8.851 Homework 4

Iain Stewart, March 5, 2003

Problem 1) HQET for Antiquarks

Do problem 1 in section 2 of the book.

Problem 2) Heavy-to-Light Form Factors in HQET

Consider heavy-to-light semileptonic decays, B to a vector meson.

a) Do the first part of problem 3 in section 2 of the book. (Do not bother with the last part which asks you to discuss the decays to ρ mesons.) Argue that

$$\langle V(p', \epsilon)|\bar{q}\Gamma Q_v|P^{(Q)}(v)\rangle = \mathrm{tr}\,\left(M_V \Gamma H_v^{(Q)}\right), \tag{1}$$

where M_V depends on p', ϵ^*, and v. Then show that the most general M_V gives no reduction in the number of vector and axial-vector form factors (which is why this problem considers flavor symmetry relations but not spin symmetry relations).

b) (For bonus only) Use your results from a) to solve problem 4 in section 2 which shows that there are spin symmetry relations for the tensor current.

8.851 Solutions 4

Michael McNeil Forbes

April 9, 2003

1 Comments

The grading for these questions refers to the following parts:

1a: Antiquark propagator.

1b: Antiquark vertex.

2a: Problem 2.3.

2b: Argument that spin symmetry does not reduce the number of form factors.

2c: Tensor results (problem 2.4.)

2 Question 1: HQET for Antiquarks

One can see that, if $p = mv + k$, where mv is on-shell ($v \cdot v$=1), then the projectors

$$P_{\pm} = \frac{1 \pm \not{v}}{2} \tag{1}$$

project out the particle and antiparticle components respectively. We can see this by considering the Dirac equation. In a covariant formulation, particles ψ_+ have momentum p^μ while antiparticles ψ_- have momentum $-p^\mu$, thus, the Dirac equation is:

$$(\pm \not{p} - m)\psi_{\pm} = 0 \qquad \Longrightarrow \qquad \pm \not{p}\psi_{\pm} = m\psi_{\pm}. \tag{2}$$

Thus,

$$P_{\pm}\psi_{\pm} = \frac{m \pm \not{p}}{2m}\psi_{\pm} = \frac{2m}{2m}\psi_{\pm} = \psi_{\pm}, \tag{3}$$

$$P_{\mp}\psi_{\pm} = \frac{m \mp \not{p}}{2m}\psi_{\pm} = \frac{0}{2m}\psi_{\pm} = 0. \tag{4}$$

We see that, to transform relations for particles to the equivalent relations for antiparticles, we simply take $P_+ \to P_-$, $p \to -p$, $v \to -v$ and $k \to -k$ everywhere and exchange appropriate vertices.

Thus, the propagator and vertex follow from the particle result:

$$P_+ \frac{i}{v\cdot k + i\epsilon} \qquad \longrightarrow \qquad P_- \frac{i}{(-v)\cdot(-k) + i\epsilon} = P_- \frac{i}{v\cdot k + i\epsilon}, \tag{5}$$

$$-ig(T^A)v_\mu \qquad \longrightarrow \qquad ig(T^A)^T v_\mu. \tag{6}$$

The transpose here results from the interpretation of the ends of the antiquark propagator: antiquarks propagate "backward" in time and so connect vertices in the reverse order. Another way of seeing this is that the free vacuum is annihilated by $\psi_+ |0\rangle = 0$ since it has no particles, however, it also has no antiparticles and thus $\bar{\psi}_- |0\rangle = 0$ since $\psi_- \sim b^\dagger$. Thus, the anti-quark propagator is $\langle \psi_- \bar{\psi}_- \rangle$ compared to $\langle \bar{\psi}_+ \psi_+ \rangle$ for the quarks. Thus, in terms of indices, the index order is reversed. The vertex still couples the same way to ψ but the propagator is "backward" leading to the transpose.

