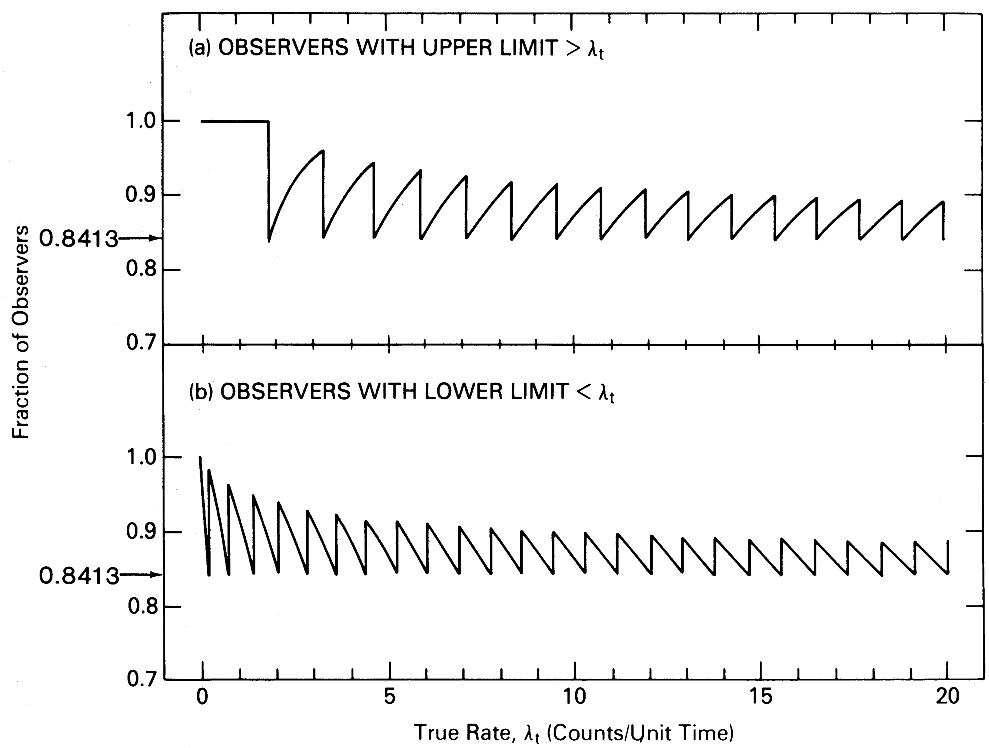
Poisson Limits Issue/Question

The Gehrels paper (which is copied after these two pages) uses the "standard" formulae for calculating upper and lower limits λ_u and λ_ℓ in the case where *n* random (Poisson-distributed) events are detected in a given observation:

$$1-CL = \sum_{x=0}^{n} \frac{\lambda_u^x e^{-\lambda_u}}{x!} \quad \text{and} \quad CL = \sum_{x=0}^{n-1} \frac{\lambda_\ell^x e^{-\lambda_u}}{x!}$$

where $CL \in (0,1)$ is the confidence level (these are Eq. 1 & 2 in Gehrels). The problem is that these limits tend to over-cover on average – the interval between them is usually too large. This can be seen in Fig. 2 in the Gehrels paper:



These plots show that the confidence intervals usually over-cover, in the specific example of a confidence level of 0.8413 for the upper and lower limits. Only at the points where the plots above reach a minimum do the limits provide correct coverage. If one wants uncertainties in which one is absolutely certain never to under-cover, but for which some over-coverage is not a problem, that is OK. But if one would like uncertainties that provide the correct coverage "on average," these uncertainties are too large.

Ultimately, this is because the Poisson distribution is discrete, and one can't detect a fraction of an event. Thus, by performing a measurement, one is in effect truncating to an integer the number of events one would "expect" to measure. If one detects 10 events, and wants to determine a true rate from such a measurement, it is often good to think of the measurement as actually implying that one has detected a "number of events" that falls in the interval [10,11).

This is in fact done when one determines the central value of the true rate from such a measurement. But when determining the uncertainties, the formulae above are really effectively assuming that the integral value *plus one* is the central value, and thus provide larger uncertainties which tend to over-cover on average.

One clearly does not want to replace this with a formula that would tend to under-cover, as would happen by effectively assuming that the integral value itself is the central value. But by taking the average between the two, I think one may obtain limits that tend to be correct "on average," and which do not bias toward over- or under-coverage. To do this, one would make the following replacements:

To calculate upper limit λ_u:

Standard/Gehrels | new | (alternate form of new)

$$1-CL = \sum_{x=0}^{n} \frac{\lambda_u^x e^{-\lambda_u}}{x!} \rightarrow 0.5\times\left[\sum_{x=0}^{n-1} \frac{\lambda_u^x e^{-\lambda_u}}{x!} + \sum_{x=0}^{n} \frac{\lambda_u^x e^{-\lambda_u}}{x!}\right] = e^{-\lambda_u}\left(\sum_{x=0}^{n-1} \frac{\lambda_u^x}{x!} + \frac{0.5\lambda_u^n}{n!}\right)$$

To calculate lower limit λ_ℓ:

Standard/Gehrels | new | (alternate form of new)

$$CL = \sum_{x=0}^{n-1} \frac{\lambda_\ell^x e^{-\lambda_\ell}}{x!} \rightarrow 0.5\times\left[\sum_{x=0}^{n-1} \frac{\lambda_\ell^x e^{-\lambda_\ell}}{x!} + \sum_{x=0}^{n} \frac{\lambda_\ell^x e^{-\lambda_\ell}}{x!}\right] = e^{-\lambda_\ell}\left(\sum_{x=0}^{n-1} \frac{\lambda_\ell^x}{x!} + \frac{0.5\lambda_\ell^n}{n!}\right)$$

Does this appear to be a reasonable thing to do? I.e., wouldn't it make the fraction of observers with upper limit greater than the true rate, and lower limit less than the true rate, be (at least approximately) equal to the CL, when averaged over the estimate of the probability density function of the true rate?

If so, it would perhaps be very worthwhile for Dag and I to calculate approximate formulae, analogous to the ones in the Gehrels paper, e.g. $\lambda_u = n + 1 + \sqrt{n + 0.75}$, which provide people with trivial-to-calculate approximations to the correct uncertainties, and which are fairly accurate (better than, e.g., the usual $\lambda_u = n + \sqrt{n}$), but which do not tend to over-cover like that $\lambda_u = n + 1 + \sqrt{n + 0.75}$ formula. (The fact that the Gehrels paper has over 800 published citations testifies to the demand for simple approximate formulae.) We would of course *need to show* in such a paper that the new approximate formulae do in fact approximate the above "new" limit formulae, just like Gehrels shows that his approximate formulae do in fact correctly approximate the "Standard/Gehrels" limits.

THE ASTROPHYSICAL JOURNAL, 303:336–346, 1986 April 1

CONFIDENCE LIMITS FOR SMALL NUMBERS OF EVENTS IN ASTROPHYSICAL DATA

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ABSTRACT

Convenient tables and approximate formulae are presented for confidence limits based on Poisson and binomial statistics. Poisson statistics apply when event rates are calculated from small numbers of observed events, and binomial statistics apply when ratios of two different event types are calculated from small numbers of observed events. The limits in the tables are given for all confidence levels commonly used in astrophysics.

Subject heading: numerical methods

I. INTRODUCTION

In many areas of astrophysics it occasionally happens that only a small number of events of interest are detected during an observation. Examples range from the number of supernovae seen in a given period of time from a cluster of galaxies to the number of gamma rays detected during a source observation. If the goal is to determine quantities such as the event rate or the ratio of different event types, then the best approach is to repeat the measurement with a longer integration time or a larger collection factor in order to obtain enough events for an accurate measurement. In some cases, for one reason or another, this is not possible or practical, and one is forced to make the best use of the data in hand. Results are then typically quoted as upper limits at a specified confidence level or as a measured value with error bars containing a specified confidence interval. Conventionally, error bars plotted in figures are 84.13% confidence upper and lower limits containing a 68.27% confidence interval.

The calculation of limits for small numbers of counts is based on standard equations derived from Poisson and binomial statistics. Although the equations are straightforward, using them directly is cumbersome and involves interpolating in tables and executing several mathematical operations. Tables of limits for a few confidence levels have been published (Pearson and Hartley 1966; Beyer 1966), but not all levels commonly used in astrophysics are included. Also, I have found that the tables most often referenced (Beyer 1966) have inaccuracies in the last decimal place. Another shortcoming in this subject is that approximate formulae for quick estimation or for use in computer programs are not available. The purpose of this paper is to present tables of Poisson and binomial limits for all confidence levels commonly used in astrophysics and to derive easy-to-use approximate formulae for calculating the limits.

