Astronomical Magnitudes, in the AB System, of Blackbody Sources and Their Reflections

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Table 1: A summary of the important *formulae* that will be ***used*** in this document.

Definition of AB magnitude[1]:

$$m_{\mathrm{AB}} \equiv -2.5\log_{10} f_\nu^{\mathrm{CGS}} - 48.60\,, \qquad (1)$$

where f_ν^{CGS} is the spectral flux density from the light source, at the location of the observer, in the CGS units of erg s^{-1} cm^{-2} Hz^{-1}.

Planck's Law of blackbody spectral radiance $B_\nu(\nu, T)$:

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT}-1}\,, \qquad (2)$$

where h (= 6.626×10^{-34} m^2 kg s^{-1}) is Planck's constant; ν is the frequency, in Hz, of the output light that is under consideration; c (= 2.998×10^{8} m s^{-1}) is the speed of light; k_B (= 1.381×10^{-23} m^2 kg s^{-2}K^{-1}) is Boltzmann's constant; T is the temperature in Kelvin of the blackbody light source; and $B_\nu(\nu, T)$ is the spectral radiance from the blackbody light source (in units of W m^{-2} sr^{-1} Hz^{-1}).

Conversion formula from f_ν^{CGS} to f_ν^{SI} :

$$f_\nu^{\mathrm{CGS}} = 1000 f_\nu^{\mathrm{SI}}\,, \qquad (3)$$

where both f_ν^{CGS} and f_ν^{SI} are the spectral flux density from the light source at the location of the observer. f_ν^{CGS} is in CGS units of erg s^{-1} cm^{-2} Hz^{-1}, and f_ν^{SI} is in the standard SI units of W m^{-2} Hz^{-1}.

Conversion formula from the spectral radiance B_ν of a spherical light source of radius R, to the spectral flux density f_ν^{SI} at a distance d from the center of the light source (where $d > R$) :

$$f_\nu^{\mathrm{SI}} = \frac{\pi R^2}{d^2} B_\nu\,, \qquad (4)$$

where both R and d are in meters; and, as previously, B_ν is the spectral radiance from the light source in units of W m^{-2} sr^{-1} Hz^{-1}, and f_ν^{SI} is the spectral flux density from the light source in units of W m^{-2} Hz^{-1}.

[1] The AB magnitude system was first defined in J. B. Oke and J. E. Gunn, *Astrophys. J.* **266**, 713 (1986), and is the only absolute standards-based (*i.e.*, invariantly convertible, per its definition, to SI-defined units) magnitude system presently in use for optical astronomy. (The Jansky, another absolute unit, is presently used in radio and microwave astronomy.)

Table 2: A summary of the important *constants* that will be ***used*** in this document.

Definition	**Symbol**	**Value and units**	
Average radius of the solar photosphere	$R_\odot$	6.963×10^{8}	m
Average temperature of the solar photosphere	$T_\odot$	5778.0	K
Average Earth-Sun distance (= 1 AU)	$d_\oplus$	1.496×10^{11}	m
Planck's constant	h	6.626×10^{-34}	$m^2\,kg\,s^{-1}$
Speed of light in vacuum	c	2.998×10^{8}	$m\,s^{-1}$
Boltzmann's constant	k_B	1.381×10^{-23}	$m^2\,kg\,s^{-2}\,K^{-1}$

Table 3: A summary of the primary *formulae* and *results* that will be ***derived*** in this document.

a) Formulae and results for ***blackbody sources viewed directly in the line-of-sight***:

