Today: Electrodynamics of Quarks and Hadrons

- Elastic $e\,p$ scattering
- Inelastic $e\,p$ scattering
  Slides from Sobie and Blokland
Elastic Electron-Proton Scattering

- This is our best probe of the internal structure of the proton.

- If the proton were structureless, we could simply recycle our result for electron-muon scattering:

\[
\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{4(p_1 - p_3)^4} \left[ 4 \left( p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2 - p_1 \cdot p_3)g^{\mu\nu} \right) \right] \\
\times \left[ 4 \left( p_2^\mu p_4^\nu + p_4^\mu p_2^\nu + (M^2 - p_2 \cdot p_4)g_{\mu\nu} \right) \right] \\
= \frac{g_e^4}{q^4} L_{\mu\nu}^{\text{electron}} L_{\mu\nu}^{\text{muon}}
\]

with \( q = p_1 - p_3 \) and

\[
L_{\mu\nu}^{\text{electron}} = 2 \left( p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + (m^2 - p_1 \cdot p_3)g^{\mu\nu} \right)
\]
But the Proton Isn’t Structureless...

- Instead of just replacing $L_{\mu\nu}^{\text{muon}}$ with $L_{\mu\nu}^{\text{proton}}$, which assumes that the proton is a true point particle, we can generically account for proton structure via

$$\langle |\mathcal{M}|^2 \rangle = \frac{g_e^4}{q^4} L_{\text{electron}}^\mu L_{\text{proton}}^\nu K_{\mu\nu}$$

- Notice that the implied proton structure does not affect the electron-photon coupling or the photon propagator. All of the complications are neatly stashed within $K_{\mu\nu}$. Pictorially,

\[ e^-, p_3 \quad \gamma \quad p, p_4 \]

\[ e^-, p_1 \quad \gamma \quad p, p_2 \]
So How Do We Calculate $K^\mu_\text{proton}_\nu$?

- Even without assuming anything about the substructure of a proton, we know that $K^\mu_\text{proton}_\nu$ is a second-rank tensor.

- In addition to $g^\mu_\nu$, we can construct tensors from the four-vectors $p_2, p_4, \text{and } q$. Since $q = p_4 - p_2$, only 2 of these four-vectors are independent, from which we choose $q$ and $p_2 = p$. Thus, our choices are

$$g^\mu_\nu, \quad p^\mu p^\nu, \quad q^\mu q^\nu, \quad (p^\mu q^\nu + p^\nu q^\mu), \quad (p^\mu q^\nu - p^\nu q^\mu)$$

For electromagnetic interactions, $L^\mu_\nu_\text{electron}$ is symmetric in $\mu$ and $\nu$, therefore we need not include $(p^\mu q^\nu - p^\nu q^\mu)$. This term would be required for weak interactions (e.g., elastic neutrino-proton scattering).
Form Factors

- Using the four symmetric second-rank tensors available to us, we write

\[ K_{\text{proton}}^{\mu\nu} = -K_1 g^{\mu\nu} + \frac{K_2}{M^2} p^\mu p^\nu + \frac{K_4}{M^2} q^\mu q^\nu + \frac{K_5}{M^2} (p^\mu q^\nu + p^\nu q^\mu) \]

where \( K_1, K_2, K_4, \) and \( K_5 \) are unknown functions which we refer to as form factors.

- The form factors can depend on \( q^2 \), the only scalar variable available to us, since \( p^2 = M^2 \) and \( p \cdot q = -q^2 / 2 \). (This last identity follows from squaring \( p_4 = p_2 + q \) and recognizing that \( p_4^2 = M^2 \) for elastic scattering.)
Simplifying Things Further

- Using the Ward identity

\[ q_\mu K_{\text{proton}}^{\mu \nu} = 0 \]

we find that there are only 2 independent form factors:

\[ K_{\text{proton}}^{\mu \nu} = K_1 \left(- g^{\mu \nu} + \frac{q^\mu q^\nu}{q^2} \right) + \frac{K_2}{M^2} \left(p^\mu + \frac{q^\mu}{2} \right) \left(p^\nu + \frac{q^\nu}{2} \right) \]

- The goal is then to measure these form factors experimentally and to try to calculate them theoretically.
Skipping a Bunch of Nasty Algebra...

