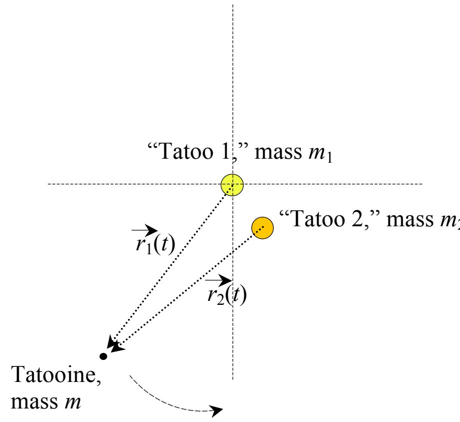
P321b Midterm SOLUTIONS

1. <u>The orbit of planet Tatooine</u>



In the Star Wars films, Tatooine (Luke Skywalker's home planet) has two suns, as pictured on the left-hand side, and as drawn on the right-hand side, above.

(a) What is the Lagrangian for the full system, in terms of m, m_1, m_2, $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\dot{\vec{r}}_1(t)$, $\dot{\vec{r}}_2(t)$, and the gravitational constant G? (Please don't forget to include the gravitational potential *between* the two suns, in addition to the potential between each of the two suns and Tatooine. And, of course, include the kinetic energy of Tatoo 2 as well as Tatooine. But no need to include the K.E. of Tatoo 1, though, since this particular coordinate system is always centred around Tatoo 1.) (40 points)

$$L = \tfrac{1}{2} m\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2} m_2(\dot{\vec{r}}_1 - \dot{\vec{r}}_2)^2 + Gmm_1/|\vec{r}_1| + Gmm_2/|\vec{r}_2| + Gm_1m_2/|\vec{r}_1 - \vec{r}_2| \,.$$

(b) In our coordinate system which we have chosen, which is centred on Tatoo 1, will the sum of the *total kinetic energy* and the *total potential energy* be conserved? And if not, then why not? – isn't total energy always conserved in a closed system? (20 points)

The conserved quantity will be $E = \Sigma_i\, \dot{q}_i(\partial L/\partial \dot{q}_i) - L$. From the Lagrangian in polar coordinates below, this will be $E = \tfrac{1}{2}m(\dot{r}_1^2 + r_1^2\dot{\phi}_1^2) + \tfrac{1}{2}m_2(\dot{r}_1^2 + r_1^2\dot{\phi}_1^2 + \dot{r}_2^2 + r_2^2\dot{\phi}_2^2 - 2(\dot{r}_1\dot{r}_2 + r_1r_2\dot{\phi}_1\dot{\phi}_2)\cos(\phi_1 - \phi_2) - 2(r_1\dot{r}_2\dot{\phi}_1 - \dot{r}_1r_2\dot{\phi}_2)\sin(\phi_1 - \phi_2)) - Gmm_1/r_1 - Gmm_2/r_2 - Gm_1m_2/\mathrm{sqrt}(r_1^2 + r_2^2 + 2r_1r_2\cos(\phi_1 - \phi_2))$, which is indeed T + U, where T is total kinetic energy, U is total potential energy, and L = T – U. So yes, the sum of the total kinetic energy and the total potential energy is indeed conserved for this system. (Sorry for the bit of a trick question!)

(c) Write down (but don't attempt to solve!!!) the resulting Lagrange's equations of motion, in terms of the polar coordinates r_1, ϕ_1, r_2, and ϕ_2. (Assume for simplicity's sake that the orbits of Tatooine and the two suns always remain within a single plane, so additional coordinates θ_1 and θ_2 are not necessary.) (40 points)

The Lagrangian in terms of these polar coordinates is $L = \frac{1}{2}m(\dot{r}_1^{\,2} + r_1^{\,2}\dot{\phi}_1^{\,2}) + \frac{1}{2}m_2(\dot{r}_1^{\,2} + r_1^{\,2}\dot{\phi}_1^{\,2} + \dot{r}_2^{\,2} + r_2^{\,2}\dot{\phi}_2^{\,2} - 2(\dot{r}_1\dot{r}_2 + r_1r_2\dot{\phi}_1\dot{\phi}_2)\cos(\phi_1 - \phi_2) - 2(r_1\dot{r}_2\dot{\phi}_1 - \dot{r}_1r_2\dot{\phi}_2)\sin(\phi_1 - \phi_2)) + \mathrm{G}mm_1/r_1 + \mathrm{G}mm_2/r_2 + \mathrm{G}m_1m_2/\mathrm{sqrt}(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))$.
Thus we have the four Lagrange's equations of motion:

$\mathrm{d}/\mathrm{d}t(\partial L/\partial\dot{r}_1) - \partial L/\partial r_1 = 0 \;\Rightarrow\; \mathrm{d}/\mathrm{d}t[(m + m_2)\dot{r}_1 - m_2\dot{r}_2\cos(\phi_1 - \phi_2) + m_2r_2\dot{\phi}_2\sin(\phi_1 - \phi_2)] = (m + m_2)r_1\dot{\phi}_1^{\,2} - m_2r_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) - m_2\dot{r}_2\dot{\phi}_1\sin(\phi_1 - \phi_2) - \mathrm{G}mm_1/r_1^{\,2} - \mathrm{G}m_1m_2(r_1 + r_2\cos(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2} \;\Rightarrow$
$(m + m_2)\ddot{r}_1 - m_2\ddot{r}_2\cos(\phi_1 - \phi_2) + m_2r_2\ddot{\phi}_2\sin(\phi_1 - \phi_2)) =$
$(m + m_2)r_1\dot{\phi}_1^{\,2} - m_2r_2(2\dot{\phi}_1 - \dot{\phi}_2)\dot{\phi}_2\cos(\phi_1 - \phi_2) - 2m_2\dot{r}_2(\dot{\phi}_1 - \dot{\phi}_2)\sin(\phi_1 - \phi_2) - \mathrm{G}mm_1/r_1^{\,2} - \mathrm{G}m_1m_2(r_1 + r_2\cos(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2}$.

