Calculations of the Output of a Diffused Planar Light Source

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This report provides analytic calculations of angular and position-space distributions of the output flux from a diffused planar light source, one example being a light source, such as a laser or LED, which is shone through a flat sheet of translucent plastic. Both simple small-angle scattering calculations, and more complex large-angle scattering formulae, are provided. Comparisons of these calculated analytic formulae with results from simple numerical Monte Carlo simulations are additionally provided.

Contents

1 Initial Assumptions and Examples

When performing analytical, and even numerical, calculations of photon scattering and absorption in a macroscopic medium, clearly a number of assumptions must be made in order to develop a model and perform such calculations. Three scattering regimes exist, in which the scattering body is respectively a) much smaller than, b) of a similar size to, and c) much larger than, the wavelength of the light being scattered; these are respectively known as a) Rayleigh scattering, b) Mie scattering, and c) Rayleigh-Gans-Debye scattering. We begin by briefly providing the results of scattering in each of the three regimes, noting that Rayleigh and Mie scattering (but not Rayleigh-Gans-Debye) are both applicable to the physical case at hand (a sheet of white plastic that both scatters and absorbs light passing through it).

In addition to typical physical approximations (such as the small angle approximation, and an assumption that scattering angles are Gaussian) we shall use at various points (but certainly not universally, we will carefully note each approximation we use in any given calculation) — we will extensively utilize two approximate models for the optical behaviour of a sheet of diffusive material, which will be denoted as the "succession-of-very-thin-layers" approximation and the "scattering centres" approximation. These two approximate models are described in sections 1.2 and 1.3, each with an example calculation utilizing the given model.

1.1 Scattering regimes: Rayleigh scattering, Mie scattering, and Rayleigh-Gans-Debye scattering

1.2 The "succession-of-very-thin-layers" (SOVTL) approximation

We first consider diffusive photon scattering in the SOVTL approximation as shown in Fig. 1.1 at the top of the following page.

Consider non-absorptive scattering in this approximation, where each of the scattering angles, as described in the figure caption as having RMS equal

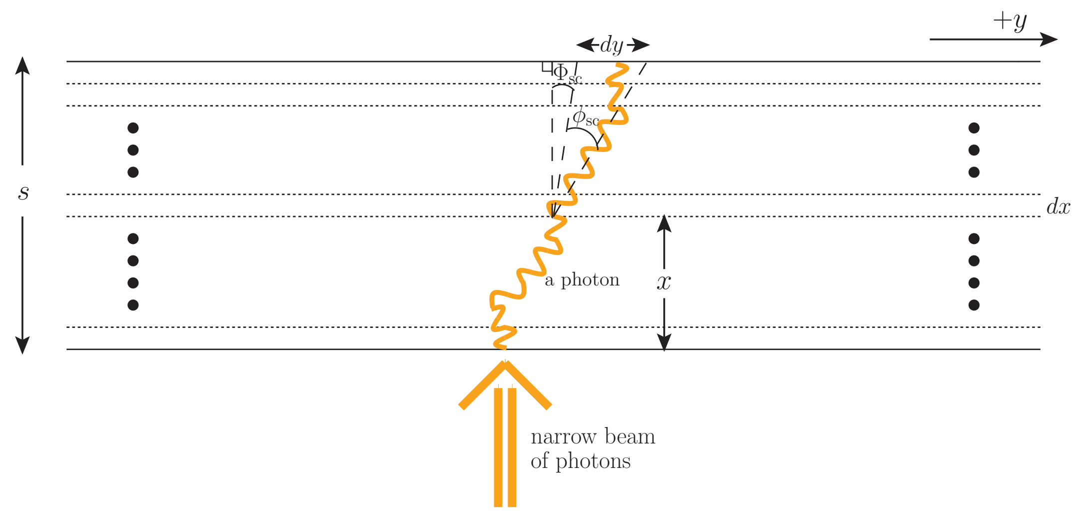

Figure 1.1: The "succession-of-very-thin-layers" (SOVTL) approximation to photon scattering in a diffusive medium. Photons in the beam can scatter at the location of each of the thin layers in the medium. We let the RMS angle of scattering for each very thin layer of thickness dx — with an example scatter being given by ϕ_{sc} above — be $\phi_{RMS}\sqrt{\frac{dx}{t_0}}$, where ϕ_{RMS} and t_0 are material-dependent parameters. The (signed) sum of all scattering angles prior to ϕ_{sc} is denoted as Φ_{sc}. The distance dy is the projected scattering distance on the upper surface of the medium due only to the single scatter ϕ_{sc}. If we are including absorption effects, we may also let the probability of non-absorption in a given layer be $e^{-dx/\tau}$, where τ is another material-dependent parameter.

