Feynman rules for QED

- The Feynman Rules for QED
- Setting up Amplitudes
- Casimir's Trick
- Trace Theorems

Slides from Sobie and Blokland

Electrons and positrons

- spinors
  $u^{(s)}$ and $v^{(s)}$ (s = spin) satisfy the Dirac equations $(\gamma^\mu p_\mu - m)u = 0$ and $(\gamma^\mu p_\mu + m)v = 0$
- adjoints
  $\overline{u} = u^\dagger \gamma^0$ and $\overline{v} = v^\dagger \gamma^0$ satisfy $\overline{u}(\gamma^\mu p_\mu - m) = 0$ and $\overline{v}(\gamma^\mu p_\mu + m) = 0$
- orthogonality
  $\overline{u}^{(1)} u^{(2)} = 0$ and $\overline{v}^{(1)} v^{(2)} = 0$
- normalization
  $\overline{u}u = 2m$ and $\overline{v}v = -2m$
- completeness
  $\sum_s u^{(s)}\overline{u}^{(s)} = \gamma^\mu p_\mu + m$ and $\sum_s v^{(s)}\overline{v}^{(s)} = \gamma^\mu p_\mu - m$

Photons

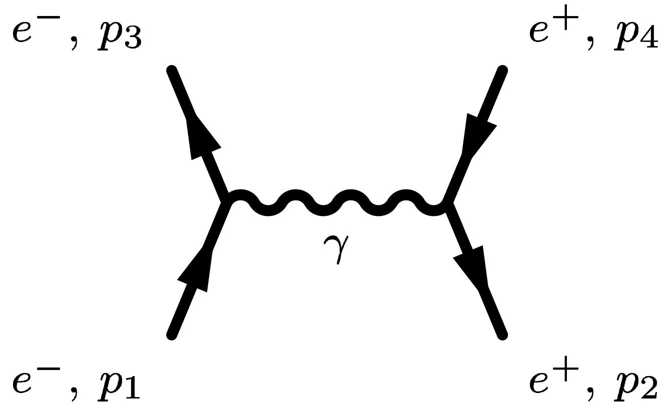
$$A^\mu(x) = a e^{-ip\cdot x} \epsilon^\mu(p)$$

- Lorentz condition
  $\epsilon^\mu p_\mu = 0$
- orthogonality
  $\epsilon^{\mu *}_{(1)} \epsilon_{\mu(2)} = 0$
- normalization
  $\epsilon^{\mu *} \epsilon_\mu = 1$
- Coulomb gauge
  $\epsilon^0 = 0$ and $\epsilon \cdot \mathbf{p} = 0$
- Completeness
  $\sum_s (\epsilon_{(s)})_i (\epsilon^*_{(s)})_j = \delta_{ij} - (p_i p_j)/p^2$

The Feynman Rules for QED

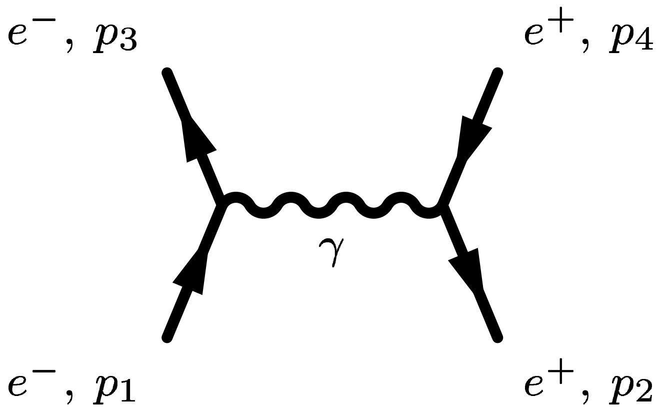
The Feynman rules provide the recipe for constructing an amplitude $\mathcal{M}$ from a Feynman diagram.

- **Step 1:** For a particular process of interest, draw a Feynman diagram with the minimum number of vertices. There may be more than one.