II. CONFIDENCE LIMITS FOR POISSON STATISTICS

a) Definitions and Numerical Solutions

We consider the case where n events are detected in a given observation. Then, based on Poisson statistics, the upper limit, λ_u, and lower limit, λ_l, of confidence level CL are defined by

$$\sum_{x=0}^{n} \frac{\lambda_u^x e^{-\lambda_u}}{x!} = 1 - \mathrm{CL} , \qquad (1)$$

$$\sum_{x=0}^{n-1} \frac{\lambda_l^x e^{-\lambda_l}}{x!} = \mathrm{CL} \qquad (n \neq 0) \qquad (2)$$

(see, e.g., Pearson and Hartley 1966; Beyer 1966). The lower limit for $n = 0$ is $\lambda_l = 0.0$. Single-sided confidence limits are defined in equations (1) and (2) and will be used throughout this paper. Double-sided intervals of level CL′ can be obtained by substituting $(1 + \mathrm{CL}')/2$ for CL in the equations. For instance, if λ_l and λ_u are each 95% single-sided limits, then the double-sided interval λ_l to λ_u has a confidence of 90%.

In the Appendix it is shown that equations (1) and (2) are reasonable definitions of Poisson confidence limits. In particular, it is illustrated that if a large number of observers measure a real physical rate, then at least 100 × CL% of them will assign upper limits based on their measurements that are greater than the real rate and at least 100 × CL% will assign lower limits that are less than the real rate, for all rates. It is also shown that the upper limits are the smallest they can be and still satisfy this condition and that the lower limits the largest they can be. The limits in equations (1) and (2) therefore optimally satisfy the definition of confidence limits (see, e.g., Cramér 1945).

It is not possible to obtain exact algebraic expressions for λ_u and λ_l from equations (1) and (2). However, as shown in the next section, approximate expressions can be found that are good to a few percent. Also, for any values of n and CL, the limits can be numerically determined to any desired accuracy. Listed in Tables 1 and 2 are exact values for λ_u and λ_l at several confidence levels for n = 0–50, determined using an iterative numerical technique (Newton's method) to solve equations (1) and (2). The confidence levels that were chosen are the obvious 0.90, 0.95, 0.99, and 0.999, as well as levels corresponding to 1, 2, and 3 σ limits for Gaussian statistics, namely 0.8413, 0.9772, and 0.9987. Also, to allow common double-sided intervals to be determined, single-sided levels of 0.975 (double-sided 0.95), 0.995 (d-s 0.99), and 0.9995 (d-s 0.999) are included.

As an example of the use of Tables 1 and 2, consider an observation where four events are detected in 10 s. The 99% confidence upper limit to the rate is then 1.160 s^{-1}, and the 99% confidence lower limit is 0.082 s^{-1}. The 99% confidence double-sided interval (CL = 0.995 column) is 0.067–1.259 s^{-1}. The measured value with error bars corresponding to 1 σ Gaussian errors is $0.40^{+0.32}_{-0.19}$ s^{-1}.

TABLE 1

POISSON SINGLE-SIDED UPPER LIMITS

	CONFIDENCE LEVEL									
n	0.8413[a]	0.90	0.95	0.975	0.9772[a]	0.99	0.995	0.9987[a]	0.999	0.9995
0	1.841	2.303	2.996	3.689	3.783	4.605	5.298	6.608	6.908	7.601
1	3.300	3.890	4.744	5.572	5.683	6.638	7.430	8.900	9.233	9.999
2	4.638	5.322	6.296	7.225	7.348	8.406	9.274	10.87	11.23	12.05
3	5.918	6.681	7.754	8.767	8.902	10.05	10.98	12.68	13.06	13.93
4	7.163	7.994	9.154	10.24	10.39	11.60	12.59	14.39	14.79	15.71
5	8.382	9.275	10.51	11.67	11.82	13.11	14.15	16.03	16.45	17.41
6	9.584	10.53	11.84	13.06	13.22	14.57	15.66	17.62	18.06	19.05
7	10.77	11.77	13.15	14.42	14.59	16.00	17.13	19.17	19.63	20.65
8	11.95	12.99	14.43	15.76	15.94	17.40	18.58	20.69	21.16	22.22
9	13.11	14.21	15.71	17.08	17.27	18.78	20.00	22.18	22.66	23.75
10	14.27	15.41	16.96	18.39	18.58	20.14	21.40	23.64	24.13	25.26
11	15.42	16.60	18.21	19.68	19.87	21.49	22.78	25.08	25.59	26.74
12	16.56	17.78	19.44	20.96	21.16	22.82	24.14	26.51	27.03	28.20
13	17.70	18.96	20.67	22.23	22.43	24.14	25.50	27.91	28.45	29.65
14	18.83	20.13	21.89	23.49	23.70	25.45	26.84	29.31	29.85	31.08
15	19.96	21.29	23.10	24.74	24.95	26.74	28.16	30.69	31.24	32.50
16	21.08	22.45	24.30	25.98	26.20	28.03	29.48	32.06	32.62	33.90
17	22.20	23.61	25.50	27.22	27.44	29.31	30.79	33.42	33.99	35.29
18	23.32	24.76	26.69	28.45	28.68	30.58	32.09	34.76	35.35	36.68
19	24.44	25.90	27.88	29.67	29.90	31.85	33.38	36.10	36.70	38.05
20	25.55	27.05	29.06	30.89	31.13	33.10	34.67	37.44	38.04	39.41
21	26.66	28.18	30.24	32.10	32.34	34.35	35.95	38.76	39.37	40.76
22	27.76	29.32	31.41	33.31	33.55	35.60	37.22	40.07	40.70	42.11
23	28.87	30.45	32.59	34.51	34.76	36.84	38.48	41.38	42.02	43.45
24	29.97	31.58	33.75	35.71	35.96	38.08	39.74	42.69	43.33	44.78
25	31.07	32.71	34.92	36.90	37.16	39.31	41.00	43.98	44.64	46.11
26	32.16	33.84	36.08	38.10	38.36	40.53	42.25	45.28	45.94	47.42
27	33.26	34.96	37.23	39.28	39.55	41.76	43.50	46.56	47.23	48.74
28	34.35	36.08	38.39	40.47	40.74	42.98	44.74	47.84	48.52	50.04
29	35.45	37.20	39.54	41.65	41.92	44.19	45.98	49.12	49.80	51.35
30	36.54	38.32	40.69	42.83	43.10	45.40	47.21	50.39	51.08	52.64
31	37.63	39.43	41.84	44.00	44.28	46.61	48.44	51.66	52.36	53.94
32	38.72	40.54	42.98	45.17	45.46	47.81	49.67	52.92	53.63	55.23
33	39.80	41.65	44.13	46.34	46.63	49.01	50.89	54.18	54.90	56.51
34	40.89	42.76	45.27	47.51	47.80	50.21	52.11	55.43	56.16	57.79
35	41.97	43.87	46.40	48.68	48.97	51.41	53.32	56.69	57.42	59.06
36	43.06	44.98	47.54	49.84	50.14	52.60	54.54	57.93	58.67	60.34
37	44.14	46.08	48.68	51.00	51.30	53.79	55.75	59.18	59.93	61.60
38	45.22	47.19	49.81	52.16	52.46	54.98	56.96	60.42	61.17	62.87
39	46.30	48.29	50.94	53.31	53.62	56.16	58.16	61.66	62.42	64.13
40	47.38	49.39	52.07	54.47	54.78	57.35	59.36	62.89	63.66	65.39
41	48.46	50.49	53.20	55.62	55.93	58.53	60.56	64.13	64.90	66.64
42	49.53	51.59	54.32	56.77	57.09	59.71	61.76	65.36	66.14	67.90
43	50.61	52.69	55.45	57.92	58.24	60.88	62.96	66.58	67.37	69.14
44	51.68	53.78	56.57	59.07	59.39	62.06	64.15	67.81	68.60	70.39
45	52.76	54.88	57.69	60.21	60.54	63.23	65.34	69.03	69.83	71.63
46	53.83	55.97	58.82	61.36	61.69	64.40	66.53	70.25	71.06	72.88
47	54.90	57.07	59.94	62.50	62.83	65.57	67.72	71.47	72.28	74.11
48	55.98	58.16	61.05	63.64	63.97	66.74	68.90	72.68	73.51	75.35
49	57.05	59.25	62.17	64.78	65.12	67.90	70.08	73.90	74.72	76.58
50	58.12	60.34	63.29	65.92	66.26	69.07	71.27	75.11	75.94	77.81
60	68.79	71.20	74.39	77.23	77.60	80.62	82.99	87.11	88.01	90.01
70	79.41	81.99	85.40	88.44	88.83	92.06	94.58	98.96	99.91	102.0
80	89.98	92.73	96.35	99.57	99.98	103.4	106.1	110.7	111.7	113.9
90	100.5	103.4	107.2	110.6	111.1	114.7	117.4	122.3	123.3	125.7
100	111.0	114.1	118.1	121.6	122.1	125.8	128.8	133.8	134.9	137.4

[a] Corresponding to Gaussian statistics 1 $\sigma = 0.8413$, 2 $\sigma = 0.9772$, and 3 $\sigma = 0.9987$.

