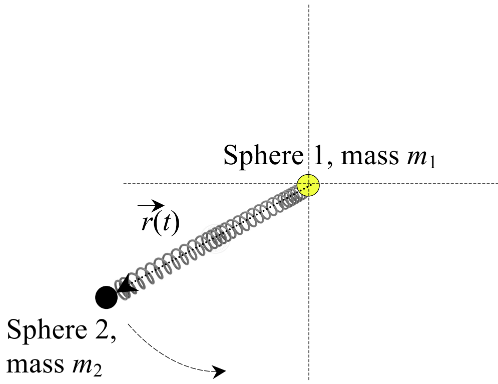
P321b Midterm Practice Problem SOLUTIONS

1. Two spheres connected by a spring



Two spheres, of masses m_1 and m_2 respectively, are connected by a spring with spring constant k (and with zero length when unextended, so that the potential energy of the spring when stretched to length r is $+\frac{1}{2}kr^2$). The spheres are orbiting around each other in a vacuum, so neither one of the spheres has a fixed position in space. Let us, however, consider a coordinate system which is non-rotating, but is always centred on Sphere 1 (i.e., rather than a coordinate system centred at the centre of mass), and thus we will only consider the position $\vec{r}(t)$ of Sphere 2 *relative* to that of Sphere 1.

(a) What is the Lagrangian for the full system, in terms of m_2, $\vec{r}(t)$, $\dot{\vec{r}}(t)$, and the spring constant k?

$$L = \tfrac{1}{2} m_2\dot{\vec{r}}^2 - \tfrac{1}{2}k\vec{r}^2 \ .$$

(b) Write down the resulting Lagrange's equations of motion, in terms of the polar coordinates r and ϕ (in the plane of motion of the system).

In polar coordinates, we have $L = \frac{1}{2}m_2(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{1}{2}kr^2$. Thus, we have $d/dt(\partial L/\partial\dot{\phi}) = 0$ $\Rightarrow m_2r^2\dot{\phi}$ = a constant angular momentum Λ, and $d/dt(\partial L/\partial\dot{r}) - (\partial L/\partial r) = 0$ $\Rightarrow$ $m_2\ddot{r} = m_2r\dot{\phi}^2 - kr$ $\Rightarrow$ $\ddot{r} = -kr/m_2 + \Lambda^2/m_2^2r^3$.

(c) What are the solutions to these equations of motion?

The differential equation above is not easy to solve. It is a complicated integral, and even though it doesn't look too horrible, it can't even be solved in Mathematica or Maple. So please don't worry about solving it – I certainly wouldn't ask you to solve a nasty differential equation like that on a real exam! But I will do it here anyway:

$\ddot{r} = d\dot{r}/dr\ dr/dt = \dot{r}\ d\dot{r}/dr = \alpha r + \beta/r^3$, where $\alpha = -k/m_2$ and $\beta = \Lambda^2/m_2^2$. Multiplying both sides by dr and integrating, we get that $\frac{1}{2}r^2 = \alpha r^2/2 - \beta/(2r^2) + C$, where C is a constant.

Multiplying both sides by two, then square-rooting, and then multiplying both sides by dt gives us that t is equal to the integral of dr/sqrt($\alpha r^2 - \beta/r^2$ + C) plus another constant of integration D. Substituting $s = r^2$ and ds = 2rdr gives us that t is equal to the integral of ds/(2 sqrt(αs^2 + C$s - \beta$)) + D. This integral can be found in big tables of integrals (like Gradshteyn & Ryzhik), and the result is that -1/(2 sqrt(β)) arcsin((C – 2βs)/($C^2 - 4\alpha\beta$)) = t + D. Substituting r back in and simplifying, one gets that
$r(t)$ = sqrt(sqrt($m_2 C^2 - 4k\Lambda$) sin(($2\Lambda t/m_2$) + D) + C) / (Λ sqrt(2)). Ugh!

(d) What is the total energy of the system?

The total energy E = $\dot{r}(\partial L/d\dot{r}) + \dot{\phi}(\partial L/d\dot{\phi}) - L \Rightarrow E = \frac{1}{2}m_2(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}kr^2$ (= T + U). Note that E can also be written in terms of the (constant) angular momentum as $\frac{1}{2}m_2\dot{r}^2 + \Lambda^2/2m_2r^2 + \frac{1}{2}kr^2$.

2. Two gravitating masses m_1 and m_2 are separated by a distance r_0 and released from rest. Write down the Lagrangian and solve Lagrange's equations of motion to show that when the separation is r ($< r_0$), the speeds are: $v_1 = m_2$ sqrt[(2G/(m_1+m_2))($1/r - 1/r_0$)] and $v_2 = m_1$ sqrt[(2G/(m_1+m_2))($1/r - 1/r_0$)] .

If x_1 is the position of the first mass and x_2 is the position of the second, the Lagrangian $L = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + Gm_1m_2/(x_2 - x_1)$. We also know that $r = x_2 - x_1$ and, if we are working in the centre of mass frame, that $m_1x_1 + m_2x_2 = 0$. Thus $x_1 = -m_2x_2/m_1$ and $x_2 = r/(1 + m_2/m_1)$, and the Lagrangian can be written purely in terms of r:
$L = m_1m_2\dot{r}^2/(m_1+m_2) + Gm_1m_2/r$. Thus we have Lagrange's equation of motion for r:
$m_1m_2\ddot{r}/(m_1+m_2) = -Gm_1m_2/r^2 \Rightarrow \ddot{r} = -G(m_1+m_2)/r^2$. One can obtain the solution to that equation of motion via $\ddot{r} = \dot{r}d\dot{r}/dr = -G(m_1+m_2)/r^2$, multiplying both sides by dr, and integrating, which gives you (defining $v \equiv \dot{r}$) that $v^2 = 2G(m_1+m_2)(1/r)$ + C. We know $v^2 = 0$ when $r = r_0$, thus $v^2 = 2G(m_1+m_2)(1/r - 1/r_0)$, and thus v = sqrt[$2G(m_1+m_2)(1/r - 1/r_0)$]. We also know that $x_2 = m_1r/(m_1 + m_2)$, thus $v_2 = m_1v/(m_1 + m_2)$, and thus
$v_2 = m_1$ sqrt[(2G/(m_1+m_2))($1/r - 1/r_0$)], and similiarly,
speed $v_1 = m_2$ sqrt[(2G/(m_1+m_2))($1/r - 1/r_0$)].