to $\phi_{RMS}\sqrt{\frac{dx}{t_0}}$, is additionally small (*i.e.*: $\phi_{RMS}\sqrt{\frac{dx}{t_0}} \ll \pi/2$) and is Gaussian-distributed with mean 0. Let us calculate 1) the RMS size of the output beam, *i.e.* the distribution of output positions of photons exiting the upper surface of the scattering medium, and also 2) the distribution of output angles of these same exiting photons.

The projected scattering distance dy from a given scatter from a layer at position x must equal $(s-x)[\tan(\Phi_{sc}+\phi_{sc})-\tan(\Phi_{sc})]$, which is approximately equal to $(s-x)[\Phi_{sc}+\phi_{sc}-\Phi_{sc}] = (s-x)\phi_{sc}$, since we are assuming all ϕ_{sc}, and thus Φ_{sc}, are small. Thus, if we consider the distribution of possible scattering distances dy due to the scatter at x, the variance of that distribution $\mathrm{var}(dy)$ must equal $(s-x)^2\mathrm{var}(\phi_{sc})$, since we are treating s and x as constants here.

We know (per the SOVTL approximation) that $\text{var}(\phi_{\text{sc}}) = \phi^2_{\text{RMS}}\frac{dx}{t_0}$, and thus $\text{var}(dy) = (s-x)^2\text{var}(\phi_{\text{sc}}) = (s-x)^2\phi^2_{\text{RMS}}(\frac{dx}{t_0})$. We would like to calculate 1) $\text{var}(y)$, where $y = \sum\limits_{\text{layers}} dy = \int_{x=0}^{x=s} dy$; as the square root of $\text{var}(y)$ will be the RMS size of the output beam, and also 2) $\text{var}(\sum\limits_{\text{layers}} \phi_{\text{sc}})$. We have that $\text{var}(y)$ $= \text{var}[\sum dy] = \text{var}[\int dy]$. From the "Bienaymé formula" (see Appendix A for a proof), we have that $\text{var}[\sum\limits_i x_i] = \sum\limits_i \text{var}(x_i)$ for <u>any</u> set of uncorrelated variables x_i, thus $\text{var}(y) = \text{var}[\int dy] = \int \text{var}(dy) = \int\limits_0^s (s-x)^2\phi^2_{\text{RMS}}(\frac{dx}{t_0}) = \frac{\phi^2_{\text{RMS}}}{t_0}\int\limits_0^s (s-x)^2 dx = \frac{\phi^2_{\text{RMS}}}{t_0}\left[-\frac{1}{3}(s-x)^3\right]_0^s = \frac{\phi^2_{\text{RMS}}s^3}{3t_0}$. Thus, 1) the RMS size of the output beam $= \sqrt{\text{var}(y)}$ must be the square root of this, *i.e.* $\frac{\phi_{\text{RMS}}s^{3/2}}{\sqrt{3t_0}}$.

To find 2), we use Bienaymé again and thus note that $\text{var}(\sum\limits_{\text{layers}} \phi_{\text{sc}}) = \sum\limits_{\text{layers}} \text{var}(\phi_{\text{sc}}) = \int_{x=0}^{x=s} \text{var}(\phi_{\text{sc}})$. This is given by $\int_{x=0}^{x=s} \phi^2_{\text{RMS}}(\frac{dx}{t_0}) = \frac{\phi^2_{\text{RMS}}}{t_0}\int_{x=0}^{x=s} dx = \frac{\phi^2_{\text{RMS}}s}{t_0}$. Thus the standard deviation of the output angles is the square root of this, *i.e.* $\phi_{\text{RMS}}\sqrt{\frac{s}{t_0}}$.

