Feynman rules for QED

- The Feynman Rules for QED
- Setting up Amplitudes
- Casimir’s Trick
- Trace Theorems

Slides from Sobie and Blokland
Electrons and positrons

- **spinors**
  \( u^{(s)} \) and \( v^{(s)} \) (s = spin) satisfy the Dirac equations
  \((\gamma^\mu p_\mu - m)u = 0\) and
  \((\gamma^\mu p_\mu + m)v = 0\)

- **adjoints**
  \( \overline{u} = u^\dagger \gamma^0 \) and \( \overline{v} = v^\dagger \gamma^0 \) satisfy
  \( \overline{u}(\gamma^\mu p_\mu - m) = 0 \) and \( \overline{v}(\gamma^\mu p_\mu + m) = 0 \)

- **orthogonality**
  \( \overline{u}^{(1)}u^{(2)} = 0 \) and \( \overline{v}^{(1)}v^{(2)} = 0 \)

- **normalization**
  \( \overline{u}u = 2m \) and \( \overline{v}v = -2m \)

- **completeness**
  \( \sum_s u^{(s)}\overline{u}^{(s)} = \gamma^\mu p_\mu + m \) and
  \( \sum_s v^{(s)}\overline{v}^{(s)} = \gamma^\mu p_\mu - m \)
Photons

\[ A^\mu(x) = ae^{-ip\cdot x} \epsilon^\mu(p) \]

- Lorentz condition
  \[ \epsilon^\mu p_\mu = 0 \]

- orthogonality
  \[ \epsilon_{\mu(1)}^\mu \epsilon_{\mu(2)} = 0 \]

- normalization
  \[ \epsilon_{\mu}^\mu = 1 \]

- Coulomb gauge
  \[ \epsilon^0 = 0 \text{ and } \epsilon \cdot p = 0 \]

- Completeness
  \[ \sum_s (\epsilon(s))^i (\epsilon(s)^*)^j = \delta_{ij} - (p_ip_j)/p^2 \]
The Feynman Rules for QED

The Feynman rules provide the recipe for constructing an amplitude $\mathcal{M}$ from a Feynman diagram.

- **Step 1:** For a particular process of interest, draw a Feynman diagram with the minimum number of vertices. There may be more than one.

  $e^-, p_3 \quad e^+, p_4$

  $\gamma$

  $e^-, p_1 \quad e^+, p_2$
The Feynman Rules for QED

- **Step 2:** For each Feynman diagram, label the four-momentum of each line, enforcing four-momentum conservation at every vertex. Note that arrows are only present on fermion lines and they represent particle flow, not momentum.

\[ e^-, p_1 \quad \gamma \quad e^+, p_2 \]

\[ e^-, p_3 \quad \gamma \quad e^+, p_4 \]
• **Step 3:** The amplitude depends on
  1. Vertex factors
  2. Propagators for internal lines
  3. Wavefunctions for external lines
Vertex Factors

- Every QED vertex, contributes a factor of $i g_e \gamma^\mu$.

- $g_e$ is a dimensionless coupling constant and is related to the fine-structure constant by

$$\alpha = \frac{g_e^2}{4\pi}$$
Propagators

- Each internal photon connects two vertices of the form \( i g_e \gamma^\mu \) and \( i g_e \gamma^\nu \), so we should expect the photon propagator to contract the indices \( \mu \) and \( \nu \).

  \[
  \text{Photon propagator: } \frac{-i g_{\mu\nu}}{q^2}
  \]

- Internal fermions have a more complicated propagator,

  \[
  \text{Fermion propagator: } \frac{i(q + m)}{q^2 - m^2}
  \]

  The sign of \( q \) matters here — we take it to be in the same direction as the fermion arrow.
External Lines

- Since both the vertex factor and the fermion propagators involve $4 \times 4$ matrices, but the amplitude must be a scalar, the external line factors must sit on the outside.

- Work *backwards* along every fermion line using:

<table>
<thead>
<tr>
<th>$e^-$ in</th>
<th>$e^-$ out</th>
<th>$e^+$ in</th>
<th>$e^+$ out</th>
<th>$\gamma$ in</th>
<th>$\gamma$ out</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>$\bar{u}$</td>
<td>$\bar{v}$</td>
<td>$v$</td>
<td>$\epsilon_\mu$</td>
<td>$\epsilon_\mu^*$</td>
</tr>
</tbody>
</table>
Matrix elements I

follow fermion lines backward to give $\mp(2)ig\gamma^\mu u(1)$
Matrix elements II

The matrix element is proportional to the two currents in the diagram below.

\[ [\bar{u}_3(i\gamma^\mu)u_1]\left(\frac{-ig_{\mu\nu}}{(p_1 - p_3)^2}\right) [\bar{v}_2(i\gamma^\nu)v_4] \]

\[ e^-, \ p_3 \quad \gamma \quad e^+, \ p_4 \]

\[ e^-, \ p_1 \quad \gamma \quad e^+, \ p_2 \]
And Finally...