Explicitly, define

$$\Psi = e^{imv\cdot x}(\psi_- + \psi_+). \tag{7}$$

Now we have the identities

$$P_\pm \gamma^\mu = \gamma^\mu P_\mp \pm v^\mu, \tag{8}$$

$$P_\pm \gamma^\mu P_\pm = \pm P_\pm v^\mu P_\pm, \tag{9}$$

$$P_\pm^\dagger \gamma^0 = \gamma^0 P_\pm, \tag{10}$$

$$P_\pm \gamma^\mu P_\mp = (\gamma^\mu \pm v^\mu) P_\mp, \tag{11}$$

$$P_\pm \not{v} P_\mp = (\not{v} \pm 1) P_\mp = \pm 2 P_\pm P_\mp = 0, \tag{12}$$

$$P_\pm \not{D} P_\mp = P_\pm \not{D}_\perp P_\mp \qquad \text{where} \qquad D_\perp^\mu = D^\mu - v^\mu (v\cdot D). \tag{13}$$

Now, the QCD Lagrangian becomes

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\Psi}(i\not{D} - m)\Psi, \tag{14}$$

$$= (\bar{\psi}_+ + \bar{\psi}_-)(i\not{D} - 2mP_+)(\psi_+ + \psi_-), \tag{15}$$

$$= \bar{\psi}_+ (iv\cdot D - 2m)\psi_+ - \bar{\psi}_- iv\cdot D \psi_- + \bar{\psi}_+ i\not{D}_\perp \psi_- + \bar{\psi}_- i\not{D}_\perp \psi_+. \tag{16}$$

The equations of motion are thus

$$(iv\cdot D - 2m)\psi_+ = -i\not{D}_\perp \psi_-, \qquad (iv\cdot D)\psi_- = -i\not{D}_\perp \psi_+. \tag{17}$$

Solving for the anti-quarks (or equivalently, "integrating out" the antiquarks at tree level), we have

$$\psi_+ = -(iv\cdot D - 2m)^{-1} i\not{D}_\perp \psi_-, \tag{18}$$

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_- \left[-iv\cdot D - i\not{D}_\perp \frac{1}{iv\cdot D - 2m} i\not{D}_\perp \right] \psi_-. \tag{19}$$

The second term is suppressed by m^{-1} and is dropped to leading order. Now we normal order the operators recalling that the vacuum is annihilated by $\chi = \psi_-^\dagger$, so we have[1]

$$\begin{aligned} L\mathcal{L}_{\mathrm{QCD}} &=: \bar{\phi}_-[-iv \cdot (\partial + igA^A\tau^A)]\psi_- : + \mathrm{O}(m^{-1}), \qquad (20) \\ &= \bar{\chi}[iv^\mu(-\partial_\mu + igA^A_\mu[\tau^A]^T)]\chi + \mathrm{O}(m^{-1}), \qquad (21) \end{aligned}$$

from which the propagator and vertex can be read off including correct signs and the transpose. The propagator has a projector appended to project out the antiparticle state.

3 Question 2: Heavy-to-Light Form Factors in HQET.

The form factors are:

$$\begin{aligned} \langle V(p',\varepsilon)|\bar{q}\gamma_\mu\gamma_5 Q|P^{(Q)}(p)\rangle &= -if^{(Q)}\varepsilon^*_\mu - i\varepsilon^* \cdot p[a^{(Q)}_+(p+p')_\mu + a^{(Q)}_-(p-p')_\mu], \\ &= -if^{(Q)}\varepsilon^*_\mu - i\varepsilon^* \cdot vm_Q[(a^{(Q)}_+ + a^{(Q)}_-)m_Q v_\mu + (a^{(Q)}_+ - a^{(Q)}_-)p'_\mu], \\ \langle V(p',\varepsilon)|\bar{q}\gamma_\mu Q|P^{(Q)}(p)\rangle &= g^{(Q)}\varepsilon_{\mu\nu\lambda\sigma}\varepsilon^{*\nu}(p+p')^\lambda(p-p')^\sigma \\ &= g^{(Q)}\varepsilon_{\mu\nu\lambda\sigma}\varepsilon^{*\nu}m_Q(p'^\lambda v^\sigma - v^\lambda p'^\sigma). \end{aligned}$$

