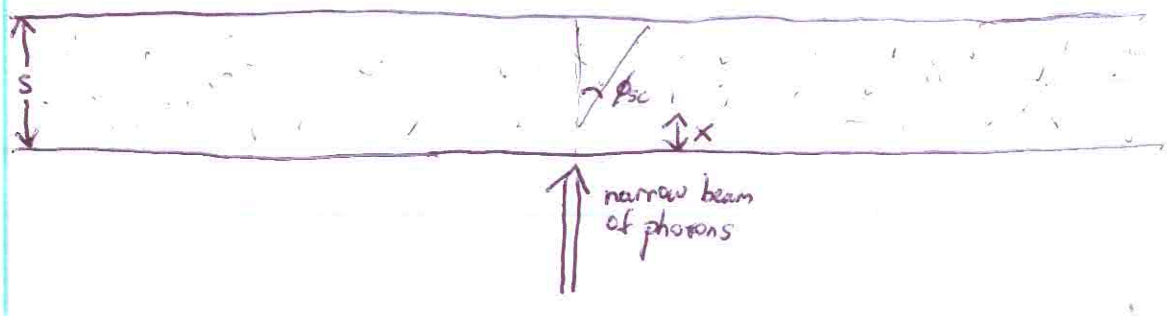
Scattering centres approximation



Let the mean number of scattering centres per unit ~~volume~~ area be n, where n is a large number. Let the RMS angle of scattering from each centre be ϕ_{RMS}, and let the (circular) scattering cross-section of each centre be t_0. Start with no absorption, and Gaussian, impact-parameter-independent scattering.

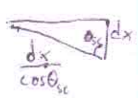
change in projected y from last scatter

$$dy = (s-x)\phi_{sc} \Rightarrow var(dy) = (s-x)^2 var(\phi_{sc}) = (s-x)^2 \phi_{RMS}^2$$

$$y = \sum dy \Rightarrow var(y) = var\left(\sum dy\right). \quad var\left(\sum x\right) = \sum var(x) \text{ (Bienaymé)}$$

$$\Rightarrow var(y) = \sum var(dy) = \sum (s-x)^2 \phi_{RMS}^2 = \int_0^s (s-x)^2 \phi_{RMS}^2 \left(\frac{\text{\# of centres hit per short distance}}{dx}\right) \times dx$$

or "chance of hitting a centre"



$$\left(\frac{\text{\# of centres hit per short distance}}{dx}\right) = \frac{\text{length of travel} \times t_0 \times n}{dx}. \quad \text{length of travel} = \frac{dx}{\cos\theta_{sc}} = \frac{dx}{\cos\sum\phi_{sc}}$$

$\Rightarrow \left(\frac{\text{\# of centres}}{\dots}\right) = \frac{t_0 n}{\cos\sum\phi_{sc}}$. If we use the small angle approx., $\cos\sum\phi_{sc} \approx 1 \Rightarrow$

$$var(y) = \int_0^s (s-x)^2 \phi_{RMS}^2 t_0 n \, dx = \phi_{RMS}^2 t_0 n \left[-\tfrac{1}{3}(s-x)^3\right]_0^s = \frac{\phi_{RMS}^2 t_0 n s^3}{3}.$$

$$\cos\left(\sum\phi_{sc}\right) \approx 1 - \frac{\left(\sum\phi_{sc}\right)^2}{2} \Rightarrow \frac{1}{\cos(\sum\phi_{sc})} \approx 1 + \frac{\left(\sum\phi_{sc}\right)^2}{2}$$

$$var(x) \quad f\left(\frac{1}{\cos x}\right)$$

$$var(\theta_{sc}) = var\left(\sum\phi_{sc}\right) = \sum var(\phi_{sc}) \text{ (Bienayme) } = \sum\phi_{RMS}^2$$

$$= \int_0^s \phi_{RMS}^2 \times \left(\frac{\#\text{ of centres hit per slant distance}}{dx}\right) \times dx = \int_0^s \frac{\phi_{RMS}^2 t_0 n\, dx}{\cos(\sum\phi_{sc})}$$

. In the small angle approx. this is just $\phi_{RMS}^2 t_0 n s$. If we go the next higher-order term in the series...; then we note that

$$var\left(\sum_i \phi_{sc_i}\right) = var\left(\phi_{sc} \times \underbrace{\text{the number of scatters}}_{\text{call this } N_{sc}}\right)$$

Note that N_{sc} and ϕ_{sc} are not necessarily independent...