- **Step 4:** The overall amplitude is the coherent sum of the individual amplitudes for each diagram:

\[ \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \ldots \]

\[ \Rightarrow |\mathcal{M}^2| = |\mathcal{M}_1 + \mathcal{M}_2 + \ldots|^2 \]

- **Step 4a:** *Antisymmetrization.* Include a minus sign between diagrams that differ only in the exchange of two identical fermions.
Examples

- There are only a handful of ways to make tree-level diagrams in QED.

- Today, we will construct amplitudes for Bhabha scattering \((e^+ e^- \rightarrow e^+ e^-)\) and Compton scattering \((e \gamma \rightarrow e \gamma)\).

- Next week, we will undertake thorough calculations for Mott scattering \((e \ell \rightarrow e \ell)\), pair annihilation \((e^+ e^- \rightarrow \gamma \gamma)\). You will examine fermion pair-production via \((e^+ e^- \rightarrow f \bar{f})\) for your assignment.
Example: Bhabha Scattering

\[ e^-, p_3 \quad \rightarrow \quad e^+, p_4 \quad \rightarrow \quad e^-, p_3 \quad \rightarrow \quad e^+, p_4 \]

\[ e^-, p_1 \quad \rightarrow \quad e^+, p_2 \quad \rightarrow \quad e^-, p_1 \quad \rightarrow \quad e^+, p_2 \]

- Antisymmetrization \( \Rightarrow M = M_t - M_s \)

\[
M_t = i \left[ \bar{u}_3 (i g_\gamma \gamma^\mu) u_1 \right] \left( \frac{-i g_{\mu\nu}}{(p_1 - p_3)^2} \right) \left[ \bar{v}_2 (i g_\gamma \gamma^\nu) v_4 \right]
\]

\[
M_s = i \left[ \bar{u}_3 (i g_\gamma \gamma^\mu) v_4 \right] \left( \frac{-i g_{\mu\nu}}{(p_1 + p_2)^2} \right) \left[ \bar{v}_2 (i g_\gamma \gamma^\nu) u_1 \right]
\]
Example: Compton Scattering

\[ \gamma, \ p_3 \quad e^-, \ p_4 \quad \gamma, \ p_3 \quad e^-, \ p_4 \]

\[ e^-, \ p_1 \quad \gamma, \ p_2 \quad e^-, \ p_1 \quad \gamma, \ p_2 \]

- No antisymmetrization \( \Rightarrow \ M = M_1 + M_2 \)

\[ M_1 = i \left[ \bar{u}_4 (i g_e \gamma^\mu) \left( \frac{i (\not{p}_1 - \not{p}_3 + m)}{(p_1 - p_3)^2 - m^2} \right) (i g_e \gamma^\nu) u_1 \right] \epsilon^*_3 \epsilon_2 \]

\[ M_2 = i \left[ \bar{u}_4 (i g_e \gamma^\mu) \left( \frac{i (\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2} \right) (i g_e \gamma^\nu) u_1 \right] \epsilon^*_3 \epsilon_2 \]
Polarized Particles

- A typical QED amplitude might look something like

\[ M \sim \left[ \bar{u}_1 \Gamma^\mu v_2 \right] \epsilon_{3\mu} \]

The Feynman rules won’t take us any further, but to get a number for \( M \) we will need to substitute explicit forms for the wavefunctions of the external particles: \( \bar{u}_1 \), \( v_2 \), and \( \epsilon_{3\mu} \).

- If all external particles have a known polarization, this might be a reasonable way to calculate things. More often, though, we are interested in unpolarized particles.
Spin-Averaged Amplitudes

- If we do not care about the polarizations of the particles then we need to
  1. *Average* over the polarizations of the initial-state particles
  2. *Sum* over the polarizations of the final-state particles in the squared amplitude $|\mathcal{M}|^2$.