In addition, we have the normalization

$$|P^{(Q)}(p)\rangle = \sqrt{m_Q}\,|P^{(Q)}(v)\rangle \qquad (22)$$

The states and operators of HQET are normalized such that, in the limit $m_{b,c} \to \infty$, one has a flavour symmetry. This relates

$$\langle V(p',\varepsilon)|\bar{q}\Gamma b_v|P^{(b)}(v)\rangle = \langle V(p',\varepsilon)|\bar{q}\Gamma c_v|P^{(c)}(v)\rangle\,. \qquad (23)$$

Collecting components, we thus have

$$\begin{aligned} \sqrt{m_b}g^b(y) &= \sqrt{m_c}g^c(y), \qquad (24) \\ \sqrt{m_b}^{-1}f^b(y) &= \sqrt{m_c}^{-1}f^c(y), \qquad (25) \\ m_b^{3/2}[a^{(b)}_+(y) + a^{(b)}_-(y)] &= m_c^{3/2}[a^{(c)}_+(y) + a^{(c)}_-(y)], \qquad (26) \\ \sqrt{m_b}[a^{(b)}_+(y) - a^{(b)}_-(y)] &= \sqrt{m_c}[a^{(c)}_+(y) - a^{(c)}_-(y)]. \qquad (27) \end{aligned}$$

Now, consider a general current $\langle V(p',\varepsilon)|\bar{q}\Gamma Q_v|P^{(Q)}(v)\rangle$. Under spin transformations,

$$\begin{aligned} Q_v &\to D(R)_Q Q_v, \\ H_v = P_+[P^{*(Q)}_v + iP^{(Q)}_v\gamma_5] &\to D(R)_Q H_v, \\ \langle V(p',\varepsilon)|\bar{q}\Gamma Q_v|P^{(Q)}(v)\rangle &\to \langle V(p',\varepsilon)|\bar{q}\Gamma D(R)_Q Q_v|P^{(Q)}(v)\rangle\,. \end{aligned}$$

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[1] Note, one sign change comes from anticommutation. Another on the kinetic term from integration by parts. The γ^0 freely commutes through all terms as there is no more Lorentz structure.

Thus, if we let $\Gamma \to \Gamma D(R)_Q^{-1}$ then the matrix element will be invariant. Thus, we can construct the most general form factor from combinations of ΓH_v and Γ that are invariant under the spurious symmetry $\Gamma \to \Gamma D(R)_Q^{-1}$. This implies that the most general form contains ΓH_v.

This quantity is still a bispinor (it has uncontracted light quark indices). To form a Lorentz covariant current, these must be contracted. The most general such contraction is

$$\langle V(p',\varepsilon)|\bar{q}\Gamma Q_v|P^{(Q)}(v)\rangle = \mathrm{Tr}\,\langle M_v \Gamma H_v^{(Q)}\rangle \tag{28}$$

where the bispinor M_v depends on the kinematic variables p', v and the polarization ε. There is a source of confusion in the book about currents and matrix elements. Expressions like this and (2.89) are actually only valid relations between matrix elements of certain states. Thus, in this case, we are taking a matrix element with a pseudoscalar state and so we should replace the operator $P_v^{*(Q)\mu} \to 0$ and $P_v^{(Q)} \to$ const in the operator H_v for this matrix element. We will absorb this constant into the definition of M_v and take it to be one.