In general, $var(ab) = cov(a^2, b^2) + (var(a) + E(a)^2)(var(b) + E(b)^2) - (cov(a,b) + E(a)E(b))^2$

$E(\phi_{sc}) = 0$. Thus

$$var(N_{sc}\phi_{sc}) = cov(N_{sc}^2, \phi_{sc}^2) + var(\phi_{sc})\left(var(N_{sc}) + E(N_{sc})^2\right) - cov^2(\phi_{sc}, N_{sc})$$

$$N_{sc} = t_0 n \int_0^s \frac{dx}{\cos(\int_0^x \phi_{sc})} \qquad E(N_{sc}) = t_0 n \int_0^s \frac{dx}{E(\cos(\int_0^x \phi_{sc}))}$$

$$\approx t_0 n \int_0^s dx\, E\left[1 + \frac{(\int_0^x \phi_{sc})^2}{2}\right] \qquad E[\textstyle\int_0^x \phi_{sc}]^2 = var(\int_0^x \phi_{sc})$$

$$\approx t_0 n \int_0^s dx \left(1 + \frac{\phi_{RMS}^2 t_0 n x}{2}\right)$$

$$= t_0 n s + \left[\frac{\phi_{RMS}^2 t_0 n x^2}{4}\right]_0^s = t_0 n s\left(1 + \frac{\phi_{RMS}^2 s}{4}\right)$$

yadda yadda yadda...

Try large angle. To do that, first try nonzero but small incident angle, then try very large incident angle, then use Bayes.

$$var(\theta_{sc}) = var\left(\theta_{in} + \sum\phi_{sc}\right) \underset{\text{angle } \theta_{in}}{\overset{\text{small}}{=}} \sum var(\phi_{sc}) \text{ (Bienayme) } = \sum\phi_{RMS}^2 = \int_0^s \frac{\phi_{RMS}^2 t_0 n\, dx}{\cos(\theta_{in} + \sum\phi_{sc})}$$

$$= \frac{\phi_{RMS}^2 t_0 n s}{\cos\theta_{in}}$$

, with $E(\theta_{sc}) = \theta_{in}$. With a very large incident angle, we have

$$var(\theta_{sc}) = var\left(\tfrac{\pi}{2} - \delta_{in} + \sum\phi_{sc}\right) = var\left(\sum\phi_{sc}\right) = \sum_{i,j} cov\left(\sum_{n=0}^{i}\phi_n, \sum_{n=0}^{j}\phi_n\right)$$

$$= \sum_{i,j} var\left(\sum_{n=0}^{\min(i,j)}\phi_n\right) = \sum_{i,j}\sum_{n=0}^{\min(i,j)} var(\phi_n) = \sum_{i,j}\sum_{n=0}^{\min(i,j)} \phi_{RMS}^2$$

$$= \int_0^s dx \int_0^s dy \int_0^{\min(x,y)} dz \frac{\phi_{RMS}^2 t_0 n}{\cos(\frac{\pi}{2} - \delta_{in} + \sum\phi_{sc})} = \int_0^s dx \int_0^s dy \int_0^{\min(x,y)} dz \frac{\phi_{RMS}^2 t_0 n}{\sin(\delta_{in} - \sum\phi_{sc})}$$

add absorption

$$\mathrm{var}(y) = \mathrm{var}\left(\sum \delta y\right) = \sum_{i,j} \mathrm{cov}(\delta y_i, \delta y_j)$$

$$\mathrm{cov}(\delta y_i, \delta y_j) = \varepsilon^2 \,\mathrm{cov}(\theta_i, \theta_j)$$

$$\cancel{\theta_\alpha = \left(\sum_{n=0}^{\alpha}\phi_n\right) * \Pi}$$

$$\mathrm{cov}(\theta_i,\theta_j) = \left(\prod_{n=0}^{\max(i,j)} e^{-t_n/\tau}\right)\mathrm{cov}\left(\sum_{n=0}^{i}\phi_n, \sum_{n=0}^{j}\phi_n\right)$$

