - V_A = IR$. Taking the derivative of the first of these two equations, we have that $I = C\mathrm{d}V_A/\mathrm{d}t$, and thus, inserting that in the second equation, we have that $V_S - V_A = RC(\mathrm{d}V_A/\mathrm{d}t)$. Let's define a variable $V' \equiv V_S - V_A$. Then $\mathrm{d}V'/\mathrm{d}t = -\mathrm{d}V_A/\mathrm{d}t$, and thus $V' = -RC(\mathrm{d}V'/\mathrm{d}t)$. We can solve this differential equation: we have $-\mathrm{d}t/(RC) = \mathrm{d}V'/V'$, and integrating gives us that $\ln(V') = -t/(RC) + \alpha$, where α is a constant of integration. That implies that $V' = Ae^{-t/(RC)}$, where A is just another way of writing the constant of integration ($A \equiv e^{\alpha}$), and thus $V_A = V_S - Ae^{-t/(RC)}$. We can assume that the capacitor is initially uncharged, and thus at $t = 0$, V_A must equal 0, and thus A must be identical to V_S, and thus $V_A = V_S(1 - e^{-t/(RC)})$. We want to know at what times t does V_A equal 10% and 90% of V_S; i.e. when does $e^{-t/(RC)}$ equal 0.9 and 0.1? This occurs when $t/(RC) = -\ln(0.9)$ and $t/(RC) = -\ln(0.1)$, i.e. when $t = 0.105RC$ and $t = 2.303RC$. The difference between those two times, i.e. the rise time, is almost exactly $2.2RC$.

4. Bugg problems 1.10.3, 1.10.6, 1.10.7, 1.10.9, 2.10.7, 2.10.8, 2.10.12, 2.10.13, 3.12.6, 3.12.13.

Solutions to these will either be posted later, and/or done during the review session on Wednesday afternoon.