TABLE 2

POISSON SINGLE-SIDED LOWER LIMITS

	CONFIDENCE LEVEL									
n	0.8413	0.90	0.95	0.975	0.9772	0.99	0.995	0.9987	0.999	0.9995
1	0.173	0.105	5.13−2[a]	2.53−2	2.30−2	1.01−2	5.01−3	1.35−3	1.00−3	5.00−4
2	0.708	0.532	0.355	0.242	0.230	0.149	0.103	5.29−2	4.54−2	3.20−2
3	1.367	1.102	0.818	0.619	0.569	0.436	0.338	0.212	0.191	0.150
4	2.086	1.745	1.366	1.090	1.058	0.823	0.672	0.465	0.429	0.355
5	2.840	2.433	1.970	1.623	1.583	1.279	1.078	0.792	0.739	0.632
6	3.620	3.152	2.613	2.202	2.153	1.785	1.537	1.175	1.107	0.967
7	4.419	3.895	3.285	2.814	2.758	2.330	2.037	1.603	1.520	1.348
8	5.232	4.656	3.981	3.454	3.391	2.906	2.571	2.068	1.971	1.768
9	6.057	5.432	4.695	4.115	4.046	3.507	3.132	2.563	2.452	2.220
10	6.891	6.221	5.425	4.795	4.719	4.130	3.717	3.084	2.961	2.699
11	7.734	7.021	6.169	5.491	5.409	4.771	4.321	3.628	3.491	3.202
12	8.585	7.829	6.924	6.201	6.113	5.428	4.943	4.191	4.042	3.726
13	9.441	8.646	7.690	6.922	6.828	6.099	5.580	4.772	4.611	4.269
14	10.30	9.470	8.464	7.654	7.555	6.782	6.231	5.367	5.195	4.828
15	11.17	10.30	9.246	8.395	8.291	7.477	6.893	5.977	5.794	5.402
16	12.04	11.14	10.04	9.145	9.036	8.181	7.567	6.599	6.405	5.990
17	12.92	11.98	10.83	9.903	9.789	8.895	8.251	7.233	7.028	6.590
18	13.80	12.82	11.63	10.67	10.55	9.616	8.943	7.877	7.662	7.201
19	14.68	13.67	12.44	11.44	11.32	10.35	9.644	8.530	8.306	7.822
20	15.57	14.53	13.25	12.22	12.09	11.08	10.35	9.193	8.958	8.453
21	16.45	15.38	14.07	13.00	12.87	11.83	11.07	9.863	9.619	9.093
22	17.35	16.24	14.89	13.79	13.65	12.57	11.79	10.54	10.29	9.741
23	18.24	17.11	15.72	14.58	14.44	13.33	12.52	11.23	10.96	10.40
24	19.14	17.97	16.55	15.38	15.23	14.09	13.26	11.92	11.65	11.06
25	20.03	18.84	17.38	16.18	16.03	14.85	14.00	12.62	12.34	11.73
26	20.93	19.72	18.22	16.98	16.83	15.62	14.74	13.32	13.03	12.41
27	21.84	20.59	19.06	17.79	17.64	16.40	15.49	14.03	13.73	13.09
28	22.74	21.47	19.90	18.61	18.45	17.17	16.25	14.75	14.44	13.78
29	23.65	22.35	20.75	19.42	19.26	17.96	17.00	15.47	15.15	14.47
30	24.55	23.23	21.59	20.24	20.07	18.74	17.77	16.19	15.87	15.17
31	25.46	24.11	22.44	21.06	20.89	19.53	18.53	16.92	16.59	15.87
32	26.37	25.00	23.30	21.89	21.71	20.32	19.30	17.65	17.32	16.58
33	27.28	25.89	24.15	22.72	22.54	21.12	20.08	18.39	18.05	17.30
34	28.20	26.77	25.01	23.55	23.36	21.92	20.86	19.13	18.78	18.01
35	29.11	27.66	25.87	24.38	24.19	22.72	21.64	19.88	19.52	18.73
36	30.03	28.56	26.73	25.21	25.03	23.53	22.42	20.63	20.26	19.46
37	30.94	29.45	27.59	26.05	25.86	24.33	23.21	21.38	21.00	20.19
38	31.86	30.34	28.46	26.89	26.70	25.14	24.00	22.14	21.75	20.92
39	32.78	31.24	29.33	27.73	27.53	25.96	24.79	22.89	22.51	21.66
40	33.70	32.14	30.20	28.58	28.38	26.77	25.59	23.66	23.26	22.40
41	34.62	33.04	31.07	29.42	29.22	27.59	26.38	24.42	24.02	23.14
42	35.55	33.94	31.94	30.27	30.06	28.41	27.18	25.19	24.78	23.88
43	36.47	34.84	32.81	31.12	30.91	29.23	27.99	25.96	25.54	24.63
44	37.39	35.74	33.69	31.97	31.76	30.05	28.79	26.73	26.31	25.38
45	38.32	36.65	34.56	32.82	32.61	30.88	29.60	27.51	27.08	26.14
46	39.24	37.55	35.44	33.68	33.46	31.70	30.41	28.29	27.85	26.89
47	40.17	38.46	36.32	34.53	34.31	32.53	31.22	29.07	28.62	27.65
48	41.10	39.36	37.20	35.39	35.17	33.36	32.03	29.85	29.40	28.42
49	42.02	40.27	38.08	36.25	36.02	34.20	32.85	30.63	30.18	29.18
50	42.95	41.18	38.96	37.11	36.88	35.03	33.66	31.42	30.96	29.95
60	52.28	50.31	47.85	45.79	45.53	43.46	41.93	39.40	38.88	37.73
70	61.65	59.51	56.83	54.57	54.29	52.02	50.33	47.54	46.96	45.70
80	71.07	68.77	65.88	63.44	63.13	60.67	58.84	55.81	55.18	53.80
90	80.53	78.08	74.98	72.37	72.04	69.41	67.44	64.18	63.51	62.02
100	90.02	87.42	84.14	81.36	81.02	78.22	76.12	72.65	71.92	70.33

[a] 5.13−2 means 5.13×10^{-2}.

b) Approximate Algebraic Expressions

The standard method for determining Poisson confidence limits makes use of the relation between the Poisson sum and the χ^2 probability function

$$\sum_{x=0}^{n-1} \frac{\lambda^x e^{-\lambda}}{x!} = 1 - P(\chi^2|\nu) \quad (3)$$

(see, e.g., Abramowitz and Stegun 1965), where $P(\chi^2|\nu)$ is the χ^2 probability for ν degrees of freedom with $\chi^2 = 2\lambda$ and $\nu = 2n$. From equation (3), the Poisson upper and lower limits of confidence CL defined by equations (1) and (2) are related to the χ^2 probability function by

$$P(2\lambda_u|2n + 2) = \text{CL} \quad (4)$$

$$P(2\lambda_l|2n) = 1 - \text{CL} . \quad (5)$$

Thus, λ_u and λ_l can be determined using tables of percentage points of the χ^2 distribution. This method is straightforward but is definitely less convenient than tables of the limits themselves. Also, since no χ^2 table covers all confidence levels commonly used, interpolations and extrapolations are often required. Equations (4) and (5) are, however, good starting points for developing approximate algebraic expressions for the limits.

i) *Upper Limits*

Several approximate inverse relations giving χ^2 in terms of $P(\chi^2|\nu)$ are listed by Abramowitz and Stegun (1965). In this section, these relations will be used to obtain several approximate expressions for λ_u. Their accuracy will then be compared and a recommended numerical procedure given. All of the equations in Abramowitz and Stegun make use of the variable y_a defined as the Gaussian integral value with the same probability as $P(\chi^2|\nu)$; i.e., if $P(\chi_a^2|\nu) = a$, then y_a is such that $G(y_a) = (2\pi)^{-1/2} \int_{-\infty}^{y_a} e^{-t^2/2} dt = a$. In our final equations we use the variable $S = |y_a|$, which is the equivalent Gaussian number of σ corresponding to the confidence level. Values of S for the common confidence levels are listed in Table 3, along with other data pertaining to the next section.

The first approximate relation is

$$\chi_a^2 \approx \tfrac{1}{2}(y_a + \sqrt{2\nu - 1})^2 , \quad (6)$$

which, combined with equation (4), gives

$$\lambda_u \approx n + S\sqrt{n + \frac{3}{4}} + \frac{S^2 + 3}{4} . \quad (7)$$

Another approximate relation is

$$\chi_a^2 \approx \nu\left(1 - \frac{2}{9\nu} + y_a\sqrt{\frac{2}{9\nu}}\right)^3 , \quad (8)$$

which gives

$$\lambda_u \approx (n + 1)\left[1 - \frac{1}{9(n + 1)} + \frac{S}{3\sqrt{n + 1}}\right]^3 . \quad (9)$$

A final approximate expression is obtained by expanding equation (9) and keeping only terms of order $(n + 1)^\alpha$ where $\alpha \geq 0$, giving

$$\lambda_u \approx n + S\sqrt{n + 1} + \frac{S^2 + 2}{3} . \quad (10)$$

In Table 4 we compare the accuracy of equations (7), (9), and (10) for several values of n at the two extreme confidence levels discussed in this paper. The table lists the exact values from Table 1 in the column under λ_u and gives the values from the approximations along with (in parentheses) the percentage error with respect to λ_u. The Gaussian limit $n + S(n)^{1/2}$ is also given for comparison. Equation (7) is quite good for CL = 0.8413 but becomes poor at higher confidence levels. Equation (9) is the most complicated of the approximate expressions but is best overall—within 3% of the true limit for all values of n and all confidence levels considered in this paper. Equation (10) is fair at low confidence levels but becomes very accurate at high confidence levels where it actually does better than equation (9) from which it was derived. Its accuracy is better than 10% for all values of n for all confidence levels and better than 1% for $n \geq 4$.