Now we can use covariance to restrict the form of M_v. In particular, it must be a bispinor constructed from p', v and ε^* and have the appropriate parity transformations

$$M_v(v_p, p'_p, \varepsilon^*_p) = -\gamma^0 M_v(v, p', \varepsilon^*)\gamma^0. \tag{29}$$

To see this, consider inserting the parity operator $\widehat{\mathbf{P}}^\dagger\widehat{\mathbf{P}} = \widehat{\mathbf{1}}$:

$$\begin{aligned}
\langle V(p',\varepsilon)|\widehat{\mathbf{1}}\bar{q}\Gamma Q_v\widehat{\mathbf{1}}|P^{(Q)}(v)\rangle &= \langle V(p',\varepsilon)|\widehat{\mathbf{P}}^\dagger\widehat{\mathbf{P}}(\bar{q}\Gamma Q_v)\widehat{\mathbf{P}}^\dagger\widehat{\mathbf{P}}|P^{(Q)}(v)\rangle\,, && (30)\\
&= \langle V(p'_p,\varepsilon_p)|(-1)(\bar{q}\gamma^0\Gamma\gamma^0 Q_v)(-1)|P^{(Q)}(v_p)\rangle\,, && (31)\\
&= \mathrm{Tr}\left(M_v(\varepsilon^*,p',v)\Gamma H_v^{(Q)}\right), && (32)\\
&= \mathrm{Tr}\left(M_v(\varepsilon^*_p,p'_p,v_p)\gamma^0\Gamma\gamma^0\frac{1+\not{v}_p}{2}i\gamma_5\right), && (33)\\
&= -tr\left(\gamma^0 M_v(\varepsilon^*_p,p'_p,v_p)\gamma^0\Gamma\frac{1+\not{v}}{2}i\gamma_5\right). && (34)
\end{aligned}$$

The minus sign comes from commuting the γ^0 past the γ^5. In addition, commuting the γ^0 past the $\not{v}_p$ changes the spatial signs: $\gamma^0\not{v}_p = \not{v}\gamma^0$.

The transformation properties of the kinetic elements are

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^0, -\vec{\gamma}), \tag{35a}$$

$$\varepsilon_p^{*\mu} = (\varepsilon^{*0}, -\vec{\varepsilon}^*), \tag{35b}$$

$$v_p^\mu = (v^0, -\vec{\mathbf{v}}), \tag{35c}$$

$$p_p'^\mu = (p'^0, -\vec{\mathbf{p}}'). \tag{35d}$$

Thus, we have the following

Scalar (Parity Even): $\not{v}$, $\not{p}'$.

Pseudo-Scalar (Parity Odd): $\not{\varepsilon}^*$, $\varepsilon^* \cdot v$, $\not{v}$, $\not{p}'$.

Note that one pseudo-scalar candidate, $\varepsilon^* \cdot p' = 0$ has been omitted.

Noting that $\not{\varepsilon}^* \not{\varepsilon}^* = 1$, $\not{v}\not{v} = 1$ and $\not{p}'\not{p}' = m^2$ etc. we see that one only need consider terms with 3 or fewer factors. One can also use the relation $H_v^Q \not{v} = -H_v^Q$ to eliminate spurious factors of $\not{v}$ that appear to the left of other terms. Thus, the most general form for M_v is

$$M_v = a\not{\varepsilon}^* + b\varepsilon^* \cdot v\not{p} + c\not{\varepsilon}^* \not{v} + d\not{\varepsilon}^* \not{p}'. \tag{36}$$

Thus, we see that there are exactly four unknown form factors. These can be related to those given in the first part of the problem by computing the traces. In particular, including spin symmetry does not reduce the number of form factors.

The results are (thanks to Dru Renner for the following):

$$f^{(Q)} = -2m_Q^{1/2}(a + c + 4d), \tag{37a}$$

$$a_+^Q + a_-^Q = 4m_Q^{-3/2}c, \tag{37b}$$

$$a_+^Q - a_-^Q = 2m_Q^{-1/2}(d - b), \tag{37c}$$

$$g^Q = 2m_Q^{-1/2}d. \tag{37d}$$

The tensor results can be obtained similarly by performing the traces to give:

$$g_+^Q + g_-^Q = -2m_Q^{-1/2}(a + c), \tag{38a}$$

$$g_+^Q - g_-^Q = -2m_Q^{1/2}d, \tag{38b}$$

$$h^Q = -m_Q^{-3/2}b, \tag{38c}$$

using the definitions of the form factors given in the text, problem 2.4. Through the constants a, b, c and d one can relate the form factors for the tensor current to those of the vector and pseudo-scalar currents, obtaining the results of problem 2.4.