$$= \left(e^{-\frac{1}{\tau}\sum_{n=0}^{\max(i,j)} \frac{\varepsilon}{\cos\left(\sum_{n=0}^{n}\phi_n\right)}}\right) \mathrm{var}\left(\sum_{n=0}^{\min(i,j)}\phi_n\right)$$

$$\approx \left(e^{-\frac{\varepsilon}{\tau}\sum_{n=0}^{\max(i,j)} 1}\right)\sum_{n=0}^{\min(i,j)}\left(\phi_{RMS}^2 \frac{\varepsilon}{t_0}\right)$$

$$\Rightarrow \mathrm{var}(y) = \sum_{i,j}\left[\varepsilon^2 \left(e^{-\frac{\varepsilon}{\tau}\sum_{n=0}^{\max(i,j)}1}\right)\left(\frac{\phi_{RMS}^2\,\varepsilon}{t_0}\sum_{n=0}^{\min(i,j)}1\right)\right]$$

$$= \phi_{RMS}^2 \frac{\varepsilon^3}{t_0}\sum_{i,j=0}^{s/\varepsilon}\left[\left(e^{-\frac{\varepsilon}{\tau}\sum_{n=0}^{\max(i,j)}1}\right)\sum_{n=0}^{\min(i,j)}1\right]$$

$$(\lim \varepsilon\to 0) \quad = \phi_{RMS}^2\frac{\varepsilon^3}{t_0}\left[\frac{\tau^3}{\varepsilon^3} - e^{-s/\tau}\left(\frac{s^2\tau}{2\varepsilon^3} + \frac{s\tau^2}{\varepsilon^3}+\frac{\tau^3}{\varepsilon^3}\right)\right]$$

$$= \frac{\phi_{RMS}^2}{t_0}\left[\tau^3 - e^{-s/\tau}\left(\frac{s^2\tau}{2}+s\tau^2+\tau^3\right)\right]$$

as $\tau\to\infty$ $[\,]\to \frac{1}{3}s^3$

$\cancel{2\pi * phi/bins}$

$\frac{bins * phi}{2\pi}$

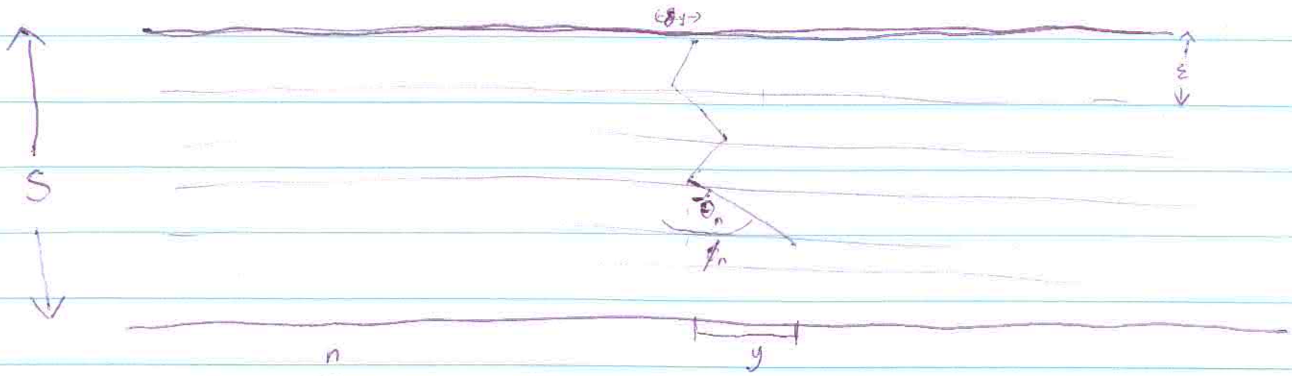
$\frac{bins}{2\pi} * \left(phi + \frac{2\pi}{2\times bins}\right) = \frac{bins * phi}{2\pi} + \frac{1}{2}$

