Fetter + Walecka 2.2

Part 1: Using Newton's laws:

From F+W (11.3), we have:

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_e = \vec{F}_g + \vec{F}' \nearrow 0 - 2m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_e - m\vec{\omega}\times(\vec{\omega}\times r)$$

$$\Rightarrow \quad m\ddot{\vec{r}} = \frac{-GM_e m}{R_e^2}\hat{r} - 2m\vec{\omega}\times\dot{\vec{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= m\vec{g} - 2m\vec{\omega}\times\dot{\vec{r}} \quad \text{where} \quad \leftarrow (11.8)$$

$$\vec{g} = \frac{-GM_e}{R_e^2}\hat{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

So we've shown (11.8). To get $\vec{g}$ in the form given in (11.6), we need to determine what $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is... Change into Cartesian coordinates (since cross products are not easy to calculate in spherical coordinates). Then

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \Rightarrow \vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{\omega}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ -\omega y & \omega x & 0 \end{vmatrix} = \begin{pmatrix} -\omega^2 x \\ -\omega^2 y \\ 0 \end{pmatrix} = \underline{-\omega^2(x\hat{x} + y\hat{y})}.$$

To convert back into spherical coordinates, use $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

and $\hat{x} = \begin{pmatrix} \sin\theta\cos\phi \\ \cos\theta\cos\phi \\ -\sin\phi \end{pmatrix}$, $\hat{y} = \begin{pmatrix} \sin\theta\sin\phi \\ \cos\theta\sin\phi \\ \cos\phi \end{pmatrix}$, $\hat{z} = \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$. Then

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\omega^2(x\hat{x}+y\hat{y}) = -\omega^2\left[r\sin\theta\cos\phi\begin{pmatrix}\sin\theta\cos\phi\\ \cos\theta\cos\phi\\ -\sin\phi\end{pmatrix} + r\sin\theta\sin\phi\begin{pmatrix}\sin\theta\sin\phi\\ \cos\theta\sin\phi\\ \cos\phi\end{pmatrix}\right]$$

$$= -\omega^2 r\begin{pmatrix}\sin^2\theta\\ \frac{1}{2}\sin2\theta\\ 0\end{pmatrix} = -\omega^2 R_e\left(\sin^2\theta\,\hat{r} + \tfrac{1}{2}\sin2\theta\,\hat{\theta}\right).$$ Thus

$$\vec{g} = \frac{-GM_e}{R_e^2}\hat{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\left(\frac{GM_e}{R_e^2} - \omega^2 R_e\sin^2\theta\right)\hat{r} + \tfrac{1}{2}\omega^2 R_e\sin2\theta\,\hat{\theta}$$

(11.6). //

Part 2: Using Lagrange's equations:

$$L(\dot{\vec{r}},\vec{r},t) = T - U$$

(T: kinetic energy; U: potential energy)

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$ (Landau + Lifshitz 4.6)

$U = \frac{-GM_e m}{r}$ (potential due to gravity)

$$\Rightarrow L(\dot{\vec{r}},\vec{r},t) = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) + \frac{GM_e m}{r}$$

We have Lagrange's eqns:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad ①$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \quad ②$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \quad ③$$

Note that, since we are at a fixed point on the surface of the Earth, we have $r = R_e$, $\dot{r} = 0$, and $\dot{\theta} = 0$.
Thus, from ①, we have $m\ddot{r} = \frac{-GM_e m}{R_e^2} + mR_e\omega^2\sin^2\theta$.
From ②, we get $mr^2\ddot{\theta} = mr^2\dot{\phi}^2\sin\theta\cos\theta \Rightarrow \ddot{\theta} = \frac{1}{2}\omega^2\sin2\theta$.
And from ③, we get $mr^2\ddot{\phi}\sin^2\theta = 0 \Rightarrow \ddot{\phi} = 0$.

$$\text{Thus } \ddot{\vec{r}} = \ddot{r}\hat{r} + r\ddot{\theta}\hat{\theta} = -\left(\frac{GM_e}{R_e^2} - \omega^2 R_e\sin^2\theta\right)\hat{r} + \tfrac{1}{2}\omega^2 R_e\sin2\theta\,\hat{\theta}$$