The AB magnitude, as a function of wavelength $\boldsymbol{\lambda}$ (in **nm**), of a spherical blackbody source of radius $\boldsymbol{R}$ and surface (or photosphere) temperature $\boldsymbol{T}$, that is in the *direct line-of-sight* of an observer located at a distance $\boldsymbol{d}$ from the center of the blackbody source (where, of course, $d > R$) :

$$m_{\mathrm{AB}}(\lambda) = -2.5\log_{10}\left[\frac{(2\pi \times 10^{27})hcR^2}{d^2\lambda^3\left(e^{\left[(10^9 hc)/(k_B T\lambda)\right]} - 1\right)}\right] - 56.10 \qquad (5)$$

$$= -2.5\log_{10}\left[\frac{R^2}{d^2\lambda^3\left(e^{\left[(1.4384\times 10^7)/(T\lambda)\right]} - 1\right)}\right] - 63.8407\,, \qquad (6)$$

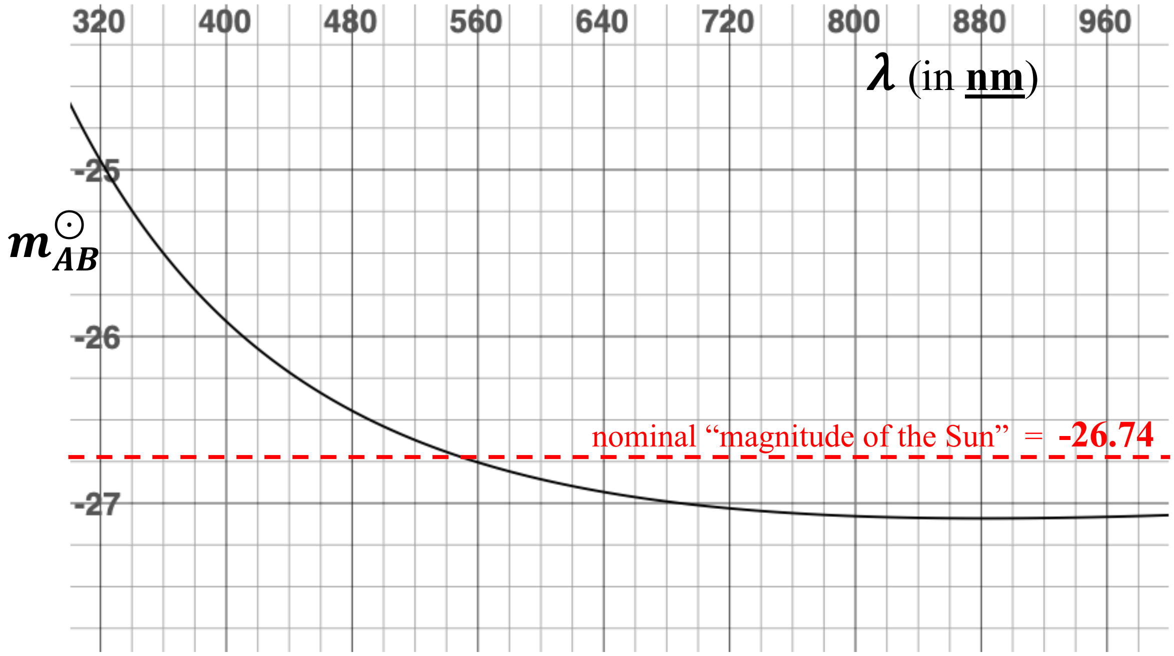
with the latter formula obtained when numerically filling in for the constants h, c, and k_B.

i) Formulae relevant for blackbody sources at the temperature $\boldsymbol{T_\odot}$ of the Sun