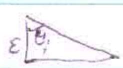
$$\mathrm{var}(\theta_{sc}) = \mathrm{var}\left(\sum\phi_{sc}\right) = \sum \mathrm{var}(\phi_{sc}) \quad (\text{Bienaymé})$$

$$= \left(\sum_{n=1}^{s/\varepsilon}\phi_{RMS}^2\frac{\varepsilon}{t_0}\right)\left(\prod_{n=1}^{s/\varepsilon} e^{-\varepsilon/\tau}\right) = \left(\frac{\phi_{RMS}^2\, s}{t_0}\right)\left(e^{-\varepsilon/\tau \sum_{n=1}^{s/\varepsilon}1}\right)$$

$$\xrightarrow{\lim \varepsilon\to 0} \frac{\phi_{RMS}^2\, s}{t_0}e^{-s/\tau} \Rightarrow \sigma_{\theta_{sc}} = \phi_{RMS}\, e^{-\frac{s}{2\tau}}\sqrt{\frac{s}{t_0}}$$

no more small angle approx., but no absorption





$$\theta_n \equiv \sum_{i=0}^{n} \phi_i \qquad var(y) = var\left(\sum \delta y\right) = \sum_{i,j} cov(\delta y_i, \delta y_j)$$

Small angle approx: $\delta y_i = \varepsilon \theta_i$. No small angle approx: $\delta y_i = \varepsilon \tan \theta_i$

$$\Rightarrow cov(\delta y_i, \delta y_j) = \varepsilon^2 cov(\tan\theta_i, \tan\theta_j) = \cancel{\varepsilon^2 cov\left(\tan\left(\sum_{n=0}^{i} \phi_n\right), \tan\left(\sum_{n=0}^{j}\phi_n\right)\right)}$$

$$var(\tan(x)) = (\text{if } x \text{ is normally distributed}) \sum_{n=0}^{\infty} \frac{\sigma^n}{(\frac{n}{2})!\, 2^{n/2}} \sum_{i=0}^{n} U_i U_{n-i} \left({}_nC_i - {}_{n/2}C_{i/2}\right)$$

for n even, 0 for n odd; but 0 for i odd

$$cov(f(x), f(y)) = E(f(x)f(y)) - Ef(x)Ef(y)$$

$$Ef(x)Ef(y) = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(Ex)}{n!} \mu_n^x\right)\left(\sum_{m=0}^{\infty} \frac{f^{(m)}(Ey)}{m!} \mu_m^y\right) = \left(\sum_{n=0}^{\infty} \frac{f^{(2n)}(Ex)}{(2n)!} \frac{(2n)!\,\sigma_x^{2n}}{n!\,2^n}\right) \times \left(\dots \right.$$

$$= \left(\sum_{n=0}^{\infty} \frac{f^{(2n)}(Ex)}{n!\,2^n} \sigma_x^{2n}\right) \times \left(\sum_{m=0}^{\infty} \frac{f^{(2m)}(Ey)}{m!\,2^m} \sigma_y^{2m}\right)$$

$$f(x)f(y) = \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(Ex)}{n!} (x - Ex)^n\right)\left(\sum_{m=0}^{\infty} \frac{f^{(m)}(Ey)}{m!} (y - Ey)^m\right)$$

$$= (\text{Cauchy product}) \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{f^{(k)}(Ex) f^{(j-k)}(Ey)}{k!\,(j-k)!} (x - Ex)^k (y - Ey)^{j-k}$$

$$\Rightarrow E(f(x)f(y)) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{f^{(k)}(Ex) f^{(j-k)}(Ey)}{k!\,(j-k)!} \mu_{k,j-k}$$

where $\mu_{k,j-k}$ is the $(k, j-k)$ bivariate central moment.

$$\frac{\alpha}{2\sigma_x^2} + \frac{1}{2\sigma_y^2} = \frac{1}{2\sigma_x^2\sigma_y^2}$$

Determining the moments and bivariate moments of Gaussians

We know that (Wikipedia) the univariate central moments

$$\mu_p \equiv E[X^p] = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \sigma^p (p-1)!! & \text{if } p \text{ is even} \end{cases} \qquad (p-1)!! = \frac{(p-1)!}{(p-2)!!} = \frac{(p-1)!}{2^{p/2-1}(\frac{p}{2}-1)!} = \frac{p!}{2^{p/2}(\frac{p}{2})!}$$

$$= \frac{\sigma^p p!}{2^{p/2}(p/2)!}$$

We want to find $\mu_{a,b} = E[X^a Y^b]$. First of all, why is this true?