Recommended approximation: For applications where upper limits good to within 10% are accurate enough, use the simple expression in equation (10), $\lambda_u \approx n + S(n + 1)^{1/2} + (S^2 + 2)/3$. If higher accuracy is required, use the exact upper limits for the smallest few n and equation (10) for all others or use the more complicated equation (9). For the special case of $S = 1$, the simple expression in equation (7), $\lambda_u \approx n + (n + \frac{3}{4})^{1/2} + 1$, is good to better than 1.5% for all n. The Gaussian statistics limit $\lambda_u \approx n + S(n)^{1/2}$ becomes accurate at the few percent level only for $n > 100$.

ii) *Lower Limits*

To obtain approximate expressions for the Poisson lower limits, the same approximate χ^2 inverse relations from the last section are used. In this case, the starting point is the relationship between the Poisson confidence lower limit and the χ^2 probability function expressed in equation (5). Combining equation (5) with equation (6) gives

$$\lambda_l \approx n - S\sqrt{n - \frac{1}{4}} + \frac{S^2 - 1}{4} , \quad (11)$$

and with equation (8) gives

$$\lambda_l \approx n\left(1 - \frac{1}{9n} - \frac{S}{3\sqrt{n}}\right)^3 . \quad (12)$$

TABLE 3

Data for Approximate Expressions for Poisson Limits

Parameter	Confidence Level									
	0.8413	0.900	0.950	0.975	0.9772	0.990	0.995	0.9987	0.999	0.9995
S[a]	1.000	1.282	1.645	1.960	2.000	2.326	2.576	3.000	3.090	3.291
β	0.0	0.010	0.031	0.058	0.062	0.103	0.141	0.222	0.241	0.287
γ	...	−4.00	−2.50	−2.22	−2.19	−2.07	−2.00	−1.88	−1.85	−1.80

[a] Number of Gaussian σ.

TABLE 4

COMPARISON OF POISSON UPPER LIMIT APPROXIMATE EXPRESSIONS

				Gaussian Limit		Equation (7)		Equation (9)		Equation (10)	
n	CL	S	λ_u	Value	(% Error[a])	Value	(% Error)	Value	(% Error)	Value	(% Error)
0..........	0.8413	1.0	1.841	0.000	(−)	1.866	(1.4)	1.826	(0.8)	2.000	(8.6)
1..........	0.8413	1.0	3.300	2.000	(39)	3.323	(0.7)	3.287	(0.4)	3.414	(3.5)
2..........	0.8413	1.0	4.638	3.414	(26)	4.658	(0.4)	4.627	(0.2)	4.732	(2.0)
3..........	0.8413	1.0	5.918	4.732	(20)	5.936	(0.3)	5.909	(0.2)	6.000	(1.4)
4..........	0.8413	1.0	7.163	6.000	(16)	7.179	(0.2)	7.154	(0.1)	7.236	(1.0)
10..........	0.8413	1.0	14.27	13.16	(7.7)	14.28	(0.08)	14.26	(0.04)	14.32	(0.3)
100..........	0.8413	1.0	111.0	110.0	(0.9)	111.0	(<)[b]	111.0	(<)	111.0	(0.01)
0..........	0.9995	3.291	7.601	0.000	(−)	6.308	(17)	7.832	(3.0)	7.568	(0.4)
1..........	0.9995	3.291	9.999	4.291	(57)	8.811	(12)	10.18	(1.8)	9.931	(0.7)
2..........	0.9995	3.291	12.05	6.654	(45)	10.92	(9.4)	12.20	(1.3)	11.98	(0.6)
3..........	0.9995	3.291	13.93	8.700	(38)	12.83	(7.9)	14.07	(1.0)	13.86	(0.5)
4..........	0.9995	3.291	15.71	10.58	(33)	14.63	(6.9)	15.83	(0.8)	15.64	(0.5)
10..........	0.9995	3.291	25.26	20.41	(19)	24.25	(4.0)	25.34	(0.3)	25.19	(0.3)
100..........	0.9995	3.291	137.4	132.9	(3.3)	136.5	(0.6)	137.4	(0.02)	137.4	(0.02)

[a] % error = $100 \times |\text{value} - \lambda_u|/\lambda_u$.
[b] Symbol " < " means less than 0.01%.

Expanding equation (12) and keeping only terms of order n^α, where $\alpha \geq 0$, results in

$$\lambda_l \approx n - S\sqrt{n} + \frac{S^2 - 1}{3} . \quad (13)$$

As will be shown below, none of the above three equations are adequate approximations for all confidence levels discussed in this paper. I have therefore devised a slightly more complex expression based on equation (12) that is quite accurate for all n and for all confidence levels in Table 3. The equation is

$$\lambda_l \approx n\left(1 - \frac{1}{9n} - \frac{S}{3\sqrt{n}} + \beta n^\gamma\right)^3 , \quad (14)$$

where β and γ are free parameters that are adjusted for each confidence level. The best values are listed in Table 3 and shown graphically in Figure 1.

In Table 5 the accuracy of equations (11), (12), (13), and (14) are compared. Equations (11) and (12) are possibilities for

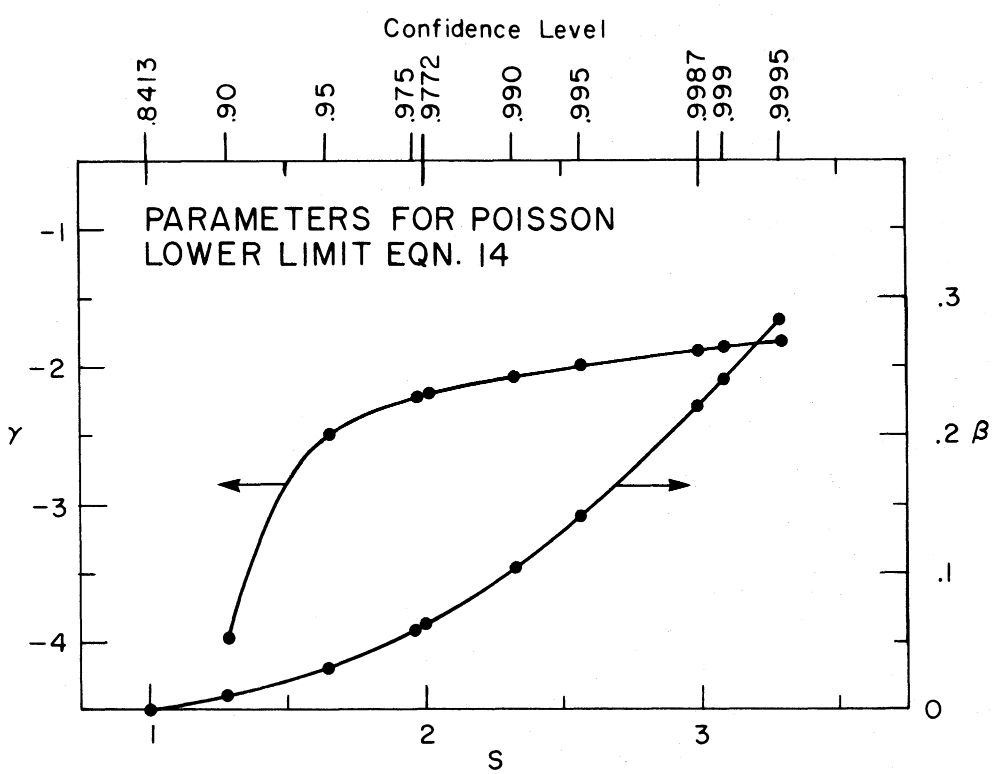

FIG. 1.—Parameters β and γ from Poisson lower limit approximation, eq. (14). Dots represent values in Table 3.

