P321(b), Assignement 1

1 Exercise 3.1 (Fetter and Walecka)

a)

The problem is that of a point mass rotating along a circle of radius a, rotating with a constant angular velocity Ω. Generally, 3 coordinates are needed to characterize the point mass, however, because the mass is constrained to the circle, and due to the constant angular velocity, the system only has one degree of freedom. Therefore, only one generalized coordinate is needed here, we take the angle θ, from the downward vertical. In the spherical coordinate system, the point mass coordinates are

$$\begin{aligned} x &= a\sin\theta\cos\phi \quad , \\ y &= a\sin\theta\sin\phi \quad , \\ z &= a\cos\theta \quad . \end{aligned} \tag{1}$$

The components of the velocity vector are obtained by differentiating the above according to time

$$\begin{aligned} \dot{x} &= a\dot{\theta}\cos\theta\cos\phi - a\Omega\sin\theta\sin\phi \quad , \\ \dot{y} &= a\dot{\theta}\cos\theta\sin\phi + a\Omega\sin\theta\cos\phi \quad , \\ \dot{z} &= -a\dot{\theta}\sin\theta \quad . \end{aligned} \tag{2}$$

Where the angular velocity is $\Omega = \dot{\phi}$. Looking at the coordinates x, y, z, it might look like we have two degrees of freedom, since we need both θ and ϕ to specify the position. However, because $\Omega = \dot{\phi}$ is fixed, we have $\phi = \Omega t$ which uniquely gives ϕ at any time t.

The Lagrangian L is the difference between the kinetic energy $T = mv^2/2$, and the potential energy $V = mgz$. We choose to take the zero potential at the bottom of the ring, we get

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{ma^2}{2}(\dot{\theta}^2 + \Omega^2\sin^2\theta) \quad , \\ V &= mga(1-\cos\theta) \quad , \\ L &= T - V = \frac{ma^2}{2}(\dot{\theta}^2 + \Omega^2\sin^2\theta) + mga(\cos\theta - 1) \quad . \end{aligned} \tag{3}$$

b)

The point of equilibrium is the point where the mass does not move with respect to the ring. We have to find the equations of motion, and look at the point where $\dot{\theta} = 0$. The equations of motion are given by the Lagrange equations. Since we only have one degree of freedom $q = \theta$, the equations are

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} = 0 \quad \Longrightarrow \quad \ddot{\theta} - \Omega^2 \sin\theta\cos\theta + \frac{g}{a}\sin\theta = 0 \quad . \tag{4}$$

At equilibrium, we have $\theta = \theta_0$, $\dot{\theta} = 0$, $\ddot{\theta}$. Plugging this into the above equations gives

$$\cos\theta_0 = \frac{g}{a\Omega^2} \quad . \tag{5}$$

It is possible to obtain the same condition using Newton's mechanics. Indeed, the equilibrium is achieved because the force of gravity $\vec{W} = m\vec{g}$, the centrifugal force $\vec{F}_c = mv^2/a\sin\theta\vec{u}_r$, and the normal force $\vec{N} = -N\vec{e}_r$, all compensate exactly to 0. Here $\vec{u}_r$ is a unit vector pointing outward, parallel to the plane (x, y), and $\vec{e}_r$ is the radial vector, and in the local rotating frame we also have the vector $\vec{e}_\theta$. In the spherical coordinate system, these vector are

$$\begin{aligned}
\vec{e}_r &= \sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y - \cos\theta\vec{e}_z \quad , \\
\vec{e}_\theta &= \dot{\theta}\cos\theta\cos\phi\vec{e}_x + \dot{\theta}\cos\theta\sin\phi\vec{e}_y + \dot{\theta}\sin\theta\vec{e}_z \quad , \\
\vec{u}_r &= \sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta \quad .
\end{aligned} \tag{6}$$