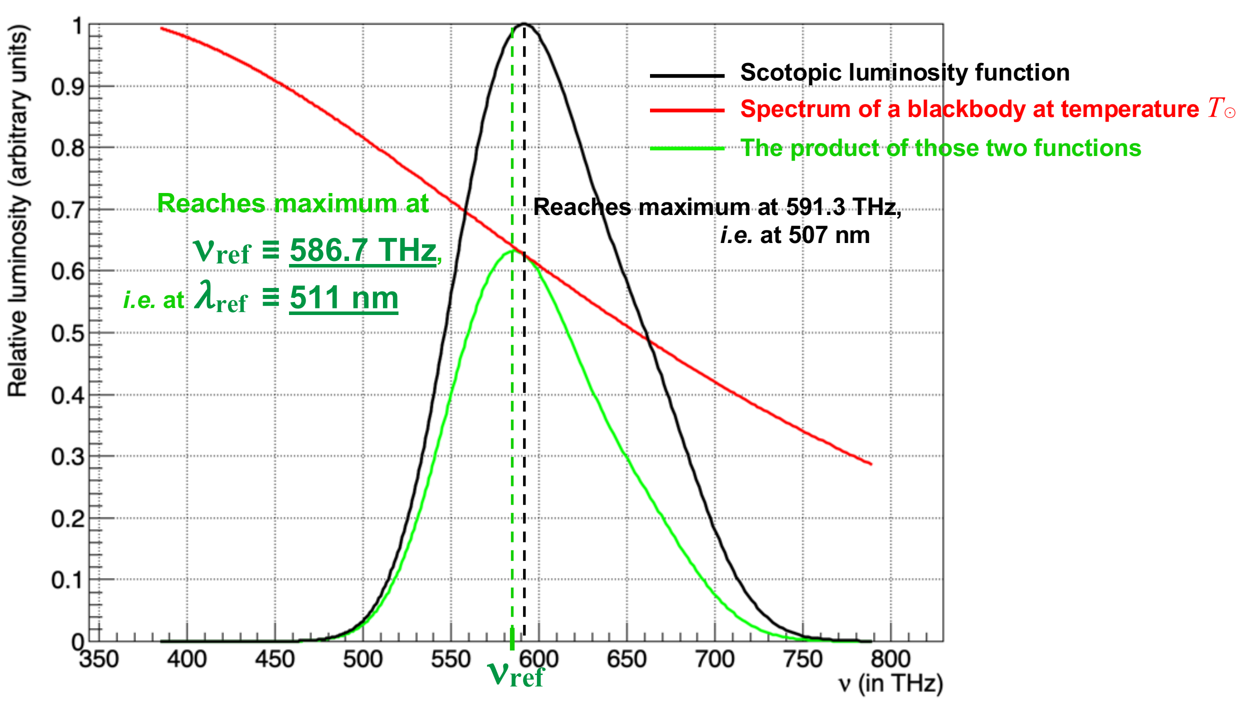
The resulting AB "magnitude of the Sun", when viewed from orbit near Earth (*i.e.*, for $\boldsymbol{R} = \boldsymbol{R_\odot}$, $\boldsymbol{T} = \boldsymbol{T_\odot}$, and $\boldsymbol{d} = \boldsymbol{d_\oplus}$), as a function of wavelength $\boldsymbol{\lambda}$ (in **nm**), is :

$$m_{AB}^{\odot}(\lambda) = -2.5\log_{10}\left[\frac{1}{\lambda^3\left(e^{\left[2489.5/\lambda\right]} - 1\right)}\right] - 52.1800\,. \qquad (7)$$

A plot of Eq. (7) above, within the optical region of the spectrum:



A plot of the **spectrum of a blackbody at the temperature $T_{\odot}$ of the Sun**, as a function of frequency ν; together with a plot of the **scotopic luminosity function** (*i.e.*, the relative sensitivity of the average human eye, when viewing at night), also when a function of frequency ν; and the **product of those two functions** (in order to obtain the frequency ν_{ref}, and thus the wavelength λ_{ref}, at which the object appears brightest to the average human eye at night):



The resulting AB "magnitude of the Sun" at $\boldsymbol{\lambda} = \lambda_{\text{ref}}$: $\boldsymbol{m_{AB}^{\odot}(\lambda = \lambda_{\text{ref}}) \; = \; -26.59}$.

ii) Formulae relevant for blackbody sources at incandescent lightbulb temperatures