$\mathrm{d}/\mathrm{d}t(\partial L/\partial\dot{r}_2) - \partial L/\partial r_2 = 0 \;\Rightarrow\; \mathrm{d}/\mathrm{d}t[m_2\dot{r}_2 - m_2\dot{r}_1\cos(\phi_1 - \phi_2) - m_2r_1\dot{\phi}_1\sin(\phi_1 - \phi_2)] = m_2r_2\dot{\phi}_2^{\,2} - m_2r_1\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) + m_2\dot{r}_1\dot{\phi}_2\sin(\phi_1 - \phi_2) - \mathrm{G}mm_2/r_2^{\,2} - \mathrm{G}m_1m_2(r_2 + r_1\cos(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2} \;\Rightarrow\; m_2\ddot{r}_2 - m_2\ddot{r}_1\cos(\phi_1 - \phi_2) - m_2r_1\ddot{\phi}_1\sin(\phi_1 - \phi_2) = m_2r_2\dot{\phi}_2^{\,2} - m_2r_1(2\dot{\phi}_1 - \dot{\phi}_2)\dot{\phi}_2\cos(\phi_1 - \phi_2) + 2m_2\dot{r}_1\dot{\phi}_2\sin(\phi_1 - \phi_2) - \mathrm{G}mm_2/r_2^{\,2} - \mathrm{G}m_1m_2(r_2 + r_1\cos(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2}$.

$\mathrm{d}/\mathrm{d}t(\partial L/\partial\dot{\phi}_1) - \partial L/\partial\phi_1 = 0 \;\Rightarrow\; \mathrm{d}/\mathrm{d}t[(m + m_2)r_1^{\,2}\dot{\phi}_1 - m_2r_1r_2\dot{\phi}_2\cos(\phi_1 - \phi_2) - m_2r_1\dot{r}_2\sin(\phi_1 - \phi_2)] = m_2(\dot{r}_1\dot{r}_2 + r_1r_2\dot{\phi}_1\dot{\phi}_2)\sin(\phi_1 - \phi_2) - m_2(r_1\dot{r}_2\dot{\phi}_1 - \dot{r}_1r_2\dot{\phi}_2)\cos(\phi_1 - \phi_2) + \mathrm{G}m_1m_2r_1r_2\sin(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2} \;\Rightarrow\; (m + m_2)r_1^{\,2}\ddot{\phi}_1 - m_2r_1r_2\ddot{\phi}_2\cos(\phi_1 - \phi_2) - m_2r_1\ddot{r}_2\sin(\phi_1 - \phi_2) = -2(m + m_2)\dot{r}_1\dot{\phi}_1 + m_2(2\dot{r}_1\dot{r}_2 + r_1r_2\dot{\phi}_2^{\,2})\sin(\phi_1 - \phi_2) + 2m_2\dot{r}_1r_2\dot{\phi}_2\cos(\phi_1 - \phi_2) + \mathrm{G}m_1m_2r_1r_2\sin(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2}$.

$\mathrm{d}/\mathrm{d}t(\partial L/\partial\dot{\phi}_2) - \partial L/\partial\phi_2 = 0 \;\Rightarrow\; \mathrm{d}/\mathrm{d}t[m_2r_2^{\,2}\dot{\phi}_2 - m_2r_1r_2\dot{\phi}_1\cos(\phi_1 - \phi_2) + m_2\dot{r}_1r_2\sin(\phi_1 - \phi_2)] = -m_2(\dot{r}_1\dot{r}_2 + r_1r_2\dot{\phi}_1\dot{\phi}_2)\sin(\phi_1 - \phi_2) + m_2(r_1\dot{r}_2\dot{\phi}_1 - \dot{r}_1r_2\dot{\phi}_2)\cos(\phi_1 - \phi_2) - \mathrm{G}m_1m_2r_1r_2\sin(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2} \;\Rightarrow\; m_2r_2^{\,2}\ddot{\phi}_2 - m_2r_1r_2\ddot{\phi}_1\cos(\phi_1 - \phi_2) + m_2\ddot{r}_1r_2\sin(\phi_1 - \phi_2) = -2m_2\dot{r}_2\dot{\phi}_2 - m_2(2\dot{r}_1\dot{r}_2 + r_1r_2\dot{\phi}_1^{\,2})\sin(\phi_1 - \phi_2) + 2m_2r_1\dot{r}_2\dot{\phi}_1\cos(\phi_1 - \phi_2) - \mathrm{G}m_1m_2r_1r_2\sin(\phi_1 - \phi_2))/(r_1^{\,2} + r_2^{\,2} + 2r_1r_2\cos(\phi_1 - \phi_2))^{3/2}$.