Formulae for angular and spatial output distributions for more complex cases than the above in the SOVTL approximation will be calculated in Chapters 2 and 3.

1.3 The "scattering centres" (SC) approximation

We next consider diffusive photon scattering in the SC approximation as shown in Fig. 1.2 below:

Consider non-absorptive scattering as in the previous section, where each of the scattering angles is small (*i.e.*: $\phi_{\text{RMS}} \ll \pi/2$) and is Gaussian-distributed with mean 0. We calculate (as above in the SOVTL approximation) 1) the RMS size of the output beam, and 2) the distribution of output angles of the same exiting photons.

The projected scattering distance dy from a given scatter from a centre at position x is (as with SOVTL) approximately equal to $(s-x)\phi_{\text{sc}}$. Thus, if we consider the distribution of possible scattering distances dy due to the scatter at x, the variance of that distribution $\text{var}(dy)$ must equal $(s-x)^2\text{var}(\phi_{\text{sc}}) =$

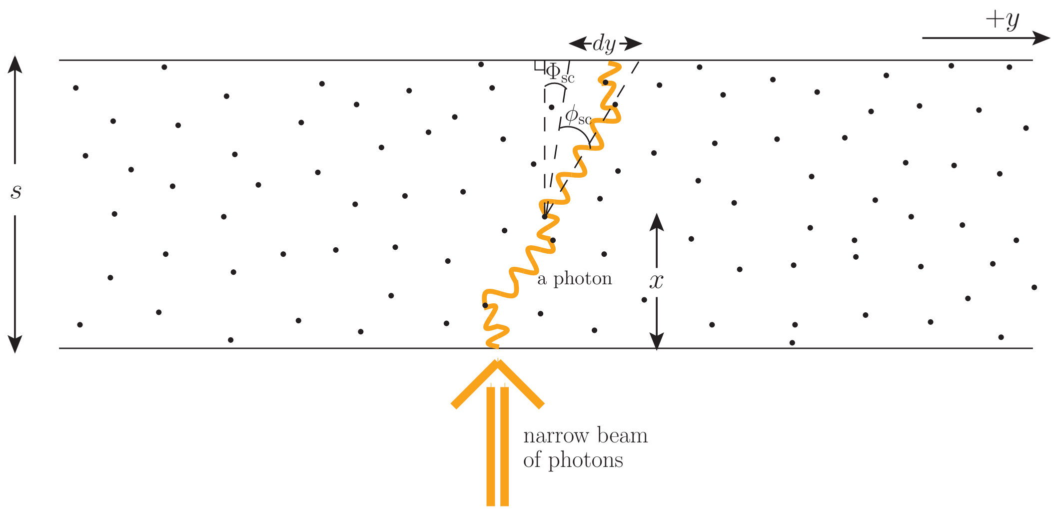

Figure 1.2: The scattering centres (SC) approximation to photon scattering. Photons in the beam can scatter at the random location of each of the scattering centres in the medium. We let the mean number of scattering centres per unit area be n (with $ns^2 \gg 1$). Let the RMS angle of scattering for each centre be ϕ_{RMS}, and let the (circular) scattering cross-section of each centre be t_0. We initially consider scattering to be impact parameter independent within the circular cross-section, and to have a Gaussian scattering angle distribution.

$(s-x)^2\phi_{\text{RMS}}^2$. Using Bienaymé here as well, we have that $\text{var}(y) = \text{var}[\sum dy]$ $= \sum \text{var}(dy) = \sum (s-x)^2\phi_{\text{RMS}}^2 = \int_0^s (s-x)^2\phi_{\text{RMS}}^2 \delta i dx$, where δi represents the number of centres hit (or the "chance of hitting a centre") per the short distance of travel dx. This quantity δi will equal $\frac{(\text{length of travel}) \times t_0 \times n}{dx}$. The length of travel will equal $\frac{dx}{\cos \Phi_{\text{sc}}}$, where Φ_{sc} is the sum of all scattering angles ϕ_{sc} for all scatters prior to the scatter under consideration. Thus we have that $\delta i = \frac{t_0 \times n}{\cos \sum \phi_{\text{sc}}}$. Using the small angle approximation, $\cos \sum \phi_{\text{sc}} \approx 1$, thus $\text{var}(y) \approx \int_0^s (s-x)^2 \phi_{\text{RMS}}^2 t_0 n dx = \phi_{\text{RMS}}^2 t_0 n \left[-\frac{1}{3}(s-x)^3\right]_0^s = \frac{\phi_{\text{RMS}}^2 t_0 n s^3}{3}$. Thus, 1) the RMS size of the output beam $= \phi_{\text{RMS}}\sqrt{\frac{t_0 n s^3}{3}}$.