- We call this the spin-averaged amplitude and we denote it by
  $$\langle |\mathcal{M}|^2 \rangle$$

- Note that the averaging over initial state polarizations involves summing over all polarizations and then dividing by the number of independent polarizations, so $\langle |\mathcal{M}|^2 \rangle$ involves a sum over the polarizations of *all* external particles.
Spin Sums

- Let’s simplify things even further and suppose that we have

\[ M \sim [\bar{u}_1 \Gamma u_2] \]

Then

\[ |M|^2 \sim [\bar{u}_1 \Gamma u_2] [\bar{u}_1 \Gamma u_2]^* \]

\[ \sim [\bar{u}_1 \Gamma u_2] \left[u_1^\dagger \gamma^0 \Gamma u_2 \right]\dagger \]

\[ \sim [\bar{u}_1 \Gamma u_2] \left[u_2^\dagger \Gamma^\dagger \gamma^0 \Gamma^\dagger u_1 \right] \]

\[ \sim [\bar{u}_1 \Gamma u_2] \left[u_2^\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 u_1 \right] \]

\[ \sim [\bar{u}_1 \Gamma u_2] \left[\bar{u}_2 \Gamma u_1 \right] \]
\[ |\mathcal{M}|^2 \sim [\bar{u}_1 \Gamma u_2] [\bar{u}_2 \bar{\Gamma} u_1] \]

- Applying the completeness relation

\[ \sum_{s_i=1,2} u_i^{s_i} \bar{u}_i^{s_i} = (\phi_i + m_i) \]

to \( u_2 \bar{u}_2 \) in the squared-amplitude above (summing over the spins of particle 2),

\[ \sum_{s_2} |\mathcal{M}|^2 \sim [\bar{u}_1 \Gamma (\phi_2 + m_2) \bar{\Gamma} u_1] \]

\[ \sim [\bar{u}_1 Q u_1] \]
• The right-hand side is just a number, but if we represent the matrix multiplication with summations over indices, we can rewrite it as

\[
\bar{u}_1 Q u_1 = (\bar{u}_1)_i Q_{ij} (u_1)_j = Q_{ij} (u_1 \bar{u}_1)_{ji} = [Q (u_1 \bar{u}_1)]_{ii} = \text{Tr} [Q(u_1 \bar{u}_1)]
\]

• Finally, we apply the completeness relation once again, so that we get

\[
\sum_{s_1} |\mathcal{M}|^2 \sim \text{Tr} [Q (\phi_1 + m_1)]
\]
In total, we have

\[ \mathcal{M} \sim [\bar{u}_1 \Gamma u_2] \]

\[ \Rightarrow \langle |\mathcal{M}|^2 \rangle \sim \frac{1}{2} \text{Tr} \left[ \Gamma (\phi_2 + m_2) \bar{\Gamma} (\phi_1 + m_1) \right] \]

The factor of \( \frac{1}{2} \) is from the averaging over initial spins, assuming exactly one of \( u_1 \) and \( u_2 \) corresponds to an initial-state particle. If they are both in the initial state (e.g., pair annihilation), the factor is \( \frac{1}{4} \). If neither is in the initial state (e.g., pair production), the factor is 1.
Casimir’s Trick

- This procedure of calculating spin-averaged amplitudes in terms of traces is known as Casimir’s Trick.

\[ \sum_{\text{all spins}} [\bar{u}_a \Gamma_1 u_b] [\bar{u}_a \Gamma_2 u_b]^* = \text{Tr} \left[ \Gamma_1 (\psi_b + m_b) \bar{\Gamma}_2 (\psi_a + m_a) \right] \]

- If antiparticle spinors (\( \nu \)) are present in the spin sum, we use the corresponding completeness relation

\[ \sum_{s_i = 1,2} \nu^s_i \bar{\nu}^s_i = (\psi_i - m_i) \]
Traces

- Because of Casimir’s Trick, we’re going to find ourselves calculating a lot of traces involving $\gamma$-matrices.

- General identities about traces:

\[
\begin{align*}
\text{Tr}(A + B) &= \text{Tr}(A) + \text{Tr}(B) \\
\text{Tr}(\alpha A) &= \alpha \text{Tr}(A) \\
\text{Tr}(AB) &= \text{Tr}(BA) \\
\text{Tr}(ABC) &= \text{Tr}(CAB) = \text{Tr}(BCA)
\end{align*}
\]
Building Blocks

- The two major identities that we will need in order to build more complicated trace identities are

\[ g_{\mu \nu} g^{\mu \nu} = 4 \]
\[ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu \nu}(\times I) \]

- You can show that \( \gamma_\mu \gamma^\mu = 4 \) and \( \gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu \lambda} \). In a similar fashion, we find that

\[ \gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu (2g^{\mu \nu} - \gamma^\mu \gamma^\nu) \]
\[ = 2\gamma^\nu - \gamma_\mu \gamma^\mu \gamma^\nu \]
\[ = 2\gamma^\nu - 4\gamma^\nu \]
\[ = -2\gamma^\nu \]
Simple Trace Identities

