Fetter + Walecka 3.8

(a) In cylindrical coordinates, $L = T - U = \frac{1}{2}m(\dot{r}^2 + \dot{z}^2 + r^2\dot{\phi}^2) - U$.

We have $z = \alpha \sin\frac{r}{R} \Rightarrow \dot{z} = (\alpha\dot{r}/R)\cos\left(\frac{r}{R}\right)$ and $U = mgz = mg\alpha\sin\left(\frac{r}{R}\right)$.

Thus $L(r,\phi;\dot{r},\dot{\phi}) = \frac{1}{2}m\left[\dot{r}^2\left(1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r}{R}\right)\right) + r^2\dot{\phi}^2\right] - mg\alpha\sin\left(\frac{r}{R}\right)$.

The first equation of motion we'll do is for ϕ. $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \cancel{\frac{\partial L}{\partial\phi}} = 0$

$\Rightarrow \frac{d}{dt}(mr^2\dot{\phi}) = 0 \Rightarrow mr^2\dot{\phi}$ = a constant angular momentum Λ.

Now for r. $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \Rightarrow$

$$\frac{d}{dt}\left[m\dot{r}\left(1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r}{R}\right)\right)\right] - mr\dot{\phi}^2 + m\dot{r}^2\frac{\alpha^2}{R^3}\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right) + \frac{mg\alpha}{R}\cos\left(\frac{r}{R}\right) = 0$$

$$\Rightarrow m\ddot{r}\left(1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r}{R}\right)\right) - 2m\dot{r}^2\frac{\alpha^2}{R^3}\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right) = mr\dot{\phi}^2 - m\dot{r}^2\frac{\alpha^2}{R^3}\sin\left(\frac{r}{R}\right)\cos\left(\frac{r}{R}\right) - \frac{mg\alpha}{R}\cos\left(\frac{r}{R}\right)$$

$$\Rightarrow \ddot{r}\left(1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r}{R}\right)\right) = \frac{\Lambda^2}{m^2r^3} + \frac{\dot{r}^2\alpha^2}{2R^3}\sin\left(\frac{2r}{R}\right) - \frac{g\alpha}{R}\cos\left(\frac{r}{R}\right).$$

(b) Stationary horizontal circular orbits have $\ddot{r} = \dot{r} = 0$

$$\Rightarrow \frac{\Lambda^2}{m^2r^3} = \frac{g\alpha}{R}\cos\left(\frac{r}{R}\right) \Rightarrow \Lambda = \sqrt{\frac{g\alpha m^2r^3}{R}\cos\left(\frac{r}{R}\right)}$$

$$\Rightarrow \omega \equiv \dot{\phi} = \sqrt{\frac{g\alpha}{rR}\cos\left(\frac{r}{R}\right)} \quad \left(\text{and thus } r = \frac{g\alpha}{\omega^2R}\cos\left(\frac{r}{R}\right)\right).$$

(c) $r(t) = r_0 + \eta(t)$, where $\eta \ll r_0$.

For the orbit to be stable, $\ddot{r}\ (=\ddot{\eta})$ should be negative for η positive, and $\ddot{r}\ (=\ddot{\eta})$ should be positive for η negative.

From the equation of motion for r, we have that

$$\ddot{\eta}\left(1 + \frac{\alpha^2}{R^2}\cos^2\left(\frac{r_0+\eta}{R}\right)\right) = \frac{L^2}{m^2(r_0+\eta)^3} + \cancelto{\sim\dot{\eta}^2}{\frac{\dot{\eta}^2\alpha^2}{2R^3}\sin\left(\frac{2r_0+2\eta}{R}\right)} - \frac{g\alpha}{R}\cos\left(\frac{r_0+\eta}{R}\right)$$