To find 2), we use Bienaymé again and thus note that $\text{var}(\sum \phi_{\text{sc}}) = \sum \text{var}(\phi_{\text{sc}}) = \int_{x=0}^{x=s} \text{var}(\phi_{\text{sc}}) = \int_{x=0}^{x=s} \phi_{\text{RMS}}^2 \delta i dx = \phi_{\text{RMS}}^2 t_0 n \int_{x=0}^{x=s} dx = \phi_{\text{RMS}}^2 t_0 n s$. Thus the standard deviation of the output angles is the square root of this, *i.e.* $\phi_{\text{RMS}}\sqrt{t_0 n s}$.

1.4 Three spatial dimensions, and the observer's perspective

2 Calculations of Small-Angle Scattering Formulae

2.1 Deviation from normal incidence angle

2.2 Calculation using non-projected “δy” rather than dy

2.3 Adding absorption

2.4 Impact-parameter-dependent SC approximation scattering

2.5 Initial considerations for non-Gaussian scattering: the example of Rayleigh scattering

3 Calculations of Large-Angle Scattering Formulae

3.1 Non-Gaussian scattering redux

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4 A Toy Monte Carlo Simulation and Its Results

4.1 Description of the Toy MC

4.2 Results with Gaussian scattering

4.3 Results with Rayleigh scattering

5 Comparison of Analytic Calculation Formulae with Toy Monte Carlo Results

A Some of the Necessary Input Formulae

A.1 A brief proof of the Bienaymé formula

We would like to show that

$$\mathrm{var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} [\mathrm{var}(X_i)] \tag{A.1}$$

for n uncorrelated variables X_i, which is known as the *Bienaymé formula*.

We start by noting that the definition of the variance $\mathrm{var}(X)$ is $E[(X - E(X])^2]$, where $E[X]$ is the expectation value of the variable X. Now consider the variance of the sum of two uncorrelated variables X and Y, *i.e.* $\mathrm{var}(X + Y)$. We have that

$$\mathrm{var}(X+Y) = E[(X+Y-E[X+Y])^2] \tag{A.2}$$

by definition. Since X and Y are uncorrelated, we note that $E[X + Y] = E[X] + E[Y]$, and thus the above equals

$$E[(X+Y-E[X]-E[Y])^2] = E[(X-E[X]+Y-E[Y])^2] = \\ E[(X-E[X])^2] + E[(Y-E[Y])^2] + 2E[(X-E[X])(Y-E[Y])]. \tag{A.3}$$

We note that since X and Y are uncorrelated, $X - E[X]$ and $Y - E[Y]$ must be uncorrelated as well, and thus

$$E[(X-E[X])(Y-E[Y])] = E[(X-E[X])]E[(Y-E[Y]). \tag{A.4}$$

But

$$E[(X-E[X])] = E[X] - E[E[X]] = E[X] - E[X] = 0, \tag{A.5}$$

which is also true for $E[(Y - E[Y])$. Thus,

$$\mathrm{var}(X+Y) = E[(X-E[X])^2] + E[(Y-E[Y])^2] = \mathrm{var}(X) + \mathrm{var}(Y). \tag{A.6}$$

Since the above is true for any two uncorrelated variables, we can just use induction to show that it works for any number n of uncorrelated variables, and thus

$$\mathrm{var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \left[\mathrm{var}(X_i)\right]. \tag{A.7}$$

A.2 The moments and bivariate moments of Gaussian distributions

B A Brief List of the Important Results Found

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