- The simplest trace identity is: $\text{Tr}(1) = 4$
- The trace of a single $\gamma$ matrix is zero, as is the trace of any odd number of $\gamma$-matrices.
- For 2 $\gamma$-matrices,
  \[
  \text{Tr}(\gamma^\mu \gamma^\nu) = \frac{\text{Tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)}{2} = \frac{\text{Tr}(2g^{\mu\nu})}{2} = g^{\mu\nu}\text{Tr}(1) = 4g^{\mu\nu}
  \]
Traces With $\gamma^5$

- The vertex factor for weak interactions involves $\gamma^5$.
- By inspection, $\text{Tr}(\gamma^5) = 0$.
- Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (an even number of $\gamma$-matrices),
  \[
  \text{Tr}(\gamma^5\gamma^\mu) = 0 \\
  \text{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) = 0
  \]
- Also,
  \[
  \text{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0
  \]
The Non-Trivial $\gamma^5$ Trace

- Only with 4 (or more) other $\gamma$-matrices can we obtain a nonzero trace involving $\gamma^5$:

$$\text{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4i \epsilon^{\mu \nu \lambda \sigma}$$

where the totally antisymmetric tensor is defined as

$$\epsilon^{\mu \nu \lambda \sigma} \equiv \begin{cases} 
-1 & \text{for even permutations of 0123} \\
+1 & \text{for odd permutations of 0123} \\
0 & \text{if any 2 indices are the same}
\end{cases}$$
Contractions of the $\epsilon$ Tensor

- Since $\varepsilon^{\mu\nu\lambda\sigma}$ is completely antisymmetric, we will get zero when we contract this with any tensor that is symmetric in 2 indices, such as $g^{\mu\nu}$ or $(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)$.

- Only contractions with another antisymmetric tensor survive:

\[
\begin{align*}
\varepsilon^{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\lambda\sigma} &= -24 \\
\varepsilon^{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\lambda\tau} &= -6 \delta^\sigma_\tau \\
\varepsilon^{\mu\nu\lambda\sigma} \varepsilon_{\mu\nu\theta\tau} &= -2 \left( \delta^\lambda_\theta \delta^\sigma_\tau - \delta^\lambda_\tau \delta^\sigma_\theta \right) \\
&\vdots
\end{align*}
\]
Example 1

• One of the traces involved in Bhabha scattering is

\[ T = \text{Tr} [\gamma^\mu (\not\! p_1 + m) \gamma^\nu (\not\! p_3 + m)] \]

We can expand this out to create 4 terms, but 2 of these terms (the ones linear in \( m \)) will involve 3 \( \gamma \)-matrices, and are therefore zero. Thus,

\[
T = \text{Tr}(\gamma^\mu \not\! p_1 \gamma^\nu \not\! p_3) + m^2 \text{Tr}(\gamma^\mu \gamma^\nu) \\
= 4 (p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - (p_1 \cdot p_3) g^{\mu\nu}) + 4m^2 g^{\mu\nu}
\]

This result will be contracted with another trace that is covariant (i.e., \( _{\mu\nu} \) as opposed to contravariant \( ^{\mu\nu} \)) in \( \mu \) and \( \nu \).
Example 2

- It isn’t always a joyous task to contract 2 traces together.
- Consider
  \[ A = \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) \text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) \]

Evaluating the traces,

\[
A = 4 \left[ p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1 \cdot p_2) g^{\mu \nu} \right] \\
\times 4 \left[ p_{1 \mu} p_{2 \nu} + p_{1 \nu} p_{2 \mu} - (p_1 \cdot p_2) g_{\mu \nu} \right] \\
= 16 \left[ 2p_1^2 p_2^2 + 2(p_1 \cdot p_2)^2 + 4(p_1 \cdot p_2)^2 - 4(p_1 \cdot p_2)^2 \right] \\
= 32 \left[ m_1^2 m_2^2 + (p_1 \cdot p_2)^2 \right]
\]
Summary

- The Feynman rules for QED provide the recipe for translating Feynman diagrams into mathematical expressions for the amplitude.

- If we are interested in the spin-averaged amplitude $\langle |M|^2 \rangle$ then we need not ever use explicit fermion spinors and photon polarization vectors.

- Instead, Casimir’s Trick allows us to calculated spin-averaged amplitudes in terms of traces of $\gamma$-matrices.

- With practice, $\gamma$-matrix traces can be taken quite quickly.