$$\mu_p \equiv \int_{-\infty}^{\infty} (x-\mu)^p \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx = (\text{dropping } \mu\text{'s}) \int_{-\infty}^{\infty} x^p \frac{1}{\sigma_x\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma_x^2}} dx$$

$$\mu_{a,b} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^a y^b \frac{1}{\sigma_x\sigma_y 2\pi} e^{\frac{-x^2}{2\sigma_x^2}} e^{\frac{-y^2}{2\sigma_y^2}} dx\,dy = \frac{1}{2\pi\sigma_x\sigma_y}\iint_{-\infty} x^a y^b e^{\frac{-(\sigma_y^2x^2+\sigma_x^2y^2)}{2\sigma_x^2\sigma_y^2}} dx\,dy$$

Let $y = x + y'$, where x is of course 100% correlated with x, and y' is uncorrelated with x.

$$\Rightarrow \mu_{a,b} = \iint_{-\infty}^{\infty} x^a (x+y')^b \frac{1}{2\pi\sigma_x\sigma_y} e^{\frac{-x^2}{2\sigma_x^2}} e^{\frac{-(x+y')^2}{2\sigma_y^2}} dx\,dy'$$

$$= \frac{1}{2\pi\sigma_x\sigma_y}\iint_{-\infty}^{\infty} x^a (x+y')^b e^{\frac{-x^2}{2}\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}\right)} e^{\frac{-xy'}{\sigma_y^2}} e^{\frac{-y'^2}{2\sigma_y^2}} dx\,dy'$$

$\cancel{r = \alpha x + \beta y' \quad r^2 = \alpha^2x^2 + \beta^2y'^2 + 2\alpha\beta xy'}$ $\qquad p = x + \alpha y' \quad q = \beta x + \gamma y'$

$$p - \frac{\alpha}{\gamma}q = x\left(1 - \frac{\alpha\beta}{\gamma}\right) \Rightarrow x = \frac{p - \frac{\alpha}{\gamma}q}{1-\frac{\alpha\beta}{\gamma}} = \frac{\gamma p - \alpha q}{\gamma - \alpha\beta} \qquad p - \frac{1}{\beta}q = y'\left(\alpha - \frac{\gamma}{\beta}\right) \Rightarrow y' = \frac{\beta p - q}{\alpha\beta - \gamma}$$

$$dx\,dy' = \frac{1}{(\alpha\beta-\gamma)^2}(\alpha dq - \gamma dp)(\beta dp - dq) = \frac{1}{(\alpha\beta-\gamma)}\left((\alpha\beta+\gamma)dp\,dq - \gamma\beta(dp)^2 - \alpha(dq)^2\right)$$

$$p^2+q^2 = \frac{x^2}{2}\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}\right) + \frac{xy'}{\sigma_y^2} + \frac{y'^2}{2\sigma_y^2} = x^2(1+\beta^2) + xy'(2\alpha + 2\beta\gamma) + y'^2(\alpha^2+\gamma^2)$$

$$\Rightarrow \beta^2 = \frac{1}{2}\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2} - 2\right)$$

$$\frac{1}{\sigma_y^2} = 2\alpha + \gamma\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right)$$

$$\frac{1}{2\sigma_y^2} = \alpha^2+\gamma^2 \Rightarrow 2\alpha^2 + 2\gamma^2 = 2\alpha + \gamma\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right)$$

$$\Rightarrow 2\gamma^2 - \gamma\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right) + (2\alpha^2 - 2\alpha) = 0$$

$$\gamma = \frac{\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2 \pm\sqrt{\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right)^2 - 16(\alpha^2-\alpha)}}{4}$$

$$\text{or } \alpha = \frac{2\pm\sqrt{4 - 8\left(2\gamma^2 - \gamma\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right)\right)}}{4} = \frac{1\pm\sqrt{1 - 4\gamma^2 + 2\gamma\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right)}}{2}$$

$$\Rightarrow \pm\sqrt{1-4\gamma^2+2\gamma\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right)} = \frac{1}{\sigma_y^2} - \gamma\left(\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2}-2\right) - \frac{1}{2} \qquad \Rightarrow$$

