P321b Final Practice Problem Solutions

1. $F(r) = \frac{-\partial U}{\partial r} \Rightarrow U = \int F(r)dr = \frac{-k}{r} - \frac{k'}{3r^3}$

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r} + \frac{k'}{3r^3}.$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow m\ddot{r} = mr\dot{\theta}^2 - \frac{k}{r^2} - \frac{k'}{r^4} = \frac{M^2}{mr^3} - \frac{k}{r^2} - \frac{k'}{r^4}$ ① with $M \equiv mr^2\dot{\theta}$. If we have a stable orbit at $r=\rho$, then $\ddot{r} = 0 \Rightarrow 0 = \frac{M^2}{m\rho^3} - \frac{k}{\rho^2} - \frac{k'}{\rho^4}$ ② $\Rightarrow M^2 = mk\rho + \frac{mk'}{\rho}$. ③ Now let $r(t) = \rho + \varepsilon(t)$, where $\varepsilon \ll \rho$. If we have a stable orbit, then $\ddot{r}(t) = -\delta\varepsilon(t)$, ④ where δ is some positive constant, to first order in ε. From ①, we have that $m\ddot{r}(t) = \frac{M^2}{m(\rho+\varepsilon)^3} - \frac{k}{(\rho+\varepsilon)^2} - \frac{k'}{(\rho+\varepsilon)^4}$ $= \frac{M^2}{m\rho^3}\left(1+\frac{\varepsilon}{\rho}\right)^{-3} - \frac{k}{\rho^2}\left(1+\frac{\varepsilon}{\rho}\right)^{-2} - \frac{k'}{\rho^4}\left(1+\frac{\varepsilon}{\rho}\right)^{-4} =$ (to first order in ε) $\frac{M^2}{m\rho^3}\left(1-\frac{3\varepsilon}{\rho}\right) - \frac{k}{\rho^2}\left(1-\frac{2\varepsilon}{\rho}\right) - \frac{k'}{\rho^4}\left(1-\frac{4\varepsilon}{\rho}\right)$. Only the term proportional to ε remains (the constant term vanishes because of ②), thus we have that $m\ddot{r}(t) = \frac{-3M^2\varepsilon}{m\rho^4} + \frac{2k\varepsilon}{\rho^3} + \frac{4k'\varepsilon}{\rho^5} = \frac{\varepsilon}{\rho^3}\left(2k - \frac{3M^2}{m\rho} + \frac{4k'}{\rho^2}\right)$. From the stability condition ④, we know $2k - \frac{3M^2}{m\rho} + \frac{4k'}{\rho^2} < 0$. Substituting ③ for M^2, we have $2k - 3k - \frac{3k'}{\rho^2} + \frac{4k'}{\rho^2} < 0$ $\Rightarrow \frac{k'}{\rho^2} - k < 0 \Rightarrow k\rho^2 > k'$, as the question asks to show.

2. The best way to do this problem is to use eqns. 5.13a and 5.13b in Fetter + Walecka, where ε in those equations is defined by $\alpha = \arccos\left(\frac{1}{\varepsilon}\right)$, with α the angle shown in the diagram (Fig. 5.2) above the equations, and $\bar{\gamma} \equiv G(m_{rocket} + m) \approx Gm$. Thus

$$\varepsilon^2 = 1 + \frac{v_\infty^4 b^2}{\bar{\gamma}^2} \quad (1) \quad \text{and}$$

$$r_{min} = b\sqrt{\frac{\varepsilon-1}{\varepsilon+1}} = R_0. \quad (2)$$ Then do some algebra:

$$(2) \Rightarrow b^2 \frac{\varepsilon-1}{\varepsilon+1} = R_0^2 \Rightarrow b^2 \frac{(\varepsilon-1)^2}{\varepsilon^2-1} = b^2 \frac{\varepsilon^2-2\varepsilon+1}{\varepsilon^2-1} = R_0^2$$

$$\Rightarrow b^2(\varepsilon^2+1) - 2\varepsilon b^2 = R_0^2(\varepsilon^2-1)$$

$$\Rightarrow 4\varepsilon^2 b^4 = \left[R_0^2(\varepsilon^2-1) - b^2(\varepsilon^2+1)\right]^2$$

$$= R_0^4(\varepsilon^2-1)^2 + \underbrace{b^4(\varepsilon^2+1)^2}_{= b^4(\varepsilon^4+2\varepsilon^2+1)} - 2R_0^2 b^2(\varepsilon^4-1)$$