(11.6). //

P321(b), Assignement 2
Ex. 3.5, Fetter & Walecka

1 Exercise 3.5 (Fetter & Walecka)

The setup of the exercise is shown in figure

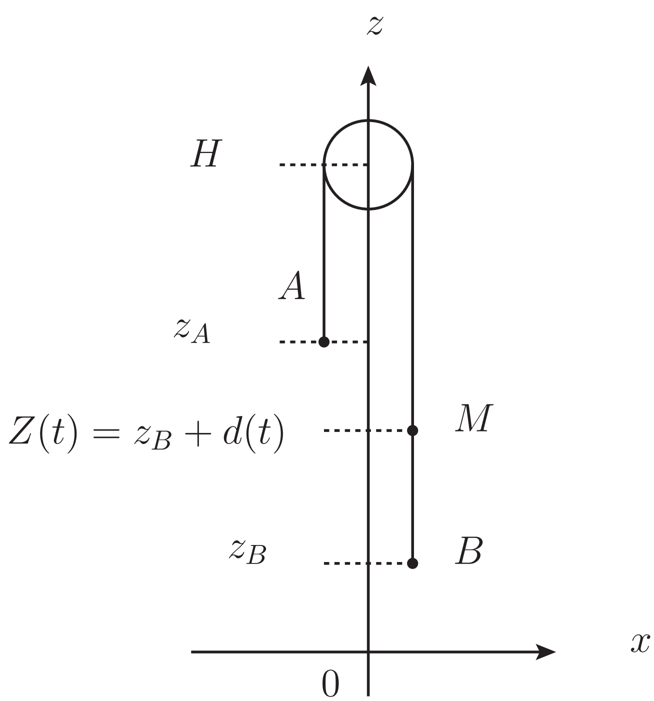

Figure 1: The monkey, of mass M, starts from point B and climbs a distance $d(t)$. The Bananas are at the point A and have a mass m. The string has a constant length L.

Equation of motion

In order to determine the equation of motion, one has to know the number of degrees of freedom. Let's count the number of degree of freedom. Each of the banana and the monkey require 3 coordinates $\{x, y, z\}$, which means a total of 6 coordinates for the whole system. However, they are both constrained to z axis, i.e. $x, y = 0$, which suppresses 4 degrees of freedom. Besides, they are linked via the string which has a constant length, so the height of the bananas is related to that of the monkey, which eliminates an extra degree of freedom. We are left with only one degree of freedom, namely the height of the monkey $Z(t) = z_B + d(t)$. We should only need one equation to describe the system.

Since the string, attached at a height H, has a constant length L, we have the following relation

$$2H = z_A + L + z_B = z_A + L - d(t) + Z(t) \quad \Leftrightarrow \quad z_A = 2H + d(t) - L - Z(t) \quad . \tag{1}$$

The initial conditions will be

$$\begin{aligned} &Z(0) = \dot{Z}(0) = 0 \quad , \quad d(0) = \dot{d}(0) = 0 \quad , \\ &z_A(0) = 2H - L \quad , \quad z_B(0) = 0 \quad . \end{aligned} \tag{2}$$

The Lagrangian for the whole system is

$$L = T_{bananas} + T_{monkey} - V_{bananas} - V_{monkey} \quad , \tag{3}$$

with

$$\begin{aligned} &T_{monkey} = \frac{M}{2}\dot{Z}^2 \quad , \\ &T_{bananas} = \frac{m}{2}\dot{z}_A^2 = \frac{m}{2}(\dot{d}^2 + \dot{Z}^2 - 2\dot{d}\dot{Z}) \quad , \\ &-V_{monkey} = -MgZ \quad , \\ &-V_{bananas} = -mgz_A = -mg(2H - L + d - Z) \quad , \end{aligned} \tag{4}$$

which leads to

$$\begin{aligned} &\frac{d}{dt}\frac{dL}{d\dot{Z}} = (M+m)\ddot{Z} - m\ddot{d} \quad , \\ &\frac{dL}{dZ} = -(M-m)g \quad . \end{aligned} \tag{5}$$

The equation of motion is derived from the Lagrange equation

$$\frac{d}{dt}\frac{dL}{d\dot{Z}} = \frac{dL}{dZ} \quad \Leftrightarrow \quad (M+m)\ddot{Z} - m\ddot{d} = (m-M)g \tag{6}$$

The equation of motion is equivalent to

$$\ddot{Z} = \frac{m}{M+m}\ddot{d} + \frac{m-M}{m+M}g \quad , \tag{7}$$

Integrate the equation of motion

We can integrate twice the equation of motion with respect to time, to obtain

$$Z(t) - \dot{Z}(0)t - Z(0) = \frac{m}{m+M}(d(t) - \dot{d}(0)t - d(0)) + \frac{m-M}{m+M}\frac{g}{2}t^2 \quad , \tag{8}$$

using the initial conditions we get

$$Z(t) = \frac{m}{m+M}d(t) + \frac{m-M}{m+M}\frac{g}{2}t^2 \quad . \tag{9}$$

Special case: $m = M$ In the case $m = M$, we get the equation

$$Z(t) = \frac{1}{2}d(t) \quad . \tag{10}$$

The height of the bananas is given by equation (1)

$$z_A(t) - z_A(0) = d(t) - Z(t) = \frac{1}{2}d(t) \quad , \tag{11}$$

in other words

$$Z(t) = z_A(t) - z_A(0) \quad , \tag{12}$$

namely, the vertical distance they travel is the exact same, hence their vertical separation remains constant!