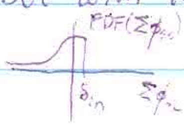
large angle w/ scattering centres cont'd

$$E(\theta_{sc}) = E(\pi/2 - \delta_{in} + \sum \phi_{sc}) = \pi/2 - \delta_{in} + E(\sum \phi_{sc}) =$$

$\pi/2 - \delta_{in} + \sum E(\phi_{sc})$. $E(\phi_{sc})$ would be 0, but we know that it has made it to the other side. ~~neg~~ Thus $\sum E(\phi_{sc}) < \delta_{in}$ and $\sum \phi_{sc} < \delta_{in}$. Without this constraint, $\sum \phi_{sc}$ would be Gaussian distributed with mean 0 and variance

$$\int_0^s dx \int_0^s dy \int_0^{\min(x,y)} dz \, \frac{\phi_{RMS}^2 \tau_0 n}{\sin \delta_{in}} = \frac{\phi_{RMS}^2 \tau_0 n}{\sin \delta_{in}} \int_0^s dx \int_0^s dy \int_0^{\min(x,y)} dz$$

$= \frac{\phi_{RMS}^2 \tau_0 n s^3}{3 \sin \delta_{in}}$. But with this constraint, we must cut off this Gaussian above δ_{in}



$$\Rightarrow PDF(\sum \phi_{sc}) \propto \begin{cases} e^{\frac{-x^2}{2\sigma^2}} & \text{for } x = \sum \phi_{sc} \leq \delta_{in} \\ 0 & \text{for } x = \sum \phi_{sc} \geq \delta_{in} \end{cases}$$

where $\sigma = \sqrt{\frac{\tau_0 n s^3 \phi_{RMS}^2}{3 \sin \delta_{in}}}$

$$\int_{-\infty}^{\delta_{in}} e^{\frac{-x^2}{2\sigma^2}} dx = \sigma \sqrt{\frac{\pi}{2}} + \sigma \sqrt{\frac{\pi}{2}} \operatorname{erf}(\delta_{in}) = \sigma \sqrt{\frac{\pi}{2}} (1 + \operatorname{erf}(\delta_{in}))$$

$\Rightarrow PDF(\sum \phi_{sc}) = N e^{\frac{-x^2}{2\sigma^2}}$ where $N = \frac{\sqrt{2}}{\sigma \sqrt{\pi}(1 + \operatorname{erf}(\delta_{in}))}$ and $\sigma =$ for $x = \sum \phi_{sc} < \delta_{in}$

The mean of that distribution is

$$N \int_{-\infty}^{\delta_{in}} x e^{\frac{-x^2}{2\sigma^2}} dx =$$

$$\left[-N\sigma^2 e^{\frac{-x^2}{2\sigma^2}} \right]_{-\infty}^{\delta_{in}} = -N\sigma^2 e^{-\frac{\delta_{in}^2}{2\sigma^2}} \; (\equiv \mu)$$

$$= \frac{-\sigma\sqrt{2}}{\sqrt{\pi}(1 + \operatorname{erf}(\delta_{in}))} e^{\frac{-\delta_{in}^2}{2\sigma^2}}$$

and the variance is

$$N \int_{-\infty}^{\delta_{in}} x^2 e^{-\frac{x^2}{2\sigma^2}} dx - \mu^2 =$$

$$\left[-xN\sigma^2 e^{\frac{-x^2}{2\sigma^2}} \right]_{-\infty}^{\delta_{in}} + N\sigma^2 \int_{-\infty}^{\delta_{in}} e^{-\frac{x^2}{2\sigma^2}} dx - \mu^2$$

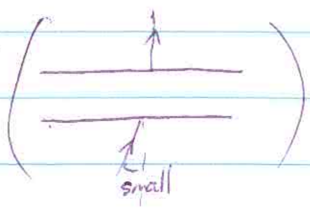
$$= \frac{-\delta_{in} \sigma \sqrt{2}}{\sqrt{\pi}(1 + \operatorname{erf}(\delta_{in}))} e^{\frac{-\delta_{in}^2}{2\sigma^2}} + \frac{\sigma\sqrt{2}}{\sqrt{\pi}(1 + \operatorname{erf}(\delta_{in}))} \left[\sigma \sqrt{\frac{\pi}{2}} (1 + \operatorname{erf}(\delta_{in})) \right] - \frac{2\sigma^2}{\pi (1 + \operatorname{erf}(\delta_{in}))^2} e^{-\frac{\delta_{in}^2}{\sigma^2}}$$