The Feynman Rules for QED

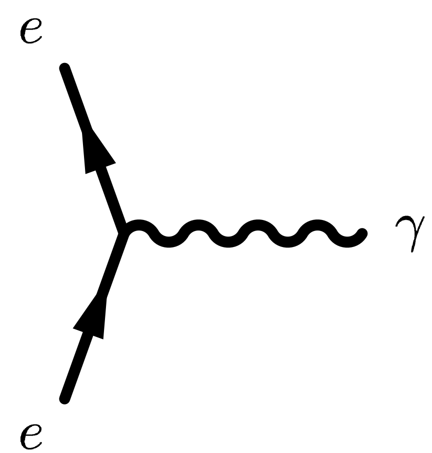
- **Step 2:** For each Feynman diagram, label the four-momentum of each line, enforcing four-momentum conservation at every vertex. Note that arrows are only present on fermion lines and they represent particle flow, not momentum.



- **Step 3:** The amplitude depends on
  1. Vertex factors
  2. Propagators for internal lines
  3. Wavefunctions for external lines

Vertex Factors

- Every QED vertex,



contributes a factor of $ig_e\gamma^\mu$.

- g_e is a dimensionless coupling constant and is related to the fine-structure constant by

$$\alpha = \frac{g_e^2}{4\pi}$$

Propagators

- Each internal photon connects two vertices of the form $ig_e\gamma^\mu$ and $ig_e\gamma^\nu$, so we should expect the photon propagator to contract the indices μ and ν.

$$\text{Photon propagator:} \quad \frac{-ig_{\mu\nu}}{q^2}$$

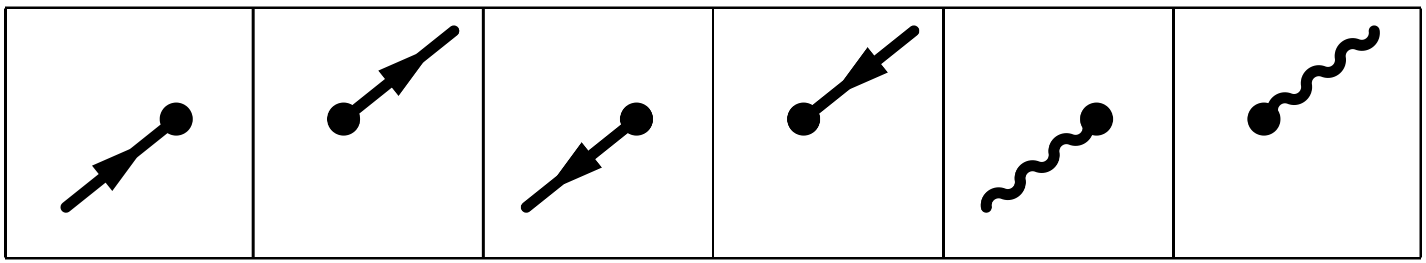
- Internal fermions have a more complicated propagator,

$$\text{Fermion propagator:} \quad \frac{i(\not{q} + m)}{q^2 - m^2}$$

The sign of q matters here — we take it to be in the same direction as the fermion arrow.

External Lines

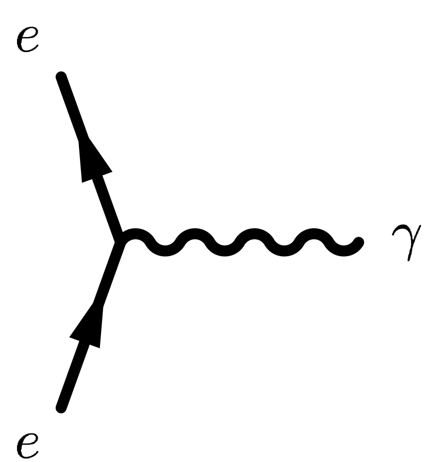
- Since both the vertex factor and the fermion propagators involve 4×4 matrices, but the amplitude must be a scalar, the external line factors must sit on the outside.
- Work *backwards* along every fermion line using:



e^- in	e^- out	e^+ in	e^+ out	γ in	γ out
u	$\bar{u}$	$\bar{v}$	v	ϵ_μ	ϵ^*_μ

Matrix elements I

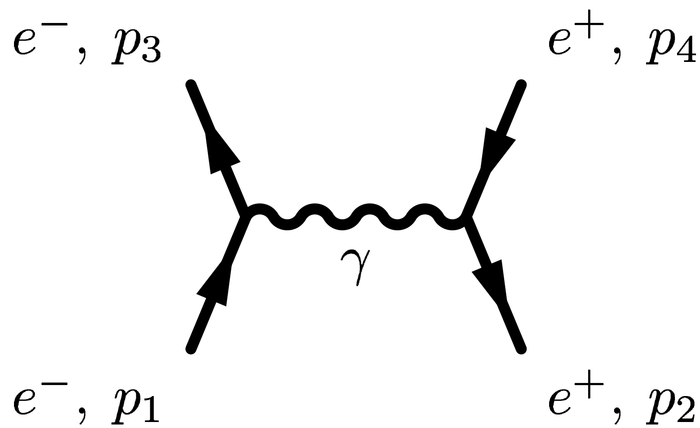
follow fermion lines backward to give $\overline{u}(2) i g \gamma^{\mu} u(1)$



