

1 Cloud Chamber Equations

We want to find the depth of the active (i.e. visible track forming) layer of a cloud chamber for a (for example) ~ 1 GeV muon that has an angle of approach that is nearly horizontal to the chamber. First, it will be informative to list the main equations governing the cloud chamber physics. These are: the Bethe-Bloch formula for the mean rate of ionization loss of a charged particle:

$$-\frac{dE}{dx} = \rho K z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\ln \frac{2m_e v^2}{I(1-\beta^2)} - \beta^2 - \frac{\delta(\beta\gamma)}{2} \right], \quad (1)$$

where ρ is the density of material (saturated air in this case), $K = 4\pi N_A r_e^2 m_e c^2$ is a constant equal to 0.307 MeV cm² mol, Z is the mean atomic number and A (in g mol⁻¹) is the mean atomic mass of the saturated air, m_e is the electron mass, v is the velocity of the particle, $\beta = v/c$, and I is the mean ionization potential of the air. The third term in the bracket is a small density correction and can be ignored for these purposes.

The next equation, important for knowing how fast droplets will fall, is Stokes' law for the viscous force F acting on a sphere of radius r falling with terminal velocity v in a fluid (i.e. saturated air) of viscosity η :

$$F = 6\pi\eta vr. \quad (2)$$

Another important equation here is the formula for the growth of a vapour droplet in a supersaturated atmosphere. This equation (6.16 in Houghton, Physics of Atmospheres, 3rd ed.) is derived from Fick's law of diffusion and the Clausius-Clapeyron equation:

$$r \frac{dr}{dt} = \frac{S - 1}{[L^2 M_r \rho_L / \lambda R T^2] + [\rho_L R T / [\rho_{\text{sat}}(T_\infty) D M_r]}, \quad (3)$$

where r is the radius of the droplet, $S = p_\infty / p_{\text{sat}}(T_\infty)$ is the saturation and $S - 1$ is the supersaturation, L is the latent heat of vapourization of the alcohol, M_r is the molecular weight (in kg) of the alcohol, ρ_L is the density of the alcohol, λ is the thermal conductivity of the saturated air, R is the ideal gas constant, T is the temperature, and D is the diffusion constant for alcohol in water.

The last important equation is the Clausius-Clapeyron relation itself. In the form that is the most useful here, it is

$$p_{\text{sat}}(T_1) = p_{\text{sat}}(T_2) e^{\frac{L M_r (T_1 - T_2)}{R T_2^2}}, \quad (4)$$

where p_{sat} is the saturation vapour pressure as a function of temperature, T_1 and T_2 are any given temperatures, L is again the latent heat of vapourization of the alcohol, M_r is again the molecular weight (in kg) of the alcohol, and R again is the ideal gas constant.

2 Calculation of Depth of Active Layer

What we want to find is the level of supersaturation in the chamber for which a line of droplets, each forming around the sites of the ions generated as per Eq. (1) above, creates a trail that is at least 1 mm wide and 10% opaque, before the droplets that are formed as per Eq. (3) either diffuse away or hit the bottom surface of the chamber (after falling as per Eq. (2)).

First we want to find the number of ions produced per unit length by the cosmic ray particle. We have $\rho \approx 10^{-3} \text{ g/cm}^{-3}$, $Z/A \approx 0.5$, and $I \approx 10 \text{ eV}$. Taking our 1 GeV muon as our example, we have that $2m_e v^2 \approx 1 \text{ MeV}$ and $1 - \beta^2 \approx 0.01$. Thus we have $-\frac{dE}{dx} \approx 1.5 \times 10^{-4} (\ln 10^7 - 1) \text{ MeV/cm} = 0.0023 \text{ MeV/cm} = 2.3 \text{ keV/cm}$, or about 230 ions produced per centimeter of track.

The ions will (I will assume) each nucleate a droplet. (They will tend to nucleate separate droplets since they are positively charged, and repel each other.) With ~ 230 droplets per cm, what size will the droplets need to be to produce a track that is at least 1 mm wide and 10% opaque? The cross-section of a droplet is πr^2 , so we have that r must be at least $\sim 120 \mu\text{m}$.

Now we need to figure out the level of supersaturation as a function of height in the chamber. Let the alcohol trough be at height h and temperature T_h , and the bottom surface be at height 0 and temperature T_0 . Just above the alcohol trough, the vapour saturation will be 100% at temperature T_h , as this is the definition of saturation vapour pressure. From Eq. (4), the saturation level as a function of temperature will be

$$S = e^{\frac{LM_r(T_h - T)}{RT_h^2}}.$$

Now we will make an assumption that can be improved later: we will assume that the temperature decreases uniformly with height in the chamber, from T_h at $z = h$ down to T_0 at $z = 0$, so $T(z) = T_0 + \frac{z}{h}(T_h - T_0)$. Thus the supersaturation as a function of height z is

$$S(z) - 1 = e^{\frac{LM_r(h-z)(T_h - T_0)}{hRT_h^2}} - 1.$$

From Eq. (2), we have $mg = 6\pi\eta vr \Rightarrow \frac{4}{3}\pi r^3\rho = -6\pi\eta\frac{dz}{dt}r \Rightarrow r = \sqrt{\frac{-9\eta}{2\rho g}\frac{dz}{dt}} \Rightarrow r\frac{dr}{dt} = \frac{-9\eta}{4\rho g}\frac{d^2z}{dt^2}$.
Thus we have that

$$\frac{-9\eta}{4\rho g}\frac{d^2z}{dt^2} = \frac{e^{\frac{LM_r(h-z)(T_h-T_0)}{hRT_h^2}} - 1}{[L^2M_r\rho_L/\lambda RT^2] + [\rho_L RT/[\rho_{\text{sat}}(T_\infty)DM_r]}.$$