Note: $\cos(\alpha+\varepsilon) \approx \cos\alpha + \varepsilon\sin\alpha$ for small ε, so $\cos^2(\alpha+\varepsilon) \approx \cos^2\alpha - \varepsilon\sin 2\alpha$

$$\Rightarrow \ddot{\eta}\left(1 + \frac{\alpha^2}{R^2}\left[\cos^2\left(\frac{r_0}{R}\right) - \cancelto{\eta^2}{\eta\sin\left(\frac{2r_0}{R}\right)}\right]\right) = \frac{\Lambda^2}{m^2r_0^3 + 3m^2\eta r_0^2} - \frac{g\alpha}{R}\left(\cos\left(\frac{r_0}{R}\right) - \frac{\eta}{R}\sin\left(\frac{r_0}{R}\right)\right)$$

$$\Rightarrow \ddot{\eta} = \frac{1}{1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r_0}{R}\right)}\left[\frac{\Lambda^2}{m^2r_0^3}\left(1 - \frac{3\eta}{r_0}\right) - \frac{g\alpha}{R}\cos\left(\frac{r_0}{R}\right) + \frac{g\alpha\eta}{R^2}\sin\left(\frac{r_0}{R}\right)\right]$$

But $\Lambda^2 = \frac{g\alpha m^2 r_0^3}{R}\cos\left(\frac{r_0}{R}\right)$ near a stable orbit $\Rightarrow$

$$\ddot{\eta} = \frac{1}{1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r_0}{R}\right)}\left[\frac{-3\eta\Lambda^2}{m^2r_0^4} + \frac{g\alpha\eta}{R^2}\sin\left(\frac{r_0}{R}\right)\right].$$

For this to be stable, $\frac{3\Lambda^2}{m^2r_0^4} > \frac{g\alpha}{R^2}\sin\left(\frac{r_0}{R}\right)$

$$\Rightarrow \frac{3m^2r_0^4\omega^2}{m^2r_0^4} > \frac{g\alpha}{R^2}\sin\left(\frac{r_0}{R}\right) \Rightarrow \omega^2 > \frac{g\alpha}{3R^2}\sin\left(\frac{r_0}{R}\right)$$

$$\Rightarrow \frac{g\alpha}{r_0R}\cos\left(\frac{r_0}{R}\right) > \frac{g\alpha}{3R^2}\sin\left(\frac{r_0}{R}\right) \Rightarrow \frac{3R}{r_0} > \tan\left(\frac{r_0}{R}\right).$$

If we define $x \equiv \frac{r_0}{R} \Rightarrow x\tan x < 3$. This is true for $x < 1.1925$ (use a graphing program) $\Rightarrow r_0 < 1.1925R$.

(d) We have $\ddot{\eta} = -f^2\eta$, where f is the frequency (in $\frac{\text{radians}}{\text{sec}}$)

$$\ddot{\eta} = \frac{-\eta}{1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r_0}{R}\right)}\left[\frac{3\Delta^2}{m^2 r_0^4} - \frac{g\alpha}{R^2}\sin\left(\frac{r_0}{R}\right)\right]$$

$$\Rightarrow f^2 = \frac{1}{1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r_0}{R}\right)}\left[\frac{3m^2 r_0^4\omega^2}{m^2 r_0^4} - \frac{g\alpha}{R^2}\sin\left(\frac{r_0}{R}\right)\right]$$

$$= \frac{1}{1+\frac{\alpha^2}{R^2}\cos^2\left(\frac{r_0}{R}\right)}\left[\frac{3g\alpha}{r_0 R}\cos\left(\frac{r_0}{R}\right) - \frac{g\alpha}{R^2}\sin\left(\frac{r_0}{R}\right)\right]$$

$$= \frac{g\alpha\cos\left(\frac{r_0}{R}\right)}{Rr_0+\frac{\alpha^2 r_0}{R}\cos^2\left(\frac{r_0}{R}\right)}\left[3-\tan\left(\frac{r_0}{R}\right)\right]$$

$$\Rightarrow f = \sqrt{\frac{g\alpha\cos\left(\frac{r_0}{R}\right)}{Rr_0+\frac{\alpha^2 r_0}{R}\cos^2\left(\frac{r_0}{R}\right)}\left[3-\tan\left(\frac{r_0}{R}\right)\right]}\ .$$