TABLE 5

COMPARISON OF POISSON LOWER LIMIT APPROXIMATE EXPRESSIONS

				GAUSSIAN LIMIT		EQUATION (11)		EQUATION (12)		EQUATION (13)		EQUATION (14)	
n	CL	S	λ_l	Value	(% Error[a])	Value	(% Error)	Value	(% Error)	Value	(% error)	Value	(% Error)
1..........	0.8413	1.0	0.173	0.000	(−)	0.134	(22)	0.171	(0.7)	0.000	(−)	0.171	(0.7)
2..........	0.8413	1.0	0.708	0.586	(17)	0.677	(4.4)	0.712	(0.5)	0.586	(17)	0.712	(0.5)
3..........	0.8413	1.0	1.367	1.268	(7.3)	1.342	(1.9)	1.372	(0.4)	1.268	(7.3)	1.372	(0.4)
4..........	0.8413	1.0	2.086	2.000	(4.1)	2.064	(1.1)	2.091	(0.3)	2.000	(4.1)	2.091	(0.3)
10..........	0.8413	1.0	6.891	6.838	(0.8)	6.878	(0.2)	6.896	(0.07)	6.838	(0.8)	6.896	(0.07)
100..........	0.8413	1.0	90.02	90.00	(0.02)	90.01	(<)[b]	90.02	(<)	90.00	(0.02)	90.02	(<)
1..........	0.9995	3.291	5.00−4	−2.29	(−)	0.608	(−)	−0.009	(−)	0.986	(−)	4.91−4	(1.8)
2..........	0.9995	3.291	3.20−2	−2.65	(−)	0.104	(−)	0.010	(70)	0.623	(−)	3.17−2	(0.9)
3..........	0.9995	3.291	0.150	−2.70	(−)	1.64−4	(−)	0.107	(28)	0.577	(−)	0.151	(1.0)
4..........	0.9995	3.291	0.355	−2.58	(−)	0.085	(76)	0.304	(14)	0.695	(−)	0.358	(0.8)
10..........	0.9995	3.291	2.699	−0.41	(−)	2.182	(19)	2.646	(2.0)	2.870	(6.3)	2.703	(0.1)
100..........	0.9995	3.291	70.33	67.09	(4.6)	69.59	(1.1)	70.30	(0.01)	70.37	(0.05)	70.32	(0.01)

[a] % error = $100 \times |\text{value} - \lambda_l|/\lambda_l$.
[b] Symbol " < " means less than 0.01%.

CL = 0.8413 but fail for higher confidence levels. Equation (13) is poor for all cases. Equations (14) is the most complex equation but is accurate to better than 2% for all values of n and all confidence levels considered in this paper. For $n \geq 10$ it is accurate to better than $\sim 0.1\%$.

Recommended approximation: For all cases use equation (14), $\lambda_l \approx n(1 - 1/9n - S/3(n)^{1/2} + \beta n^{\gamma})^3$, where β and γ are given in Table 3 ($\beta = 0$ for the special case of $S = 1$). The results are accurate to better than 2%. For the general S, the Gaussian statistics limit $\lambda_l \approx n - S(n)^{1/2}$ becomes accurate at the few percent level only for $n > 100$, although for $S = 1$ it is accurate to better than 1% for $n \geq 8$.

III. CONFIDENCE LIMITS FOR BINOMIAL STATISTICS

a) Relation to Abundance Measurements

We consider next the case where an observer is measuring two different kinds of distinguishable events. It is assumed that the source of each event type is random, so that the number of events of each type detected in a given observation time is distributed according to Poisson statistics. The objective is to obtain confidence limits on the ratio of the two event rates based on the measurement of a small number of events. Examples of this kind of study are elemental and isotopic abundance ratios in cosmic-ray astrophysics, spectral-line ratios in X-ray and γ-ray astronomy, and supernova type ratios in stellar astronomy.

The joint probability function for the two event types can be related to a combination of Poisson and binomial distributions. If n_1 and n_2 are the number of observed events of types 1 and 2, distributed according to Poisson statistics with true rates of λ_1 and λ_2, then the joint probability $f(n_1, n_2)$ of observing n_1 and n_2 is

$$f(n_1, n_2) = \frac{\lambda_1^{n_1} e^{-\lambda_1}}{n_1!} \frac{\lambda_2^{n_2} e^{-\lambda_2}}{n_2!} = \frac{(\lambda_1 + \lambda_2)^{n_1+n_2} e^{-(\lambda_1+\lambda_2)}}{(n_1 + n_2)!} \binom{n_1 + n_2}{n_1} p_1^{n_1}(1 - p_1)^{n_2} , \quad (15)$$

where $p_1 = \lambda_1/(\lambda_1 + \lambda_2)$ and $\binom{n_1+n_2}{n_1}$ is the binomial coefficient $(n_1 + n_2)!/n_1!\, n_2!$. Thus, the joint probability is equal to the Poisson probability for the combined rate times the binomial probability for obtaining specifically n_1 and n_2 given that the combined number of events observed was $n_1 + n_2$. Another way of viewing this is as follows: a source randomly emitting two types of events, each with its own average rate, is equivalent to a source randomly emitting events of unknown type at an average rate equal to the sum of the individual rates, with a weighted coin flipped for each event to determine its type.

The goal in the following sections will be to find confidence limits for p_1 in equation (15) or equivalently for the ratio of event rates, $r = \lambda_1/\lambda_2 = p_1/(1 - p_1)$. Binomial statistics will be used since, as shown above, they determine the event ratios.

b) Definitions and Numerical Solutions

For a binomial distribution, the upper limit, p_{1u}, and lower limit, p_{1l}, for the ratio of type 1 events to total events (type 1 plus type 2) are defined by

$$\sum_{x=0}^{n_1} \binom{n}{x} p_{1u}^x (1 - p_{1u})^{n-x} = 1 - \text{CL} \qquad (n_1 \neq n) , \quad (16)$$

$$\sum_{x=0}^{n_1-1} \binom{n}{x} p_{1l}^x (1 - p_{1l})^{n-x} = \text{CL} \qquad (n_1 \neq 0) \quad (17)$$

(see, e.g., Pearson and Hartley 1966; Beyer 1966), where CL is the single-sided confidence level of the limits and $n = n_1 + n_2$. For $n_1 = n$ $(n_2 = 0)$ $p_{1u} = 1.0$ and for $n_1 = 0$ $p_{1l} = 0.0$.

Equations (16) and (17) were derived for measurements with fixed n, which is not the case for ratios computed from two random rates. For two random rates that are both small, these equations give somewhat conservative limits. As an example, for all cases with both true rates less than 5.0, the fraction of observers with upper limits greater than the true ratio is always more than 0.98 for CL = 0.95. The fraction does approach 0.95 for large rates, and the limits defined here do satisfy the definition of confidence limits (see Appendix) for all rates and confidence levels. As far as I know, these limits are the best one can do for abundance measurements.

As was the case for the Poisson limits, exact general algebraic expressions for p_{1u} and p_{1l} can not be obtained from equations (16) and (17). However, as will be discussed below, there are exact solutions for some special cases, and general approximate algebraic expressions can be derived. Also, for

TABLE 6

BINOMIAL SINGLE-SIDED UPPER LIMITS[a]