The AB magnitude, as a function of wavelength $\boldsymbol{\lambda}$ (in nm), of an incandescent lamp with filament at temperature $\boldsymbol{T}$ (in Kelvin) that is radiatively dissipating electrical power $\boldsymbol{P}$ (in watts), and that is in the *direct line-of-sight* of an observer located at a distance $\boldsymbol{d}$ from the lamp filament (under the approximations that the lamp filament is a perfect isotropic blackbody, and that there is no light absorption in the glass or quartz envelope of the lamp, nor anywhere else between the filament and the observer):

$$\boldsymbol{m_{\text{AB}}(\lambda) = -2.5\log_{10}\left[\frac{(3.75\times10^{27})h^4c^3P}{\pi^5 k_B^4 T^4 d^2 \lambda^3\left(e^{\left[(10^9 hc)/(k_B T\lambda)\right]} - 1\right)}\right] - 56.10} \qquad (8)$$

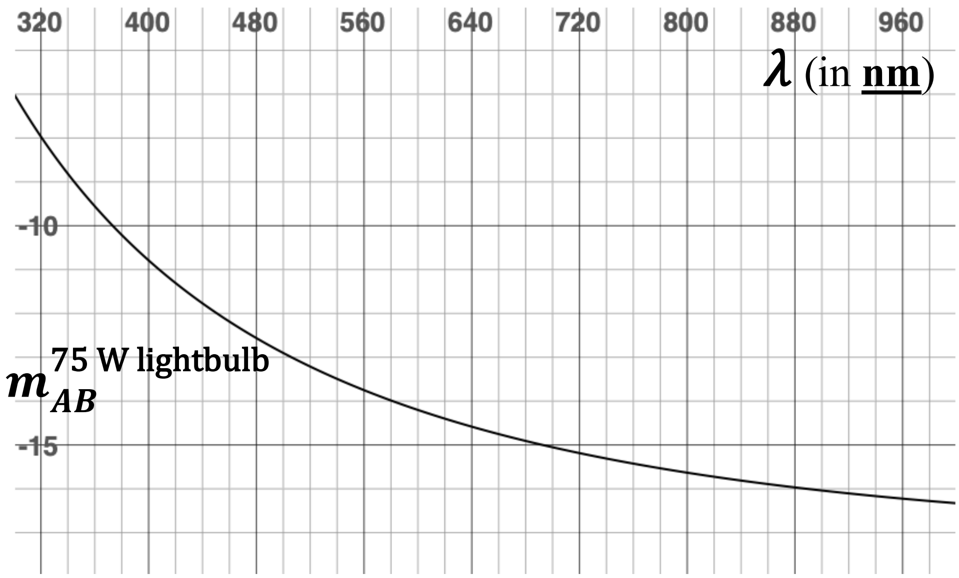
$$\boldsymbol{= -2.5\log_{10}\left[\frac{P}{T^4 d^2 \lambda^3\left(e^{\left[(1.4384\times10^7)/(T\lambda)\right]} - 1\right)}\right] - 79.2075} \,, \qquad (9)$$

with the latter formula obtained when numerically filling in for the constants h, c, and k_{B}.

For a $\boldsymbol{P}$ = **75 watt** incandescent lightbulb, with an idealized non-light-absorbing glass or quartz envelope and a filament at a typical temperature of $\boldsymbol{T}$ = **2750 K**, at a line-of-sight distance of $\boldsymbol{d}$ = **10 meters**, the resulting AB magnitude would be :

$$\boldsymbol{m_{AB}^{\text{75 W lightbulb}}(\lambda) = -2.5\log_{10}\left[\frac{1}{\lambda^3\left(e^{\left[5230.0/\lambda\right]} - 1\right)}\right] - 44.5018} \,. \qquad (10)$$