Matrix elements II

The matrix element is proportional to the two currents in the diagram below.

$$[\bar{u}_3(ig_e\gamma^\mu)u_1]\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)[\bar{v}_2(ig_e\gamma^\nu)v_4]$$



And Finally...

- **Step 4:** The overall amplitude is the coherent sum of the individual amplitudes for each diagram:

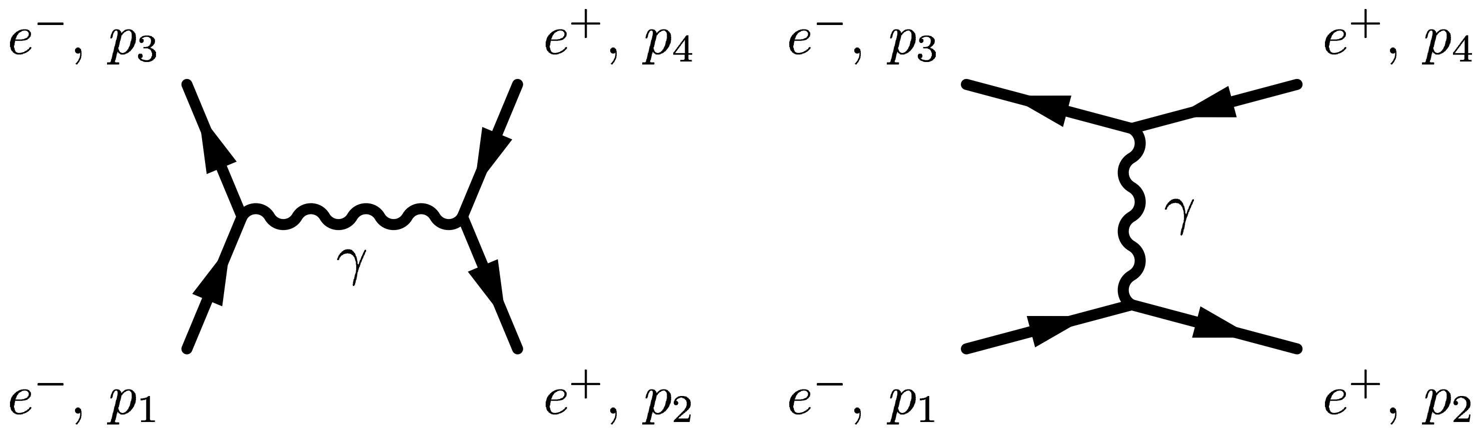
$$
\begin{aligned}
\mathcal{M} &= \mathcal{M}_1 + \mathcal{M}_2 + \ldots \\
\Rightarrow \ |\mathcal{M}^2| &= |\mathcal{M}_1 + \mathcal{M}_2 + \ldots|^2
\end{aligned}
$$

- **Step 4a:** *Antisymmetrization*. Include a minus sign between diagrams that differ only in the exchange of two identical fermions.

Examples

- There are only a handful of ways to make tree-level diagrams in QED.
- Today, we will construct amplitudes for Bhabha scattering $(e^+ \, e^- \to e^+ \, e^-)$ and Compton scattering $(e \, \gamma \to e \, \gamma)$.
- Next week, we will undertake thorough calculations for Mott scattering $(e \, \ell \to e \, \ell)$, pair annihilation $(e^+ \, e^- \to \gamma \, \gamma)$. You will examine fermion pair-production via $(e^+ \, e^- \to f \, \bar{f})$ for your assignment.