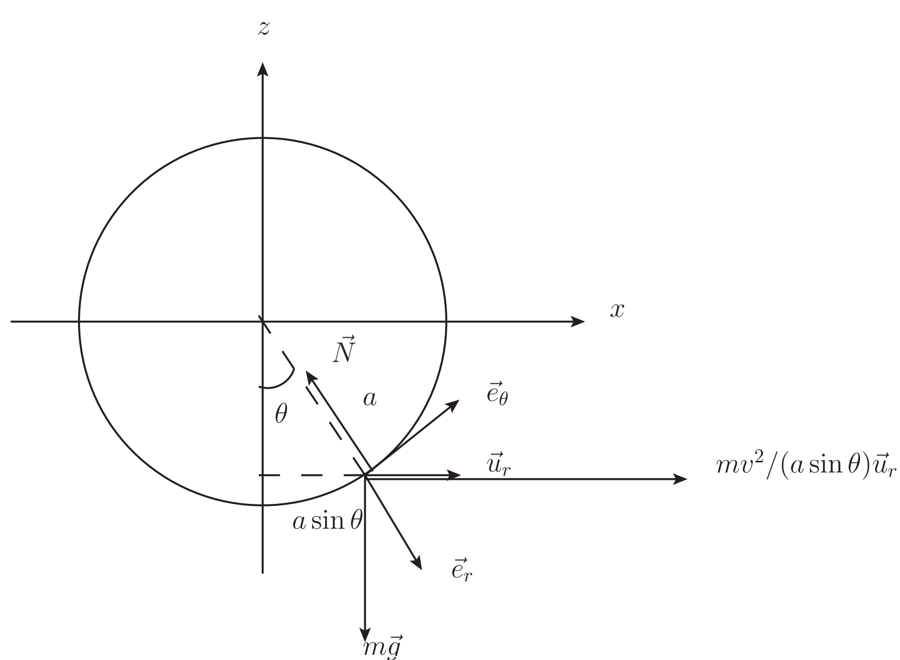

Figure 1: Local frame

The total force $\sum \vec{F} = 0$ at equilibrium, and when split in the directions $\vec{e}_r$, $\vec{e}_\theta$ give the equations

$$\begin{aligned} \vec{e}_r \quad &: \quad N = \frac{mv^2}{a} + mg\cos\theta_0 = ma\Omega^2 \sin^2\theta_0 + mg\cos\theta_0 \quad , \\ \vec{e}_\theta \quad &: \quad \frac{mv^2}{a}\frac{\cos\theta_0}{\sin\theta_0} - mg\sin\theta_0 = 0 \quad \Longleftrightarrow \quad ma\Omega^2 \sin\theta_0\cos\theta_0 - mg\sin\theta_0 = 0 \quad . \end{aligned} \tag{7}$$

The last equation is equivalent to

$$\cos\theta_0 = \frac{g}{a\Omega^2} \quad . \tag{8}$$

c)

In order to investigate the stability of the equilibrium point, we have to perturb the equation of motion around the equilibrium angle, and look at the solution. An oscillatory solution (in general, non-increasing function) implies small oscillation amplitudes around equilibrium, hence a stable point. On the other hand, an exponential (in general increasing function) implies instability. We take $\theta(t) = \theta_0 + \eta(t)$, with $\eta(t) \ll \theta_0$, and plug this in the equations of motion,

$$\ddot{\eta} - \Omega^2 \sin(\theta_0 + \eta)\cos(\theta_0 + \eta) + \frac{g}{a}\sin(\theta_0 + \eta) = 0 \quad . \tag{9}$$

Given that η is small, we can use the Taylor expansions $\cos(\theta_0 + \eta) \approx \cos\theta_0 - \eta\sin\theta_0$, and $\sin(\theta_0 + \eta) \approx \sin\theta_0 + \eta\cos\theta_0$. We obtain the expanded equation as follows

$$\ddot{\eta} + \eta\Omega^2 \sin^2\theta_0 = 0 \quad . \tag{10}$$

This is clearly an equation for harmonic oscillator, with frequency $\omega^2 = \Omega^2\sin^2\theta_0$. The equilibrium point is stable.

d)

The equilibrium condition was $\cos\theta_0 = \frac{g}{a\Omega^2}$. If $g > a\Omega^2$, then $\cos\theta_0 > 1$, which is not possible. There exist no equilibrium in this case. The physical interpretation is that the angular velocity is too small, and the centrifugal force is not enough to keep the mass at a given height. The mass rolls down to the bottom of the ring, which is the only equilibrium point.

2 Exercise 3.2 (Fetter and Walecka)

This problem is very similar to the previous one.

a)

Here again, since the angle between the wire and the vertical is fixed $\theta = \theta_0$, and because the angular velocity Ω is fixed, there is only one degree of freedom, and only one generalized coordinate which is the distance l between the mass and the origin. Because of the spherical symmetry, the coordinates of the point mass are

$$\begin{aligned} x &= l \sin\theta_0 \cos\phi \quad , \\ y &= l \sin\theta_0 \sin\phi \quad , \\ z &= l \cos\theta_0 \quad . \end{aligned} \tag{11}$$