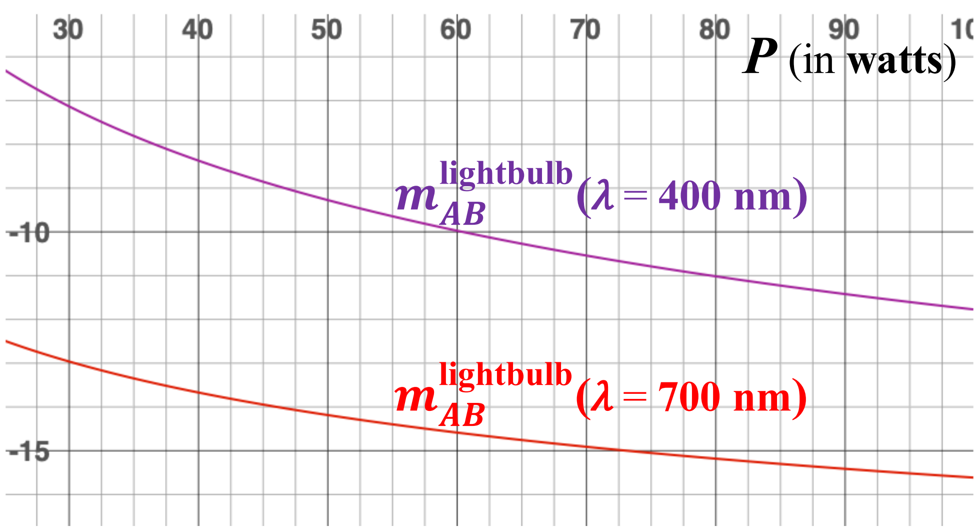
A plot of Eq. (10) above, within the optical region of the spectrum:



Considering this same lightbulb's AB magnitude as a function of both wavelength $\boldsymbol{\lambda}$ (in nm) *and* its dissipated input electrical power $\boldsymbol{P}$ (again, in watts), with an observer at a line-of-sight distance $\boldsymbol{d}$ (in meters), and with the resulting filament temperature varying with the input power, we have :

$$m_{AB}^{75\text{ W lightbulb}}(\lambda, P, d) = -2.5\log_{10}\left[\frac{1}{d^2\lambda^3\left(e^{\left[15392.7/\lambda P^{0.25}\right]} - 1\right)}\right] - 49.50\,. \quad (11)$$

Resulting plots of magnitude from Eq. (11) as a function of input electrical power $\boldsymbol{P}$, at reference wavelengths of $\boldsymbol{\lambda}$ = **400 nm** and $\boldsymbol{\lambda}$ = **700 nm**, from an observing distance of $\boldsymbol{d}$ = **10 meters**:



b) Formulae and results for ***specular reflections of* perfectly-pointlike *blackbody sources*** (i.e., the below formulae are NOT realistic, as no source is perfectly-pointlike, and the Sun certainly is not perfectly pointlike. The closest somewhat realistic applicability of equations (12) – (14) would be for distant stars.) :

The AB magnitude, as a function of wavelength $\boldsymbol{\lambda}$ (in **nm**), of the *specular reflection* from a flat surface with specular reflection coeffient $\boldsymbol{C}_\mathrm{s}$ (for the case that this reflectivity is constant across the entire relevant wavelength range, *i.e.* a geometrically flat, neutral density reflecting surface), when viewed from a distance $\boldsymbol{\ell}$ away from the reflecting surface (*e.g.*, if the reflecting surface is at orbit height $\boldsymbol{\ell}$, when viewed at the zenith from the ground) of a very distant, perfectly-pointlike blackbody source of radius $\boldsymbol{R}$ and surface (or photosphere) temperature $\boldsymbol{T}$ that is at a distance $\boldsymbol{d}$ from the reflecting surface (where, of course, $d > R$) :

$$m_{\mathrm{AB}}(\lambda) = -2.5\log_{10}\left[\frac{(2\pi \times 10^{27})hcR^2C_s}{(d+\ell)^2\lambda^3\left(e^{\left[(10^9 hc)/(k_B T\lambda)\right]} - 1\right)}\right] - 56.10 \qquad (12)$$

$$= -2.5\log_{10}\left[\frac{R^2C_s}{(d+\ell)^2\lambda^3\left(e^{\left[(1.4384\times 10^7)/(T\lambda)\right]} - 1\right)}\right] - 63.8407\,, \qquad (13)$$

with the latter formula obtained when numerically filling in for the constants h, c, and k_B.