- Working in the lab frame with the target proton at rest, we assume that the energy of the incident electron, $E$, is sufficiently large that $m$ can be ignored. Both $q^2$ and the energy of the scattered electron, $E'$, is fixed by the scattering angle:

$$E' = \frac{E}{1 + (2E/M) \sin^2(\theta/2)}$$

- We can obtain the differential cross section in terms of the form factors $K_1$ and $K_2$:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4ME \sin^2(\theta/2)}\right)^2 \frac{E'}{E} \left[2K_1 \sin^2(\theta/2) + K_2 \cos^2(\theta/2)\right]$$

This is the Rosenbluth formula.
Inelastic Electron-Proton Scattering

- If the incident electron is sufficiently energetic, it is quite unlikely that the proton will stay intact. Instead, we should be considering the more general inelastic process \( e + p \rightarrow e + X \).
Masking Our Ignorance

- As with the elastic case, we introduce a second-rank tensor $W_{\mu\nu}$ to describe the unknown details about the subprocess $\gamma + p \rightarrow X$. The electron vertex and the photon propagator are known, therefore
  \[
  \left\langle |\mathcal{M}|^2 \right\rangle = \frac{g_e^4}{q^4} L^\mu\nu_{\text{electron}} W_{\mu\nu}(X)
  \]

- We can insert this spin-averaged amplitude into the Golden Rule for scattering in order to compute a differential cross section:
  \[
  d\sigma = \frac{S \left\langle |\mathcal{M}|^2 \right\rangle}{4 \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \times \left( \frac{d^3 p_3}{(2\pi)^3 2E_3} \right) \left( \frac{d^3 p_4}{(2\pi)^3 2E_4} \right) \cdots \left( \frac{d^3 p_n}{(2\pi)^3 2E_n} \right) \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots - p_n)
  \]
Inclusive Cross Sections

- It is not feasible to measure every single piece of hadronic shrapnel and to compute a cross section for each possible set of final state particles. Instead, we typically measure only the scattered electron.

- By integrating over all accessible final states $X$ with all possible momenta, we obtain the inclusive cross section:

$$d\sigma = \frac{g_e^4 L^{\mu\nu}}{4q^4 \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \left( \frac{d^3 p_3}{(2\pi)^3 2E_3} \right) 4\pi M W_{\mu\nu}$$

$$W_{\mu\nu} \equiv \frac{1}{4\pi M} \sum_X \int W_{\mu\nu}(X) \left( \frac{d^3 p_4}{(2\pi)^3 2E_4} \right) \cdots \left( \frac{d^3 p_n}{(2\pi)^3 2E_n} \right) \times (2\pi)^4 \delta^4(q + p - p_4 - \ldots - p_n)$$
\[ d\sigma = \frac{g_e^4 L^{\mu\nu}}{4q^4 \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \left( \frac{d^3 p_3}{(2\pi)^3 2E_3} \right) 4\pi M W_{\mu\nu} \]

- With an initial electron energy of \( E \), whose mass we will neglect,
  \[ \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = ME \]

- The outgoing electron has energy \( E' \) and
  \[ \frac{d^3 p_3}{E_3} = \frac{|p_3|^2 d|p_3| d\Omega}{E'} = E' dE' d\Omega \]

- Substituting these results, the differential cross section simplifies to
  \[ \frac{d\sigma}{dE' d\Omega} = \frac{\alpha^2}{q^4} \frac{E'}{E} L^{\mu\nu} W_{\mu\nu} \]
\[ \frac{d\sigma}{dE'\ d\Omega} = \frac{\alpha^2}{q^4} \frac{E'}{E} L^\mu\nu W_{\mu\nu} \]

- Note that, unlike elastic scattering, \( E' \) is not kinematically fixed by \( E \) and \( \theta \) because the outgoing hadrons can have a range of masses. Equivalently, the total momentum,

\[ p_{\text{total}} = p_4 + p_5 + \cdots + p_n \]

is not constrained by the condition \( p_{\text{total}}^2 = M^2 \)

- This leaves us with 2 independent variables, for a given incident energy:

  Experimentalist: \( E' , \ \theta \)

  Theorist: \( q^2 , \ x \quad (x = -\frac{q^2}{2q \cdot p}) \)

Many other choices exist (\( \nu , y , Q^2 \), etc.).
Structure Functions

- From here, we proceed as with the elastic scattering case and write the most general tensor $W_{\mu\nu}$ that depends on $q$, $p$, and satisfies the Ward identity $q_\mu W^{\mu\nu}$.

- This leads to an expression for the differential cross section in terms of the two structure functions $W_1(q^2, x)$ and $W_2(q^2, x)$:

$$\frac{d\sigma}{dE' \ d\Omega} = \left( \frac{\alpha}{2E \sin^2(\theta/2)} \right)^2 \left[ 2W_1 \sin^2(\theta/2) + W_2 \cos^2(\theta/2) \right]$$

The structure functions are the inelastic generalization of the elastic form factors $K_1(q^2)$ and $K_2(q^2)$. 
Summary

- Electron-proton scattering experiments provide us with a great deal of information about the structure of the proton.

- For elastic electron-proton scattering, we can write the cross section (the Rosenbluth formula) in terms of 2 form factors: $K_1(q^2)$ and $K_2(q^2)$.

- For inelastic electron-proton scattering, we write an inclusive cross section in terms of 2 structure functions: $W_1(q^2, x)$ and $W_2(q^2, x)$. 