		CONFIDENCE LEVEL									
n_1	n_2	0.8413	0.90	0.95	0.975	0.9772	0.99	0.995	0.9987	0.999	0.9995
0..........	1	0.841	0.9000	0.9500	0.9750	0.9772	$0.9^{2}00$[b]	$0.9^{2}50$	$0.9^{2}87$	$0.9^{3}00$	$0.9^{3}50$
	2	0.602	0.684	0.776	0.842	0.849	0.9000	0.9293	0.9633	0.9684	0.9776
	3	0.459	0.536	0.632	0.708	0.717	0.785	0.829	0.889	0.9000	0.9206
	4	0.369	0.438	0.527	0.602	0.612	0.684	0.734	0.808	0.822	0.850
	5	0.308	0.369	0.451	0.522	0.531	0.602	0.653	0.733	0.749	0.781
	6	0.264	0.319	0.393	0.459	0.468	0.536	0.586	0.668	0.684	0.718
	7	0.231	0.280	0.348	0.410	0.418	0.482	0.531	0.611	0.627	0.662
	8	0.206	0.250	0.312	0.369	0.377	0.438	0.484	0.562	0.578	0.613
	9	0.185	0.226	0.283	0.336	0.343	0.401	0.445	0.520	0.536	0.570
	10	0.168	0.206	0.259	0.308	0.315	0.369	0.411	0.484	0.499	0.532
	100	0.0182	0.0228	0.0295	0.0362	0.0371	0.0450	0.0516	0.0639	0.0667	0.0732
1..........	1	0.9172	0.9487	0.9747	0.9874	0.9886	$0.9^{2}50$	$0.9^{2}75$	$0.9^{3}32$	$0.9^{3}50$	$0.9^{3}75$
	2	0.748	0.804	0.865	0.9057	0.9102	0.9411	0.9586	0.9786	0.9816	0.9870
	3	0.618	0.680	0.751	0.806	0.812	0.859	0.889	0.9291	0.9360	0.9493
	4	0.524	0.584	0.657	0.716	0.724	0.778	0.815	0.868	0.878	0.898
	5	0.454	0.510	0.582	0.641	0.649	0.706	0.746	0.807	0.819	0.843
	6	0.400	0.453	0.521	0.579	0.586	0.643	0.685	0.750	0.763	0.789
	7	0.357	0.406	0.471	0.527	0.534	0.590	0.632	0.698	0.711	0.740
	8	0.323	0.368	0.429	0.482	0.489	0.544	0.585	0.652	0.665	0.694
	9	0.294	0.337	0.394	0.445	0.452	0.504	0.544	0.610	0.624	0.653
	10	0.270	0.310	0.364	0.413	0.419	0.470	0.509	0.573	0.587	0.616
	100	0.0323	0.0380	0.0461	0.0539	0.0550	0.0639	0.0713	0.0848	0.0878	0.0947
2..........	1	0.9440	0.9655	0.9830	$0.9^{2}16$	$0.9^{2}24$	$0.9^{2}67$	$0.9^{2}83$	$0.9^{3}55$	$0.9^{3}67$	$0.9^{3}83$
	2	0.815	0.857	0.9024	0.9324	0.9357	0.9580	0.9706	0.9848	0.9870	$0.9^{2}08$
	3	0.703	0.753	0.811	0.853	0.858	0.894	0.9172	0.9473	0.9524	0.9625
	4	0.615	0.667	0.729	0.777	0.783	0.827	0.856	0.898	0.9060	0.9215
	5	0.546	0.596	0.659	0.710	0.716	0.764	0.797	0.847	0.856	0.876
	6	0.490	0.538	0.600	0.651	0.657	0.707	0.742	0.797	0.807	0.830
	7	0.444	0.490	0.550	0.600	0.606	0.656	0.693	0.750	0.761	0.785
	8	0.405	0.450	0.507	0.556	0.562	0.612	0.648	0.707	0.718	0.744
	9	0.373	0.415	0.470	0.518	0.524	0.572	0.608	0.667	0.679	0.705
	10	0.346	0.386	0.438	0.484	0.490	0.537	0.573	0.632	0.644	0.670
	100	0.0449	0.0513	0.0604	0.0690	0.0702	0.0799	0.0877	0.102	0.105	0.112
3..........	1	0.9577	0.9740	0.9873	$0.9^{2}37$	$0.9^{2}43$	$0.9^{2}75$	$0.9^{2}87$	$0.9^{3}66$	$0.9^{3}75$	$0.9^{3}87$
	2	0.853	0.888	0.9236	0.9473	0.9498	0.9673	0.9771	0.9882	0.9899	$0.9^{2}29$
	3	0.757	0.799	0.847	0.882	0.886	0.9153	0.9337	0.9580	0.9621	0.9701
	4	0.676	0.721	0.775	0.816	0.821	0.858	0.882	0.9170	0.9233	0.9361
	5	0.610	0.655	0.711	0.755	0.761	0.802	0.830	0.872	0.880	0.897
	6	0.555	0.599	0.655	0.701	0.706	0.750	0.781	0.828	0.837	0.856
	7	0.508	0.552	0.607	0.652	0.658	0.703	0.735	0.785	0.795	0.816
	8	0.469	0.511	0.564	0.610	0.615	0.660	0.693	0.745	0.756	0.778
	9	0.435	0.475	0.527	0.572	0.577	0.622	0.655	0.708	0.719	0.742
	10	0.405	0.444	0.495	0.538	0.544	0.588	0.621	0.674	0.685	0.709
	100	0.0566	0.0637	0.0736	0.0828	0.0840	0.0942	0.103	0.117	0.121	0.128
4..........	1	0.9660	0.9791	0.9898	$0.9^{2}49$	$0.9^{2}54$	$0.9^{2}80$	$0.9^{3}00$	$0.9^{3}73$	$0.9^{3}80$	$0.9^{4}00$
	2	0.879	0.9074	0.9372	0.9567	0.9588	0.9732	0.9813	$0.9^{2}04$	$0.9^{2}17$	$0.9^{2}42$
	3	0.794	0.830	0.871	0.9010	0.9044	0.9292	0.9447	0.9650	0.9684	0.9751
	4	0.720	0.760	0.807	0.843	0.847	0.879	0.9001	0.9298	0.9352	0.9460
	5	0.658	0.699	0.749	0.788	0.793	0.829	0.854	0.891	0.897	0.9117
	6	0.605	0.646	0.696	0.738	0.743	0.782	0.809	0.851	0.859	0.875
	7	0.559	0.599	0.650	0.692	0.697	0.738	0.767	0.812	0.821	0.839
	8	0.519	0.559	0.609	0.651	0.656	0.698	0.728	0.775	0.784	0.804
	9	0.485	0.523	0.573	0.614	0.619	0.661	0.691	0.740	0.750	0.771
	10	0.455	0.492	0.540	0.581	0.586	0.627	0.658	0.707	0.717	0.739
	100	0.0678	0.0754	0.0859	0.0956	0.0968	0.108	0.116	0.132	0.135	0.143
5..........	1	0.9716	0.9826	$0.9^{2}15$	$0.9^{2}58$	$0.9^{2}62$	$0.9^{2}83$	$0.9^{3}16$	$0.9^{3}77$	$0.9^{3}83$	$0.9^{4}17$
	2	0.896	0.9212	0.9466	0.9633	0.9651	0.9773	0.9842	$0.9^{2}19$	$0.9^{2}30$	$0.9^{2}51$
	3	0.821	0.853	0.889	0.9148	0.9177	0.9392	0.9525	0.9700	0.9730	0.9787
	4	0.754	0.790	0.831	0.863	0.867	0.895	0.9132	0.9391	0.9438	0.9532
	5	0.695	0.733	0.778	0.813	0.817	0.850	0.872	0.9041	0.9102	0.9227
	6	0.644	0.682	0.729	0.766	0.771	0.806	0.831	0.868	0.875	0.890
	7	0.600	0.638	0.685	0.723	0.728	0.765	0.791	0.832	0.840	0.857
	8	0.561	0.598	0.645	0.684	0.689	0.727	0.755	0.798	0.806	0.824
	9	0.526	0.563	0.610	0.649	0.653	0.692	0.720	0.765	0.774	0.793
	10	0.496	0.532	0.577	0.616	0.621	0.660	0.688	0.734	0.743	0.763
	100	0.0785	0.0865	0.0975	0.108	0.109	0.120	0.129	0.145	0.148	0.156