Equation (13), for the typical case where $\boldsymbol{\ell} \ll \boldsymbol{d}$:

$$m_{\mathrm{AB}}(\lambda) = -2.5\log_{10}\left[\frac{R^2C_s}{d^2\lambda^3\left(e^{\left[(1.4384\times 10^7)/(T\lambda)\right]} - 1\right)}\right] - 63.8407\,. \qquad (14)$$

c) Formulae and results for ***specular reflections of* extended *blackbody sources*** (*e.g.*, specular reflections of the Sun) :

A much more realistic approximation of the specular reflection properties of the Sun (or alternatively of other blackbody sources that also do not appear as being nearly perfectly pointlike) is given below, where the specular reflectance angle $\boldsymbol{\theta}_\mathrm{s}$ denotes the angle between 1) a ray from the observer to the actual satellite surface; and 2) a ray from the observer to a planar hypothetical continuation of the satellite surface that, if this ray were reflected from the satellite surface, would reflect directly to/from the center of the Sun (or of other extended blackbody source). (Please note again that these equations refer only to specular reflection; we will consider the much broader diffuse reflectance angular distribution on the following page.) The possible range of $\boldsymbol{\theta}_s$ will be (in radians) $-\frac{R}{d+\ell} < \boldsymbol{\theta}_s < \frac{R}{d+\ell}$ (**otherwise** specular reflection will just be **identically equal to zero** (and in that case, only diffuse reflections could possibly contribute).

For this case of specular reflection with reflectance angle $\boldsymbol{\theta}_s$, the above approximate Eqs. (12) & (13) become the following approximate Eqs. (15) & (16), with $\boldsymbol{A}$ being the area, in m^2, of the orbiting surface:

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda}) = -2.5\log_{10}\left[\frac{(2\pi\times10^{27})hcC_sR\sqrt{A}\cos\left(\frac{\pi(d+\ell)\theta_s}{2R}\right)}{(d+\ell)^2\lambda^3\left(e^{\left[(10^9hc)/(k_BT\lambda)\right]}-1\right)}\right] - 56.10 \qquad (15)$$

$$= -2.5\log_{10}\left[\frac{C_sR\sqrt{A}\cos\left(\frac{\pi(d+\ell)\theta_s}{2R}\right)}{(d+\ell)^2\lambda^3\left(e^{\left[(1.4384\times10^7)/(T\lambda)\right]}-1\right)}\right] - 63.8407, \qquad (16)$$

with the latter formula obtained when numerically filling in for the constants h, c, and k_B.

For the case where the reflected blackbody is, in fact, the Sun (*i.e.*, for $\boldsymbol{T} = \boldsymbol{T}_\odot$, $\boldsymbol{d} = \boldsymbol{d}_\oplus$, $\boldsymbol{R} = \boldsymbol{R}_\odot$, and $\boldsymbol{\ell} \ll \boldsymbol{d}_\oplus$):

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda}) = -2.5\log_{10}\left[\frac{C_s\sqrt{A}\cos\left(\frac{\pi d_\oplus\theta_s}{2R_\odot}\right)}{\lambda^3\left(e^{\left[2489.5/\lambda\right]}-1\right)}\right] - 30.073\,. \qquad (17)$$

Equation (17), for the case where $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathrm{ref}} = \mathbf{511\ nm}$:

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathrm{ref}}) = -2.5\log_{10}\left[C_s\sqrt{A}\cos\left(\frac{\pi d_\oplus\theta_s}{2R_\odot}\right)\right] - 4.483\,. \qquad (18)$$