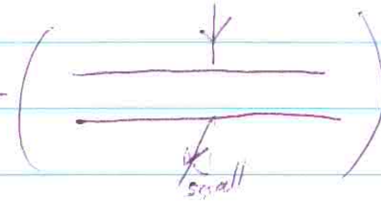
$$= \sigma^2 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}(1 + \operatorname{erf}(\delta_{in}))} e^{-\frac{\delta_{in}^2}{2\sigma^2}} \left[\delta_{in} + \frac{\sigma\sqrt{2}}{\sqrt{\pi}(1 + \operatorname{erf}(\delta_{in}))} e^{-\frac{\delta_{in}^2}{2\sigma^2}} \right]$$

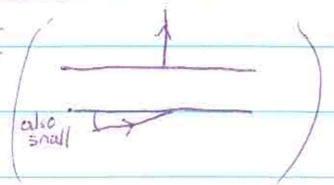
For small δ_{in}, this is a mean of $-\sqrt{\frac{2t_0 n s^3 \phi_{RMS}^2}{3\pi \delta_{in}}}$ and variance of $\frac{t_0 n s^3 \phi_{RMS}^2}{3\delta_{in}}$, essentially a flat distribution.

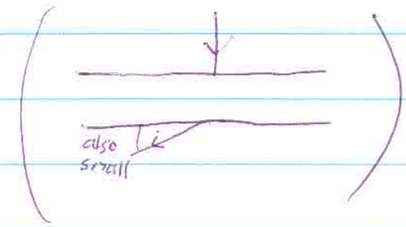
Now, we use Bayes' Theorem and thus note that:

$$\frac{\text{Likelihood of }\left(\text{diagram A}\right)}{\text{Likelihood of }\left(\text{diagram B}\right)} = \frac{L\left(\text{diagram C}\right)}{L\left(\text{diagram D}\right)}$$









Let "small" = "also small" = θ_{in}. Then the LHS of this is ≈

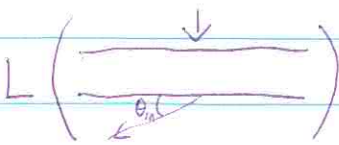
$$\frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\theta_{in}^2}{2\sigma^2}}}{\frac{1}{\pi}} \quad \text{where } \sigma = \sqrt{\frac{\phi_{RMS}^2 t_0 n s}{\cos\theta_{in}}}$$

$$= \sqrt{\frac{\pi \cos\theta_{in}}{2\phi_{RMS}^2 t_0 n s}}\, e^{-\frac{\theta_{in}^2 \cos\theta_{in}}{2\phi_{RMS}^2 t_0 n s}} \quad ①$$

The numerator of the RHS is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\theta_{in}^2}{2\sigma^2}} \quad \text{where } \sigma = \sqrt{\phi_{RMS}^2 t_0 n s}$$

$$= \sqrt{\frac{1}{2\pi\phi_{RMS}^2 t_0 n s}}\, e^{\frac{-\theta_{in}^2}{2\phi_{RMS}^2 t_0 n s}} \quad ②$$



$$\Rightarrow L\left(\text{diagram}\right) \propto \frac{②}{①} = \sqrt{\frac{1}{\pi^2\cos\theta_{in}}}\, e^{\frac{-\theta_{in}^2}{2\phi_{RMS}^2 t_0 n s}(1-\cos\theta_{in})}. \quad \cos\theta_{in} \approx 1 - \frac{\theta_{in}^2}{2}$$

$\left(1-\frac{\theta_{in}^2}{2}\right)^{-1/2} \approx 1 + \frac{\theta_{in}^2}{4}$. Thus this is $\approx \frac{1}{\pi}\left(1+\frac{\theta_{in}^2}{4}\right) e^{\frac{-\theta_{in}^4}{4\phi_{RMS}^2 t_0 n s}} \approx \frac{1}{\pi}\left(1+\frac{\theta_{in}^2}{4}\right)$