Example: Bhabha Scattering

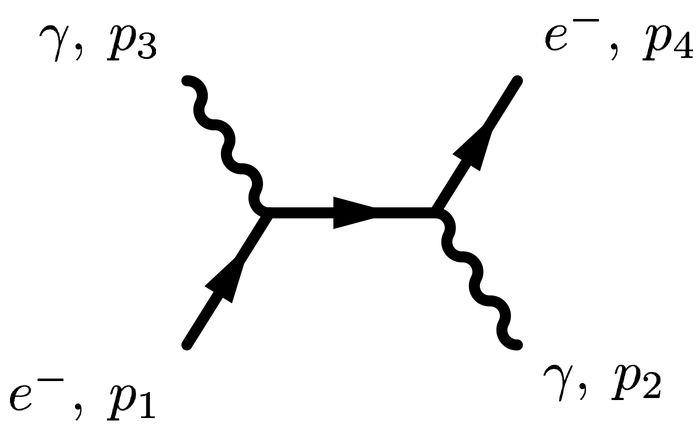


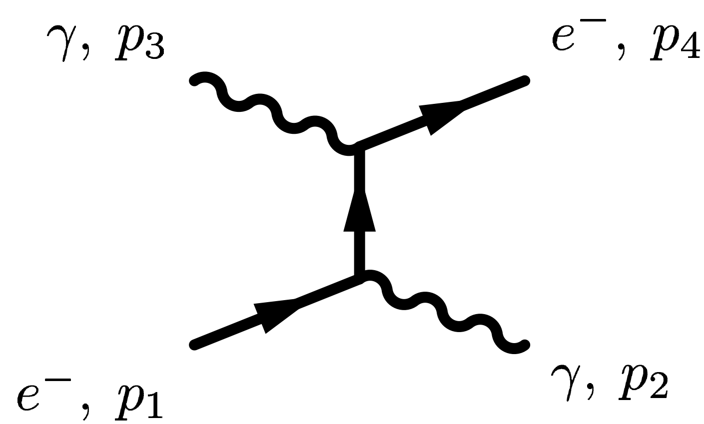
- Antisymmetrization $\Rightarrow\ \mathcal{M} = \mathcal{M}_t - \mathcal{M}_s$

$$\mathcal{M}_t = i\left[\bar{u}_3(ig_e\gamma^\mu)u_1\right]\left(\frac{-ig_{\mu\nu}}{(p_1-p_3)^2}\right)\left[\bar{v}_2(ig_e\gamma^\nu)v_4\right]$$

$$\mathcal{M}_s = i\left[\bar{u}_3(ig_e\gamma^\mu)v_4\right]\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}\right)\left[\bar{v}_2(ig_e\gamma^\nu)u_1\right]$$

Example: Compton Scattering





- No antisymmetrization $\Rightarrow\ \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$

$$\mathcal{M}_1 = i\left[\bar{u}_4(ig_e\gamma^\mu)\left(\frac{i(\not{p}_1 - \not{p}_3 + m)}{(p_1 - p_3)^2 - m^2}\right)(ig_e\gamma^\nu)u_1\right]\epsilon^*_{3\nu}\epsilon_{2\mu}$$

$$\mathcal{M}_2 = i\left[\bar{u}_4(ig_e\gamma^\mu)\left(\frac{i(\not{p}_1 + \not{p}_2 + m)}{(p_1 + p_2)^2 - m^2}\right)(ig_e\gamma^\nu)u_1\right]\epsilon^*_{3\mu}\epsilon_{2\nu}$$

Polarized Particles

- A typical QED amplitude might look something like

$$\mathcal{M} \sim [\bar{u}_1 \Gamma^\mu v_2]\, \epsilon_{3\mu}$$

  The Feynman rules won't take us any further, but to get a number for $\mathcal{M}$ we will need to substitute explicit forms for the wavefunctions of the external particles: $\bar{u}_1$, v_2, and $\epsilon_{3\mu}$.

- If all external particles have a known polarization, this might be a reasonable way to calculate things. More often, though, we are interested in unpolarized particles.

Spin-Averaged Amplitudes

- If we do not care about the polarizations of the particles then we need to

  1. *Average* over the polarizations of the initial-state particles
  2. *Sum* over the polarizations of the final-state particles

  in the squared amplitude $|\mathcal{M}|^2$.

- We call this the spin-averaged amplitude and we denote it by

$$\left\langle |\mathcal{M}|^2 \right\rangle$$

- Note that the averaging over initial state polarizations involves summing over all polarizations and then dividing by the number of independent polarizations, so $\left\langle |\mathcal{M}|^2 \right\rangle$ involves a sum over the polarizations of *all* external particles.