TABLE 6—*Continued*

		Confidence Level									
n_1	n_2	0.8413	0.90	0.95	0.975	0.9772	0.99	0.995	0.9987	0.999	0.9995
6..........	1	0.9756	0.9851	$0.9^{2}27$	$0.9^{2}64$	$0.9^{2}67$	$0.9^{2}86$	$0.9^{3}28$	$0.9^{3}81$	$0.9^{3}86$	$0.9^{4}29$
	2	0.9097	0.9314	0.9536	0.9681	0.9697	0.9803	0.9863	$0.9^{2}30$	$0.9^{2}40$	$0.9^{2}57$
	3	0.842	0.871	0.9023	0.9251	0.9278	0.9467	0.9584	0.9737	0.9763	0.9814
	4	0.780	0.812	0.850	0.878	0.882	0.9068	0.9232	0.9462	0.9504	0.9587
	5	0.725	0.759	0.800	0.833	0.836	0.866	0.886	0.9146	0.9201	0.9312
	10	0.531	0.565	0.609	0.646	0.650	0.687	0.713	0.756	0.764	0.783
	100	0.0889	0.0972	0.109	0.119	0.121	0.132	0.141	0.157	0.161	0.169
7..........	1	0.9786	0.9869	$0.9^{2}36$	$0.9^{2}68$	$0.9^{2}71$	$0.9^{2}87$	$0.9^{3}37$	$0.9^{3}83$	$0.9^{3}87$	$0.9^{4}37$
	2	0.9199	0.9392	0.9590	0.9719	0.9732	0.9826	0.9879	$0.9^{2}38$	$0.9^{2}47$	$0.9^{2}62$
	3	0.858	0.884	0.9127	0.9333	0.9356	0.9525	0.9630	0.9767	0.9790	0.9834
	4	0.801	0.831	0.865	0.891	0.894	0.9163	0.9312	0.9518	0.9556	0.9630
	5	0.750	0.781	0.819	0.848	0.852	0.879	0.897	0.9230	0.9279	0.9380
	10	0.562	0.594	0.636	0.671	0.675	0.709	0.734	0.774	0.782	0.800
	100	0.0989	0.108	0.119	0.130	0.132	0.143	0.153	0.169	0.173	0.181
8..........	1	0.9810	0.9884	$0.9^{2}43$	$0.9^{2}72$	$0.9^{2}74$	$0.9^{2}89$	$0.9^{3}44$	$0.9^{3}85$	$0.9^{3}89$	$0.9^{4}44$
	2	0.9280	0.9455	0.9632	0.9748	0.9760	0.9845	0.9891	$0.9^{2}44$	$0.9^{2}52$	$0.9^{2}66$
	3	0.872	0.895	0.9212	0.9398	0.9419	0.9572	0.9667	0.9790	0.9811	0.9851
	4	0.819	0.846	0.877	0.9008	0.9035	0.9241	0.9376	0.9564	0.9598	0.9665
	5	0.770	0.799	0.834	0.861	0.865	0.889	0.9058	0.9299	0.9344	0.9436
	10	0.588	0.620	0.659	0.692	0.697	0.729	0.753	0.790	0.798	0.814
	100	0.109	0.118	0.130	0.141	0.142	0.154	0.164	0.181	0.184	0.193
9..........	1	0.9829	0.9895	$0.9^{2}49$	$0.9^{2}75$	$0.9^{2}77$	$0.9^{3}00$	$0.9^{3}50$	$0.9^{3}86$	$0.9^{4}00$	$0.9^{4}50$
	2	0.9347	0.9505	0.9667	0.9772	0.9783	0.9859	$0.9^{2}02$	$0.9^{2}50$	$0.9^{2}57$	$0.9^{2}70$
	3	0.883	0.9043	0.9281	0.9451	0.9471	0.9610	0.9697	0.9809	0.9828	0.9864
	4	0.833	0.858	0.887	0.9091	0.9116	0.9305	0.9429	0.9601	0.9632	0.9694
	5	0.787	0.815	0.847	0.872	0.875	0.898	0.9134	0.9356	0.9398	0.9483
	10	0.612	0.642	0.680	0.711	0.715	0.746	0.768	0.804	0.811	0.826
	100	0.118	0.127	0.140	0.151	0.152	0.165	0.174	0.191	0.195	0.204
10..........	1	0.9844	$0.9^{2}05$	$0.9^{2}53$	$0.9^{2}77$	$0.9^{2}79$	$0.9^{3}09$	$0.9^{3}54$	$0.9^{3}88$	$0.9^{4}09$	$0.9^{4}55$
	2	0.9402	0.9548	0.9695	0.9791	0.9802	0.9872	$0.9^{2}10$	$0.9^{2}54$	$0.9^{2}61$	$0.9^{2}72$
	3	0.892	0.9120	0.9340	0.9496	0.9514	0.9642	0.9722	0.9825	0.9842	0.9876
	4	0.846	0.869	0.896	0.9161	0.9185	0.9360	0.9474	0.9633	0.9662	0.9719
	5	0.802	0.828	0.858	0.882	0.885	0.9056	0.9199	0.9405	0.9444	0.9522
	6	0.762	0.790	0.822	0.848	0.851	0.875	0.891	0.9159	0.9206	0.9302
	7	0.726	0.754	0.788	0.816	0.819	0.845	0.863	0.891	0.896	0.9071
	8	0.692	0.721	0.756	0.785	0.788	0.816	0.835	0.865	0.871	0.884
	9	0.661	0.690	0.726	0.756	0.759	0.788	0.808	0.840	0.847	0.860
	10	0.633	0.662	0.698	0.728	0.732	0.761	0.782	0.816	0.823	0.837
	100	0.127	0.137	0.149	0.161	0.162	0.175	0.185	0.202	0.206	0.214
100..........	1	$0.9^{2}83$	$0.9^{2}90$	$0.9^{3}49$	$0.9^{3}75$	$0.9^{3}77$	$0.9^{4}00$	$0.9^{4}50$	$0.9^{4}87$	$0.9^{5}01$	$0.9^{5}50$
	2	$0.9^{2}30$	$0.9^{2}48$	$0.9^{2}65$	$0.9^{2}76$	$0.9^{2}77$	$0.9^{2}85$	$0.9^{2}90$	$0.9^{3}48$	$0.9^{3}55$	$0.9^{3}69$
	3	0.9867	0.9893	$0.9^{2}20$	$0.9^{2}40$	$0.9^{2}42$	$0.9^{2}57$	$0.9^{2}67$	$0.9^{2}79$	$0.9^{2}81$	$0.9^{2}85$
	4	0.9799	0.9831	0.9868	0.9894	0.9897	$0.9^{2}20$	$0.9^{2}35$	$0.9^{2}55$	$0.9^{2}58$	$0.9^{2}65$
	5	0.9728	0.9767	0.9811	0.9844	0.9847	0.9877	0.9896	$0.9^{2}23$	$0.9^{2}28$	$0.9^{2}39$
	6	0.9656	0.9700	0.9751	0.9789	0.9794	0.9829	0.9853	0.9887	0.9894	$0.9^{2}07$
	7	0.9584	0.9632	0.9689	0.9733	0.9738	0.9778	0.9806	0.9847	0.9855	0.9871
	8	0.9512	0.9564	0.9626	0.9675	0.9681	0.9726	0.9757	0.9804	0.9813	0.9832
	9	0.9439	0.9496	0.9562	0.9615	0.9622	0.9671	0.9706	0.9759	0.9769	0.9791
	10	0.9367	0.9427	0.9498	0.9555	0.9562	0.9616	0.9654	0.9712	0.9723	0.9747
	100	0.538	0.548	0.560	0.571	0.573	0.584	0.593	0.607	0.610	0.617

[a] Upper limit for ratio type 1 event to type 1 + type 2. Calculate lower limit from this table with lower limit $(n_1, n_2) = 1 -$ upper limit (n_2, n_1).

[b] $0.9^{2}00$ means 0.9900; and in general for 9^n.

any specific values of n_1, n_2, and CL, exact numerical solutions can be computed to arbitrary accuracy.

Table 6 lists exact numerically determined values for p_{1u} at the previously chosen confidence levels for a number of values of n_1 and n_2. A separate table is not required for the lower limits because they can be simply obtained from the upper limits. The relationship between the two, derived from equations (16) and (17), is

$$p_{1l} = 1 - p_{2u} , \qquad (18)$$

where p_{2u} is the upper limit for the ratio of type 2 events to total events and can be obtained from Table 6 by switching n_1 and n_2. For example, if the observed number of events are $n_1 = 6$ and $n_2 = 4$, the best guess for the ratio of type 1 to total is 0.6, and the 99% confidence lower limit is 1 minus the upper limit from the table for $n_1 = 4$, $n_2 = 6$, or $1 - 0.782 = 0.218$. The 99% confidence upper limit is 0.907 directly from Table 6, and the 99% confidence double-sided interval (CL = 0.995 column) is 0.191–0.923.

In many cases, the quantity of interest is the ratio of type 1 to type 2 event, r_{12}, instead of type 1 to total. The upper and lower limits for r_{12} are related to p_{1u} and p_{1l} by

$$r_{12\,u} = \frac{p_{1u}}{1 - p_{1u}} , \quad (19)$$

$$r_{12\,l} = \frac{p_{1l}}{1 - p_{1l}} = \frac{1 - p_{2u}}{p_{2u}} . \quad (20)$$

Using the above example of $n_1 = 6$ and $n_2 = 4$, the best guess for r_{12} is 1.5, and the 99% confidence upper and lower limits (single-sided) are 9.73 and 0.279.

c) *Algebraic Expressions*

There are two special cases where exact expressions for p_{1u} can be derived from equation (16). These are

$$p_{1u} = \begin{cases} \mathrm{CL}^{1/(n_1+n_2)} & (n_2 = 1) \\ 1 - (1 - \mathrm{CL})^{1/n_2} & (n_1 = 0) \end{cases} . \quad (21)$$

Lower limits for $n_1 = 1$ and $n_2 = 0$ can be obtained from these expressions using equation (18). For instance, $p_{1l}(n_2 = 0) = (1 - \mathrm{CL})^{1/n_1}$.

For other values of n_1 and n_2, we make use of the relation between the binomial sum and the incomplete beta function, $I(p\,|\,\alpha, \beta) \equiv I_p(\alpha, \beta)$,

$$\sum_{x=\alpha}^{n} \binom{n}{x} p^x (1 - p)^{n-x} = I(p\,|\,\alpha, n - \alpha + 1) \quad (22)$$

(see, e.g., Abramowitz and Stegun 1965). Combining this relation with equations (16) and (17) gives

$$I(p_{1u}\,|\,n_1 + 1, n_2) = \mathrm{CL} , \quad (23)$$

$$I(p_{1l}\,|\,n_1, n_2 + 1) = 1 - \mathrm{CL} . \quad (24)$$

Thus, p_{1u} and p_{1l} can be obtained using tables of percentage points of the incomplete β-function (see, e.g., Beyer 1966) in the same manner that χ^2 tables are used to give Poisson limits. As with the Poisson limits, the technique is straightforward but cumbersome since several algebraic steps are involved and, in many cases, interpolations in the tables are required.

Equation (23) is useful for developing an approximate expression for p_{1u}. An approximate inverse function for the incomplete β-function is given by Abramowitz and Stegun (1965) as follows: if $I(x_a\,|\,\alpha, \beta) = 1 - a$ and y_a is defined such that $G(y_a) = a$ [see § II*b*(i)], then

$$x_a \approx \frac{\alpha}{\alpha + \beta e^{2w}} , \quad (25)$$

where

$$w = \frac{y_a(h + \lambda)^{1/2}}{h} - \left(\frac{1}{2\beta - 1} - \frac{1}{2\alpha - 1}\right)\left(\lambda + \frac{5}{6} - \frac{2}{3h}\right) ,$$

$$h = 2\left(\frac{1}{2\beta - 1} + \frac{1}{2\alpha - 1}\right)^{-1} ,$$

and

$$\lambda = \frac{y_a^2 - 3}{6} .$$

Combining equations (23) and (25), adding a correction term, and using the variable S defined in § II*b*(i) (see Table 3), we derive an approximate expression for p_{1u},

$$p_{1u} \approx \frac{(n_1 + 1)e^{2w} + \epsilon n_2}{(n_1 + 1)e^{2w} + n_2} , \quad (26)$$

where

$$w = \frac{S(h + \lambda)^{1/2}}{h} + \left(\frac{1}{2n_2 - 1} - \frac{1}{2n_1 + 1}\right)\left(\lambda + \frac{5}{6} - \frac{2}{3h}\right) ,$$

$$h = 2\left(\frac{1}{2n_2 - 1} + \frac{1}{2n_1 + 1}\right)^{-1} ,$$

$$\lambda = \frac{S^2 - 3}{6} ,$$

and

$$\epsilon = 0.64(1 - S)e^{-n_2} .$$

I was not able to find a simpler expression than equation (26) that is generally useful for all confidence levels and all values of n_1 and n_2. For cases where equations (21) do not apply (i.e., for $n_2 \geq 2$ and $n_1 \geq 1$), equation (26) gives upper limits accurate to better than 4% for all confidence levels. If $n_1 \geq 4$ and $n_2 \geq 4$, equation (26) is accurate to better than 1%. These two percentage accuracies also hold for $1 - p_{1u}$, which is the quantity used to calculate lower limits. In the Gaussian limit of large n_1 and n_2, the upper limit is given by

$$p_{1u} = \frac{n_1}{n_1 + n_2} + \frac{S\sqrt{n_1 n_2}}{(n_1 + n_2)^{3/2}} . \quad (27)$$

This expression becomes accurate at the 1% level only for n_1 and n_2 both greater than ~ 50.