$$\Rightarrow 0 = R_0^4(\varepsilon^2-1)^2 + b^4(\varepsilon^4-2\varepsilon+1) - 2R_0^2 b^2(\varepsilon^4-1)$$

$$= (R_0^4 + b^4)(\varepsilon^2-1)^2 - 2R_0^2 b^2(\varepsilon^4-1)$$

$$= (R_0^4 + b^4)(\varepsilon^2-1)^2 - 2R_0^2 b^2(\varepsilon^4 - 2\varepsilon^2 + 1 + 2\varepsilon^2 - 2)$$

$$\Rightarrow 0 = (R_0^4 + b^4)(\varepsilon^2-1)^2 - 2R_0^2 b^2\left[(\varepsilon^2-1)^2 + 2(\varepsilon^2-1)\right]$$

$\varepsilon = 1$ is not a solution, so $\Rightarrow 0 = (R_0^4 + b^4 - 2R_0^2 b^2)(\varepsilon^2-1) - 4R_0^2 b^2$

$$= (R_0^2 - b^2)^2(\varepsilon^2-1) - 4R_0^2 b^2$$

Now use (1) $\Rightarrow \varepsilon^2 - 1 = \frac{v_\infty^4 b^2}{\bar{\gamma}^2} \Rightarrow 0 = (R_0^2 - b^2)^2 \frac{v_\infty^4 b^2}{\bar{\gamma}^2} - 4R_0^2 b^2$

$b = 0$ is not a solution, so $\Rightarrow 0 = (R_0^2 - b^2)^2 \frac{v_\infty^4}{\bar{\gamma}^2} - 4R_0^2$

$$\Rightarrow (R_0^2 - b^2)^2 = 4R_0^2 \frac{\bar{\gamma}^2}{v_\infty^4} \Rightarrow R_0^2 - b^2 = \pm 2R_0 \frac{\bar{\gamma}}{v_\infty^2}$$

$\Rightarrow b^2 = R_0^2 \mp 2R\frac{\bar{\gamma}}{v_\infty^2}$. We must have $b > R_0$, thus

$$\Rightarrow b = R_0\sqrt{1 + \frac{2\bar{\gamma}}{R_0 v_\infty^2}} \approx R_0\sqrt{1 + \frac{2Gm}{R_0 v_\infty^2}}.$$

To get the deflection angle, use Fetter + Walecka 5.14 $\Rightarrow$

$$\Theta = 2\,\text{arccot}\left(\frac{v_\infty^2 b}{\bar{\gamma}}\right) = 2\,\text{arccot}\left(\frac{R_0 v_\infty^2}{Gm}\sqrt{1 + \frac{2Gm}{R_0 v_\infty^2}}\right).$$

3. Rutherford formula: $\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{zZe^2}{4E\sin^2\theta/2}\right)^2$ (F+W 5.28, L+L 19.3)

Total cross-section $\sigma_{tot} = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega$

$$= \int \left(\frac{zZe^2}{4E\sin^2\theta/2}\right)^2 d\Omega$$

$$d\Omega = 2\pi \sin(\theta) d\theta$$

Conservation of energy (glancing collision): $T_\infty = T + V$

$$\Rightarrow \tfrac{1}{2} m v_\infty^2 = \tfrac{1}{2} m v^2 + \frac{zZe^2}{R} \quad (1)$$

Conservation of angular momentum (glancing collision): $m v_\infty b = m v R$

$$\Rightarrow v = \frac{v_\infty b}{R} \quad (2)$$

Putting (2) in (1) $\Rightarrow \tfrac{1}{2} m v_\infty^2 = \tfrac{1}{2} m \left(\frac{v_\infty b}{R}\right)^2 + \frac{zZe^2}{R}$

$$\Rightarrow \tfrac{1}{2} m v_\infty^2 \left(1 - \frac{b^2}{R^2}\right) = \frac{zZe^2}{R}$$

$$\Rightarrow \frac{b^2}{R^2} = 1 - \frac{2zZe^2}{m v_\infty^2 R}$$

$$\Rightarrow b = R\left[1 - \frac{2zZe^2}{m v_\infty^2 R}\right]^{1/2}$$ = impact parameter at which incident particle <u>just</u> strikes the nucleus (b_{max})

$$\sigma_r = 2\pi \int_0^{b_{max}} b\, db = \pi b_{max}^2$$

$$\Rightarrow \sigma_r = \pi R^2 \left(1 - \frac{2zZe^2}{m v_\infty^2 R}\right)$$