Spin Sums

- Let's simplify things even further and suppose that we have

$$\mathcal{M} \sim [\bar{u}_1 \Gamma u_2]$$

Then

$$\begin{aligned}
|\mathcal{M}|^2 &\sim [\bar{u}_1 \Gamma u_2] [\bar{u}_1 \Gamma u_2]^* \\
&\sim [\bar{u}_1 \Gamma u_2] \left[u_1^\dagger \gamma^0 \Gamma u_2 \right]^\dagger \\
&\sim [\bar{u}_1 \Gamma u_2] \left[u_2^\dagger \Gamma^\dagger \gamma^{0\dagger} u_1 \right] \\
&\sim [\bar{u}_1 \Gamma u_2] \left[u_2^\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 u_1 \right] \\
&\sim [\bar{u}_1 \Gamma u_2] \left[\bar{u}_2 \bar{\Gamma} u_1 \right]
\end{aligned}$$

$$|\mathcal{M}|^2 \quad \sim \quad [\bar{u}_1 \Gamma u_2]\,[\bar{u}_2 \bar{\Gamma} u_1]$$

- Applying the completeness relation

$$\sum_{s_i=1,2} u_i^{s_i} \bar{u}_i^{s_i} = (\not{p}_i + m_i)$$

to $u_2 \bar{u}_2$ in the squared-amplitude above (summing over the spins of paticle 2),

$$\begin{aligned} \sum_{s_2} |\mathcal{M}|^2 \quad &\sim \quad [\bar{u}_1 \Gamma (\not{p}_2 + m_2) \bar{\Gamma} u_1] \\ &\sim \quad [\bar{u}_1 Q u_1] \end{aligned}$$

- The right-hand side is just a number, but if we represent the matrix multiplication with summations over indices, we can rewrite it as

$$
\begin{aligned}
[\bar{u}_1 Q u_1] &= (\bar{u}_1)_i \, Q_{ij} \, (u_1)_j \\
&= Q_{ij} \, (u_1 \bar{u}_1)_{ji} \\
&= [Q \, (u_1 \bar{u}_1)]_{ii} \\
&= \mathrm{Tr} \, [Q(u_1 \bar{u}_1)]
\end{aligned}
$$

- Finally, we apply the completeness relation once again, so that we get

$$\sum_{s_1} |\mathcal{M}|^2 \sim \mathrm{Tr} \, [Q(\not{p}_1 + m_1)]$$

- In total, we have

$$\begin{aligned} \mathcal{M} &\sim [\bar{u}_1 \Gamma u_2] \\ \Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle &\sim \frac{1}{2} \operatorname{Tr} \left[\Gamma(\not{p}_2 + m_2) \bar{\Gamma}(\not{p}_1 + m_1) \right] \end{aligned}$$

The factor of $\frac{1}{2}$ is from the averaging over initial spins, assuming exactly one of u_1 and u_2 corresponds to an initial-state particle. If they are both in the initial state (e.g., pair annihilation), the factor is $\frac{1}{4}$. If neither is in the initial state (e.g., pair production), the factor is 1.

Casimir's Trick

- This procedure of calculating spin-averaged amplitudes in terms of traces is known as Casimir's Trick.

$$\sum_{\text{all spins}} \left[\bar{u}_a \Gamma_1 u_b\right] \left[\bar{u}_a \Gamma_2 u_b\right]^* = \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 (\not{p}_a + m_a)\right]$$

- If antiparticle spinors (v) are present in the spin sum, we use the corresponding completeness relation

$$\sum_{s_i=1,2} v_i^{s_i} \bar{v}_i^{s_i} = (\not{p}_i - m_i)$$

Traces

- Because of Casimir's Trick, we're going to find ourselves calculating a lot of traces involving γ-matrices.
- General identities about traces:

$$\begin{aligned} \mathrm{Tr}(A+B) &= \mathrm{Tr}(A) + \mathrm{Tr}(B) \\ \mathrm{Tr}(\alpha A) &= \alpha \mathrm{Tr}(A) \\ \mathrm{Tr}(AB) &= \mathrm{Tr}(BA) \\ \mathrm{Tr}(ABC) &= \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA) \end{aligned}$$

Building Blocks

- The two major identities that we will need in order to build more complicated trace identities are

$$
\begin{aligned}
g_{\mu\nu} g^{\mu\nu} &= 4 \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}(\times \mathbf{I})
\end{aligned}
$$

- You can show that $\gamma_\mu \gamma^\mu = 4$ and $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$. In a similar fashion, we find that