Equation (18), for an example case where $\boldsymbol{A} = \mathbf{1\ m^2}$, $\boldsymbol{\theta}_s = \mathbf{0°}$, and $\boldsymbol{C}_s = \mathbf{0.13}$:

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathrm{ref}}) = -2.27\,. \qquad (19)$$

However, this would only be the **very brightest** approximate apparent magnitude of the satellite reflection for the **very** brief period (of order 1 second or less) when one happens to be directly within the solar specular reflection area on the ground from the satellite (*i.e.*, within a spot moving across the Earth's surface at approximately the same ground track speed as the satellite, that is *only approximately 6 km in diameter*).

In the far more typical case when one is within the very large area on the Earth's surface that can view the **diffuse** solar reflection from the satellite surface, but one is not within the much smaller solar specular reflection area on the Earth's surface, then the satellite will, of course, appear much dimmer. Rather than the characteristic angular span of $\boldsymbol{\theta_s^{max}} \equiv \frac{\boldsymbol{R}}{\boldsymbol{d+\ell}}$ for specular reflection, the characteristic angular deviation of the diffuse reflection is of order **1 radian** in radius (*i.e.*, typically of order 10^3 times larger than that maximum value of $\boldsymbol{\theta}_s$). Thus, the brightest approximate apparent magnitudes of the **diffuse** reflection reflection from the satellite surface would replace Eqs. (15) – (19) with the following Eqs. (20) – (24), for a flat satellite surface of area $\boldsymbol{A}$ with **diffuse** reflection coeffient $\boldsymbol{C}_{\mathbf{d}}$:

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda}) = -2.5\log_{10}\left[\frac{(2\times 10^{27})hcR^2C_{\mathrm{d}}A\ \cos\theta\cos\phi}{d^2\ell^2\lambda^3\left(e^{\left[{(10^9hc)}/{(k_BT\lambda)}\right]}-1\right)}\right] - 56.10\,, \tag{20}$$

$$= -2.5\log_{10}\left[\frac{R^2C_{\mathrm{d}}A\ \cos\theta\cos\phi}{4d^2\ell^2\lambda^3\left(e^{\left[{(1.4384\times 10^7)}/{(T\lambda)}\right]}-1\right)}\right] - 64.103\,, \tag{21}$$

where $\boldsymbol{\theta}$ is the angle between the surface normal and a ray from the surface center to the observer, and $\boldsymbol{\phi}$ is the angle between the surface normal and a ray from the surface center towards the Sun; and with the latter formula obtained when numerically filling in for the constants h, c, and k_{B}. For the case where the reflected blackbody is, in fact, the Sun (*i.e.*, for $\boldsymbol{R} = \boldsymbol{R}_\odot$, $\boldsymbol{T} = \boldsymbol{T}_\odot$, and $\boldsymbol{d} = \boldsymbol{d}_\oplus$) :

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda}) = -2.5\log_{10}\left[\frac{C_{\mathrm{d}}A\ \cos\theta\cos\phi}{4\ell^2\lambda^3\left(e^{\left[{2489.5}/{\lambda}\right]}-1\right)}\right] - 52.442\,. \tag{22}$$

Equation (22), for the case where $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathbf{ref}}$ **= 511 nm** :

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathrm{ref}}) = -2.5\log_{10}\left[\frac{C_{\mathrm{d}}A\ \cos\theta\cos\phi}{4\ell^2}\right] - 26.820\,. \tag{23}$$

Equation (23), for an example case where $\boldsymbol{\theta}$ **= 0°,** $\boldsymbol{\phi}$ **= 60°,** $\boldsymbol{A}$ **= 1 m²**, $\boldsymbol{\ell}$ = 700 km **= 700000 m**, and $\boldsymbol{C}_{\mathbf{d}}$ **= 0.52** :

$$\boldsymbol{m}_{\mathrm{AB}}(\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathrm{ref}}) = +5.37\,. \tag{24}$$