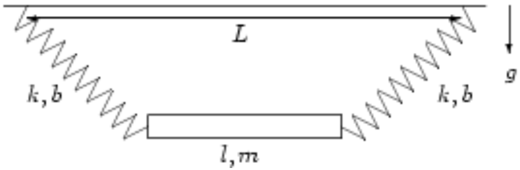
$V_c \equiv \frac{zZe^2}{R} \quad \Rightarrow \quad \sigma_r = \pi R^2 \left(1 - \frac{2V_c}{m v_\infty^2}\right)$

$E = \tfrac{1}{2} m v_\infty^2 \quad \Rightarrow \quad \sigma_r = \pi R^2 \left(1 - \frac{V_c}{E}\right)$. //

4. ~~Use as the three generalized coordinates: x, y, and ϕ where x and y are (respectively) the horizontal and vertical positions (away from the origin being the equilibrium position) of the centre of mass of the bar, and ϕ being the angle (counterclockwize from horizontal) of the bar.~~

The Problem

A uniform thin rod of length l and mass m is suspended by two equal springs of equilibrium length b, force constant k and negligible mass, as shown in the diagram. The points of attachment of the springs are separated by a distance $L > l$.



Find the normal modes of small oscillations in the plane containing the springs. Include gravity.

The Solution

The rod has three degrees of freedom, the horizontal (x) position of the center of mass, the vertical position of the center of mass (y) and the tilt of the rod θ which we will measure from the horizontal in a counterclockwise direction.

Since everything is symmetric, we can take the equilibrium position of x and θ to be zero. We will measure the y-coordinate from the plane that the springs are attached to.

Kinetic Energy

The kinetic energy of the rod is given by

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$$

where $I = \frac{1}{12}ml^2$.

Potential Energy

To get the potential energy we will need to calculate the position of the ends of the rods. The left end is given by

$$x_l = x - \frac{l}{2}\cos\theta, y_l = y - \frac{l}{2}\sin\theta$$

and the right end is given by

$$x_r = x + \frac{l}{2}\cos\theta, y_l = y + \frac{l}{2}\sin\theta$$

The length of the left spring is given by

$$l_l = \sqrt{\left(x - \frac{l}{2}\cos\theta + \frac{L}{2}\right)^2 + \left(y - \frac{l}{2}\sin\theta\right)^2}$$

The length of the right spring is given by

$$l_r = \sqrt{\left(x + \frac{l}{2}\cos\theta - \frac{L}{2}\right)^2 + \left(y + \frac{l}{2}\sin\theta\right)^2}$$

The potential energy is

$$V = mgy + \frac{1}{2}k\left[(l_l - b)^2 + (l_r - b)^2\right]$$

Equilibrium Positions

By symmetry we have $x_0 = 0$, $\theta_0 = 0$. We don't need to find the equilibrium value of y so let's take it to be y_0.

Let's take the first derivatives of V to find any relationship between y_0 and l_0, the equilibrium length of the springs.

First we have

$$\frac{\partial V}{\partial x} = k\left[\frac{l_l - b}{l_l}\left(x - \frac{l}{2}\cos\theta + \frac{L}{2}\right) + \frac{l_r - b}{l_r}\left(x + \frac{l}{2}\cos\theta - \frac{L}{2}\right)\right],$$

$$\frac{\partial V}{\partial y} = mg + k\left[\frac{l_l - b}{l_l}\left(y - \frac{l}{2}\sin\theta\right) + \frac{l_r - b}{l_r}\left(y + \frac{l}{2}\sin\theta\right)\right],$$

and

$$\frac{\partial V}{\partial \theta} = \frac{kl}{2}\left\{ \sin\theta \left[\frac{l_l - b}{l_l}\left(x - \frac{l}{2}\cos\theta + \frac{L}{2}\right) - \frac{l_r - b}{l_r}\left(x + \frac{l}{2}\cos\theta - \frac{L}{2}\right)\right] \right.$$
$$\left. + \cos\theta \left[-\frac{l_l - b}{l_l}\left(y - \frac{l}{2}\sin\theta\right) + \frac{l_r - b}{l_r}\left(y + \frac{l}{2}\sin\theta\right)\right] \right\}$$

By symmetry we know that in equilibrium $l_l = l_r = l_0, \theta = 0, x = 0$, so let's look at the equation for *y*,

$$\frac{\partial V}{\partial y} = mg + 2k\frac{l_0 - b}{l_0}\left[2y_0\right] = 0,$$

so

$$y_0 = -\frac{mg}{4k}\frac{l_0}{l_0 - b}$$

We will write our perturbed solutions as

$$x = \eta_1, y = y_0 + \eta_2, \theta = \eta_3$$

Calculating the matrices

To proceed we need to determine the matrices T_{ij},V_{ij}. Both matrices are three-by-three because we have three degrees of freedom. The kinetic energy matrix is pretty simple. We have

$$T_{ij} = \begin{bmatrix} m & & \\ & m & \\ & & I \end{bmatrix}$$

where we have the coordinates in the order *x*,*y* and θ.