Recommended expressions: For $n_1 = 0$ or $n_2 = 1$, use the exact equations (21) for the binomial upper limits. For other values of n_1 and n_2 use equation (26), which gives 4% accuracy. Calculate the lower limits from these upper limit expressions using $p_{1l} = 1 - p_{2u}$, where p_{2u} is the upper limit with n_1 and n_2 switched. This relationship combined with equations (21) gives exact lower limits for $n_1 = 1$ or $n_2 = 0$ and combined with equation (26) gives lower limits accurate to within 4% for all other values.

IV. SUMMARY

Poisson statistics apply when event rates are calculated from small numbers of observed events. For n observed events, the single-sided upper limit, λ_u, and single-sided lower limit, λ_l, of confidence level CL are given by equations (1) and (2). Previous techniques for obtaining actual limit values from these equations were cumbersome. This paper has presented tables of upper and lower limits (Tables 1 and 2) for $n = 0$–50 for all common confidence levels used in astrophysics. Also, convenient approximate expressions for λ_u and λ_l were developed, the most useful of which are $\lambda_u \approx n + S(n + 1)^{1/2} + (S^2 + 2)/3$ (good to better than 10% for all n and CL and better than to 1% for $n \geq 4$) and $\lambda_l \approx n[1 - 1/9n - S/3(n)^{1/2} + \beta n^{\gamma}]^3$ (good to better than 2% for all n and CL), where S, β, and γ are given in Table 3.

Binomial statistics apply when ratios of rates of two different event types are calculated from small numbers of observed events. For n_1 observed events of type 1 and n_2 of type 2, the single-sided upper limit, p_{1u}, and single-sided lower limit, p_{1l}, of confidence level CL to the ratio of type 1 event rate to type 1 plus type 2 are given by equations (16) and (17). Again, in this case, previous techniques for obtaining values from these equations were cumbersome. Considering upper limits first, Table 6 lists values for p_{1u} for a number of small n_1 and n_2 for all common confidence levels used in astrophysics. There are two special cases with exact expressions for the upper limits: $p_{1u} =$

CL^{1/n_1+n_2} ($n_2 = 1$) and $p_{1u} = 1 - (1 - CL)^{1/n_2}$ ($n_1 = 0$). The approximate expression in equation (26) gives values for p_{1u} accurate to better than 4% for all other n_1 and n_2, and to better than 1% for $n_1 \geq 4$ and $n_2 \geq 4$. Lower limits can be obtained directly from the upper limits in the table or in the above equations by taking 1 minus the upper limit value, for n_1 and n_2 exchanged.

I greatly appreciate the comments and advice of Martin H. Israel. This paper is a development from an unpublished report by Dr. Israel, in which equations (7) and (11) were derived. Thanks are also due to William A. Wheaton for useful discussions concerning Poisson statistics. I am grateful for E. Schronce for typing the manuscript of this paper.

APPENDIX

NUMERICAL VERIFICATION OF POISSON LIMIT EQUATIONS

The purpose of this appendix is to demonstrate that the Poisson upper and lower limits defined by equations (1) and (2) in § II satisfy the definition of confidence limits and do so in an optimum manner. To simplify the discussion we will first consider only upper limits and then generalize the results to include lower limits.

For a quantity with a continuous distribution function such as a Gaussian distribution, upper limits are defined to be confidence limits of confidence level CL if they satisfy the following condition; for a large number of observers measuring a given physical quantity (with measurements distributed about the true value according to the distribution function) and each observer assigning an upper limit based on his or her measured value, 100 × CL% of them will have upper limits greater than the true value (see, e.g., Cramér 1945). For a quantity with a discrete distribution function, such as a Poisson distribution, it is not possible to generate upper limits that satisfy this condition for all parameter values, so the definition becomes: *at least* 100 × CL% of the observers have upper limits greater than the true value. The lower limit condition is that at least 100 × CL% of the observers have lower limits less than the true value.

Calculations have been performed to verify that the Poisson upper and lower limits in equations (1) and (2) satisfy this definition of confidence limits. A finely spaced sample of assumed true event rates, λ_t, was chosen. For each λ_t, a distribution of observed values was then determined from the Poisson distribution function

$$P(n) = \frac{\lambda_t^n e^{-\lambda_t}}{n!} . \tag{A1}$$

For each observed value, upper and lower limits of confidence level 0.8413 (corresponding to 1 σ Gaussian limits) were obtained from Tables 1 and 2. A tally was then taken of all observed values to determine the fraction of observers with upper limits greater than the true rate and the fraction with lower limits less than the true rate. The results are plotted as functions of true rate in Figure 2. As expected for a discrete distribution, the fractions vary as functions of true rate, dipping to exactly 0.8413 each time the true rate

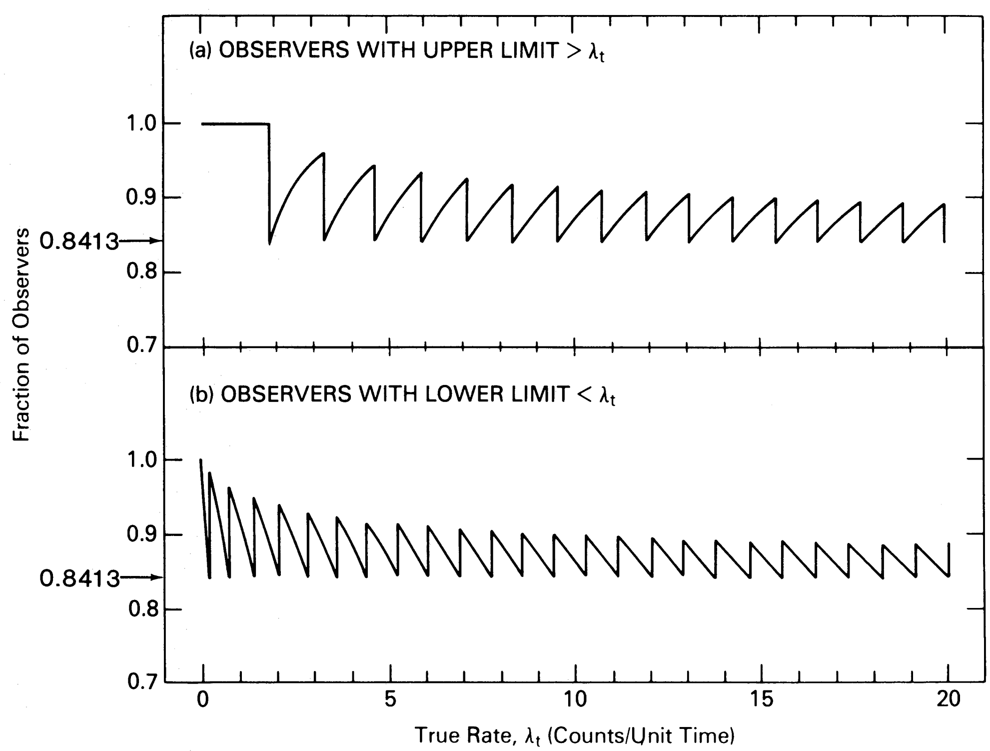

FIG. 2.—Fraction of observers with (*a*) upper limit greater than the true rate λ_t and (*b*) lower limit less than λ_t, as functions of λ_t. For all values of λ_t, the fraction is greater than or equal to the confidence level of 0.8413.

passes one of the limit values. The size of the variation decreases with increasing λ_t, approaching a constant value of 0.8413 in the limit. This is as expected since the discrete Poisson distribution approaches the continuous Gaussian distribution in the limit of large λ_t. The confidence limit definitions are satisfied for all rates; at least 84.13% of the observers have upper limits greater than the true rate and lower limits less than the true rate, for all λ_t. The limits are optimum in the sense that the fractions are exactly 0.8413 at some discrete values of the true rate. If any one of the upper limits in Table 1 were decreased or lower limits in Table 2 increased, the fraction would dip below 0.8413 for some true rate.

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