The components of the velocity vector are obtained by differentiating the above according to time

$$\begin{aligned} \dot{x} &= \dot{l} \sin\theta_0 \cos\phi - l\Omega \sin\theta_0 \sin\phi \quad , \\ \dot{y} &= \dot{l} \sin\theta_0 \sin\phi + l\Omega \sin\theta_0 \cos\phi \quad , \\ \dot{z} &= \dot{l} \cos\theta_0 \quad . \end{aligned} \tag{12}$$

We can readily write the kinetic and potential energies, as well as the Lagrangian

$$\begin{aligned} T &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}\left(\dot{l}^2 + l^2\Omega^2 \sin^2\theta_0\right) \quad , \\ V &= mgl \cos\theta_0 \quad , \\ L &= \frac{m}{2}\left(\dot{l}^2 + l^2\Omega^2 \sin^2\theta_0\right) - mgl\cos\theta_0 \quad . \end{aligned} \tag{13}$$

b)

Again, in order to find the equilibrium point, we must look at the equations of motion, which is obtained from the Lagrange equations

$$\frac{d}{dt}\frac{dL}{d\dot{l}} - \frac{dL}{dl} = 0 \quad \Longrightarrow \quad \ddot{l} - l\Omega^2 \sin^2\theta_0 + g\cos\theta_0 = 0 \quad . \tag{14}$$

The point mass will be in equilibrium at $l = l_0$, provided that $\dot{l} = \ddot{l} = 0$. When these conditions are plugged in the above equation, we find

$$l_0 = \frac{g\cos\theta_0}{\Omega^2 \sin^2\theta_0} \quad . \tag{15}$$

Similarly to the previous exercise, we can find the same condition from Newton's forces. There is the weight, the normal force and the centrifugal force. We decompose each on the basis vector $\vec{e}_x$, and $\vec{e}_z$, and we get two equations

$$\begin{aligned} \vec{e}_x \quad &: \quad -N\cos\theta_0 + ml_0\Omega^2\sin\theta_0 = 0 \quad \Longrightarrow \quad N = \frac{ml_0\Omega^2\sin\theta_0}{\cos\theta_0} \quad , \\ \vec{e}_y \quad &: \quad N\sin\theta_0 = mg \quad \Longrightarrow \quad l_0 = \frac{g\cos\theta_0}{\Omega^2\sin^2\theta_0} \quad . \end{aligned} \tag{16}$$

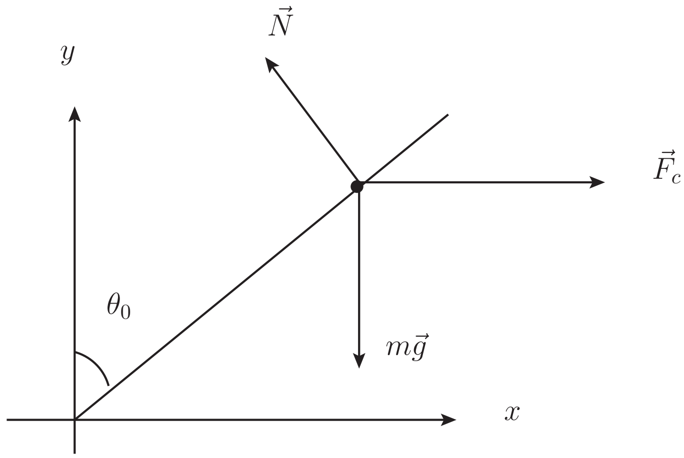

Figure 2: Straight wire local frame

c)

In order to investigate the stability, again we perturb the system around the equilibrium point, and look at the effect on the motion. We take $l(t) = l_0 + \epsilon(t)$, with $\epsilon(t) \ll l_0$, and plug this in the equations of motion,

$$\ddot{\epsilon} - (l_0 + \epsilon)\Omega^2 \sin^2\theta_0 + g\cos\theta_0 = 0 \quad \Longleftrightarrow \quad \ddot{\epsilon} = \epsilon\Omega^2 \sin^2\theta_0 \quad , \tag{17}$$

where we used the equilibrium condition $g\cos\theta_0 = l\Omega^2\sin^2\theta_0$. Since $\Omega^2\sin^2\theta_0 > 0$, it is clear that the solution is a linear combination of exponentials

$$\epsilon(t) = Ae^{\Omega^2\sin^2\theta_0 t} + Be^{-\Omega^2\sin^2\theta_0 t} \to \infty \quad . \tag{18}$$

This solution is highly unstable due to the growing exponential. The equilibrium point is an unstable equilibrium point. In the previous exercise, the equilibrium point was stable.