The potential energy matrix is a bit less straightforward, we have in general

$$V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$$

x-terms

It is crucial to keep organized. Let's do the *x* terms,

$$\frac{\partial^2 V}{\partial x^2} = k\left[\frac{l_l - b}{l_l} + \frac{l_r - b}{l_r} + \frac{b}{l_l^3}\left(x - \frac{l}{2}\cos\theta + \frac{L}{2}\right)^2 + \frac{b}{l_r^3}\left(x + \frac{l}{2}\cos\theta - \frac{L}{2}\right)^2\right]$$

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_0 = k\left[2\frac{l_0 - b}{l_0} + \frac{1}{2}\frac{b}{l_0^3}\left(l - L\right)^2\right]$$

$$\frac{\partial V}{\partial x \partial y} = k\left[\frac{b}{l_l^3}\left(x - \frac{l}{2}\cos\theta + \frac{L}{2}\right)\left(y - \frac{l}{2}\sin\theta\right) + \frac{b}{l_r^3}\left(x + \frac{l}{2}\cos\theta - \frac{L}{2}\right)\left(y + \frac{l}{2}\sin\theta\right)\right],$$

$$\left.\frac{\partial V}{\partial x \partial y}\right|_0 = 2k\left[\frac{b}{l_0^3}\left(-\frac{l_0}{2}+\frac{L}{2}\right)y_0 + \frac{b}{l_0^2}\left(\frac{l}{2}-\frac{L}{2}\right)y_0\right] = 0,$$

$$\frac{\partial V}{\partial \theta \partial x} = kl\left\{\sin\theta\left[\frac{l_l-b}{l_l}-\frac{l_r-b}{l_r}\right] + \sin\theta\left[\frac{b}{l_l^3}\left(x-\frac{l}{2}\cos\theta+\frac{L}{2}\right)^2 - \frac{b}{l_r^3}\left(x+\frac{l}{2}\cos\theta-\frac{L}{2}\right)^2\right]\right.$$
$$\left.+\cos\theta\left[-\frac{b}{l_l^3}\left(y-\frac{l}{2}\sin\theta\right)\left(x-\frac{l}{2}\cos\theta+\frac{L}{2}\right)+\frac{b}{l_r^3}\left(y+\frac{l}{2}\sin\theta\right)\left(x+\frac{l}{2}\cos\theta-\frac{L}{2}\right)\right]\right\}$$

$$\left.\frac{\partial V}{\partial \theta \partial x}\right|_0 = \frac{kl}{2}\left\{+\left[-\frac{b}{l_0^3}(y_0)\left(-\frac{l}{2}+\frac{L}{2}\right)+\frac{b}{l_0^3}(y_0)\left(\frac{l}{2}-\frac{L}{2}\right)\right]\right\} = kbly_0\frac{l-L}{2l_0^3}$$

y-terms

$$\frac{\partial V}{\partial y^2} = k\left[\frac{l_l-b}{l_l}+\frac{l_r-b}{l_r}+\frac{b}{l_l^3}\left(y-\frac{l}{2}\sin\theta\right)^2+\frac{b}{l_r^3}\left(y+\frac{l}{2}\sin\theta\right)^2\right],$$

$$\left.\frac{\partial V}{\partial y^2}\right|_0 = 2k\left[\frac{l_0-b}{l_0}+\frac{b}{l_0^3}y_0^2\right],$$

$$\frac{\partial V}{\partial \theta \partial y} = \frac{kl}{2}\left\{\sin\theta\left[\frac{b}{l_l^3}\left(x-\frac{l}{2}\cos\theta+\frac{L}{2}\right)\left(y-\frac{l}{2}\sin\theta\right)-\frac{b}{l_r^3}\left(x+\frac{l}{2}\cos\theta-\frac{L}{2}\right)\left(y+\frac{l}{2}\sin\theta\right)\right]\right.$$
$$\left.+\cos\theta\left[-\frac{l_l-b}{l_l}+\frac{l_r-b}{l_r}-\frac{b}{l_l^3}\left(y-\frac{l}{2}\sin\theta\right)^2+\frac{b}{l_r^3}\left(y+\frac{l}{2}\sin\theta\right)^2\right]\right\}$$

$$\left.\frac{\partial V}{\partial \theta \partial y}\right|_0 = 0$$

θ terms

As we saw earlier many of the terms in θ derivatives vanish because sinθ = 0 in equilibrium, so let's try to avoid calculating these terms.