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\mu &= \gamma_\mu \left(2g^{\mu\nu} - \gamma^\mu \gamma^\nu\right) \\
&= 2\gamma^\nu - \gamma_\mu \gamma^\mu \gamma^\nu \\
&= 2\gamma^\nu - 4\gamma^\nu \\
&= -2\gamma^\nu
\end{aligned}
$$

Simple Trace Identities

- The simplest trace identity is: $\mathrm{Tr}(1) = 4$
- The trace of a single γ matrix is zero, as is the trace of *any* odd number of γ-matrices.
- For 2 γ-matrices,

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= \mathrm{Tr}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)/2 \\ &= \mathrm{Tr}(2g^{\mu\nu})/2 \\ &= g^{\mu\nu}\mathrm{Tr}(1) \\ &= 4g^{\mu\nu}\end{aligned}$$

-

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right)$$

Traces With γ^5

- The vertex factor for weak interactions involves γ^5.
- By inspection, $\mathrm{Tr}(\gamma^5) = 0$.
- Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ (an even number of γ-matrices),

$$\begin{aligned} \mathrm{Tr}(\gamma^5\gamma^\mu) &= 0 \\ \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) &= 0 \end{aligned}$$

- Also,

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

The Non-Trivial γ^5 Trace

- Only with 4 (or more) other γ-matrices can we obtain a nonzero trace involving γ^5:

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$$

where the totally antisymmetric tensor is defined as

$$\epsilon^{\mu\nu\lambda\sigma} \equiv \begin{cases} -1 & \text{for } \textit{even} \text{ permutations of } 0123 \\ +1 & \text{for } \textit{odd} \text{ permutations of } 0123 \\ 0 & \text{if any 2 indices are the same} \end{cases}$$

Contractions of the ϵ Tensor

- Since $\epsilon^{\mu\nu\lambda\sigma}$ is completely antisymmetric, we will get zero when we contract this with any tensor that is symmetric in 2 indices, such as $g^{\mu\nu}$ or $(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)$.
- Only contractions with another antisymmetric tensor survive:

$$
\begin{aligned}
\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\sigma} &= -24 \\
\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\lambda\tau} &= -6\,\delta^\sigma_\tau \\
\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\theta\tau} &= -2\left(\delta^\lambda_\theta\delta^\sigma_\tau - \delta^\lambda_\tau\delta^\sigma_\theta\right) \\
&\vdots
\end{aligned}
$$

Example 1

- One of the traces involved in Bhabha scattering is

$$T = \mathrm{Tr}\left[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_3 + m)\right]$$

We can expand this out to create 4 terms, but 2 of these terms (the ones linear in m) will involve 3 γ-matrices, and are therefore zero. Thus,

$$\begin{aligned} T &= \mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) + m^2 \mathrm{Tr}(\gamma^\mu \gamma^\nu) \\ &= 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu - (p_1 \cdot p_3) g^{\mu\nu}\right) + 4m^2 g^{\mu\nu} \end{aligned}$$

This result will be contracted with another trace that is covariant (i.e., $_{\mu\nu}$ as opposed to contravariant $^{\mu\nu}$) in μ and ν.

Example 2

- It isn't always a joyous task to contract 2 traces together.
- Consider $$\mathcal{A} = \mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2)\, \mathrm{Tr}(\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2)$$

  Evaluating the traces,

$$\begin{aligned}
\mathcal{A} &= 4\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1 \cdot p_2) g^{\mu\nu}\right] \\
&\quad \times 4\left[p_{1\mu} p_{2\nu} + p_{1\nu} p_{2\mu} - (p_1 \cdot p_2) g_{\mu\nu}\right] \\
&= 16\left[2p_1^2 p_2^2 + 2(p_1 \cdot p_2)^2 + 4(p_1 \cdot p_2)^2 - 4(p_1 \cdot p_2)^2\right] \\
&= 32\left[m_1^2 m_2^2 + (p_1 \cdot p_2)^2\right]
\end{aligned}$$

Summary

- The Feynman rules for QED provide the recipe for translating Feynman diagrams into mathematical expressions for the amplitude.
- If we are interested in the spin-averaged amplitude $\left\langle |\mathcal{M}|^2 \right\rangle$ then we need not ever use explicit fermion spinors and photon polarization vectors.
- Instead, Casimir's Trick allows us to calculated spin-averaged amplitudes in terms of traces of γ-matrices.
- With practice, γ-matrix traces can be taken quite quickly.