Let's start with the first derivative. We have

$$\frac{\partial V}{\partial \theta} = \frac{kl}{2}\left\{\sin\theta\left[\frac{l_l-b}{l_l}\left(x-\frac{l}{2}\cos\theta+\frac{L}{2}\right)-\frac{l_r-b}{l_r}\left(x+\frac{l}{2}\cos\theta-\frac{L}{2}\right)\right]\right.$$
$$\left.+\cos\theta\left[-\frac{l_l-b}{l_l}\left(y-\frac{l}{2}\sin\theta\right)+\frac{l_r-b}{l_r}\left(y+\frac{l}{2}\sin\theta\right)\right]\right\}$$

so

$$\left.\frac{\partial^2 V}{\partial \theta^2}\right|_0 = kl\left[\frac{l_0-b}{l_0}L + \frac{by_0}{l_0^2}\left(-\frac{\partial l_l}{\partial\theta} + \frac{\partial l_r}{\partial\theta}\right)\right] = \frac{kl}{2}\left[\frac{l_0-b}{l_0}L + \frac{y_0^2 l}{l_0^3}\left(2L(l_0-b) - lb\right)\right]$$

The Matrix

Let's put everything together and write out the matrix

$$V_{ij} = \begin{bmatrix} k\left[2\frac{l_0-b}{l_0} + \frac{1}{2}\frac{b}{l_0^3}\left(l-L\right)^2\right] & 0 & kbly_0\frac{l-L}{2l_0^3} \\ 0 & 2k\left[\frac{l_0-b}{l_0} + \frac{b}{l_0^3}y_0^2\right] & 0 \\ kbly_0\frac{l-L}{2l_0^3} & 0 & \frac{kl}{2}\left[\frac{l_0-b}{l_0}L + \frac{y_0^2 l}{l_0^3}\left(2L(l_0-b)-lb\right)\right] \end{bmatrix}$$

We can immediately see that the *y*-oscillations don't mix with the other coordinates so we have one eigenvector $\begin{bmatrix}0\\1\\0\end{bmatrix}$ with the eigenvalue $\omega^2 = 2\frac{k}{m}\left[\frac{l_0-b}{l_0} + \frac{b}{l_0^3}y_0^2\right]$.

We are left with the following two-by-two matrix

$$\begin{bmatrix} k\left[2\frac{l_0-b}{l_0} + \frac{1}{2}\frac{b}{l_0^3}\left(l-L\right)^2\right] & kbly_0\frac{l-L}{2l_0^3} \\ kbly_0\frac{l-L}{2l_0^3} & \frac{kl}{2}\left[\frac{l_0-b}{l_0}L + \frac{y_0^2 l}{l_0^3}\left(2L(l_0-b)-lb\right)\right] \end{bmatrix}$$

Eigenvalues

The characteristic equation that we must solve to find the frequencies of the oscillations is

$$\begin{vmatrix} k\left[2\frac{l_0-b}{l_0} + \frac{1}{2}\frac{b}{l_0^3}\left(l-L\right)^2\right] - \omega^2 m & kbly_0\frac{l-L}{2l_0^3} \\ kbly_0\frac{l-L}{2l_0^3} & \frac{kl}{2}\left[\frac{l_0-b}{l_0}L + \frac{y_0^2 l}{l_0^3}\left(2L(l_0-b)-lb\right)\right] - \omega^2 I \end{vmatrix} = 0$$

where $I = \frac{1}{12}ml^2$

5. This is identical to the worked problem on pages 86-87 of L+L, starting with the equation of motion in 28.9, and with $\alpha = -\lambda$ and $\beta = 0$. Thus, if we write

$$x_0(t) = a\cos\omega t$$

then we will have (as per 28.12)

$$\lambda x_1(t) = \frac{\lambda a^2}{\omega_0^2}\left(\frac{1}{2} - \frac{1}{6}\cos 2\omega t\right)$$

and

$$\lambda^2 x_2(t) = \frac{\lambda^2 a^3}{48\,\omega_0^4}\cos 3\omega t$$

with

$$\omega = \omega_0 - \frac{5\lambda^2 a^2}{12\,\omega_0^3}$$ (and see the L+L text for the associated calculations).