3.1 References

- Manohar and Wise, *Heavy Quark Physics.* Problems 3 and 4 in Chapter 2.

8.851 Homework 5

Iain Stewart, March 12, 2003

Problem 1) Renormalization of $c_F(\mu)$

Draw the diagrams needed to compute the anomalous dimension of the coefficient $c_F(\mu)$ which multiplies the magnetic moment Lagrangian $\mathcal{L}_F^{(1)}$. Discuss whether the kinetic energy Lagrangian $\mathcal{L}_K^{(1)}$ mixes with $\mathcal{L}_F^{(1)}$ under renormalization. Argue that the anomalous dimension will be proportional to the adjoint Casmir C_A without computing integrals (Hint: think about Coulomb gauge).

Problem 2) Heavy-to-Light Currents in HQET

Consider the $\mathcal{O}(1/m_Q)$ heavy-to-light vector currents

$$\begin{aligned} O_1 &= \bar{q}\,\gamma^\mu i\not{D} Q_v\,, \qquad O_4 = \bar{q}\,(-iv\cdot\overleftarrow{D})\gamma^\mu Q_v\,,\\ O_2 &= \bar{q}\,v^\mu i\not{D} Q_v\,, \qquad O_5 = \bar{q}\,(-iv\cdot\overleftarrow{D})v^\mu Q_v\,,\\ O_3 &= \bar{q}\,iD^\mu Q_v\,, \qquad O_6 = \bar{q}\,(-i\overleftarrow{D}^\mu)Q_v\,, \end{aligned} \tag{1}$$

with coefficients B_1 to B_6. Using reparameterization invariance determine which of these coefficients are fixed by the coefficients C_1 and C_2 of the leading order vector heavy-to-light currents, $\bar{q}\gamma^\mu Q_v$ and $\bar{q}v^\mu Q_v$.

Problem 3) Non-perturbative form factor corrections

In Chapter 4 of the book do problem 3 parts a), b), and c). Feel free to use results given in problems 6-9 in Chapter 2.

currents are

$$\bar{q}\gamma^\mu \tilde{Q}_v = \bar{q}\gamma^\mu \left(1 + \frac{i\not{D}}{2m_Q}\right) Q_v + \mathrm{O}(\Lambda^2/m_Q^2), \tag{11a}$$

$$= \bar{q}\gamma^\mu Q_v + \frac{1}{2m_Q}\bar{q}\gamma^\mu i\not{D} Q_v + \mathrm{O}(\Lambda^2/m_Q^2), \tag{11b}$$

$$= J_1^\mu + \frac{1}{2m_Q}\mathcal{O}_1 + \mathrm{O}(\Lambda^2/m_Q^2), \tag{11c}$$

$$\bar{q}\mathcal{V}^\mu \tilde{Q}_v = \bar{q}\left(v^\mu + \frac{iD^\mu}{m_Q}\right)\left(1 + \frac{i\not{D}}{2m_Q}\right) Q_v + \mathrm{O}(\Lambda^2/m_Q^2), \tag{11d}$$

$$= \bar{q}v^\mu Q_v + + \frac{1}{2m_Q} 2\bar{q}iD^\mu Q_v + \frac{1}{2m_Q}\bar{q}v^\mu i\not{D} Q_v + \mathrm{O}(\Lambda^2/m_Q^2), \tag{11e}$$

$$= J_2^\mu + \frac{1}{2m_Q}(2\mathcal{O}_3 + \mathcal{O}_2) + \mathrm{O}(\Lambda^2/m_Q^2). \tag{11f}$$

Thus, in order to have combinations of the operators and currents that are reparametrisation invariant to order Λ^2/m_Q^2 as in (10), one must have

$$B_1(\mu) = C_1(\mu), \tag{12a}$$

$$B_2(\mu) = C_2(\mu), \tag{12b}$$

$$\frac{1}{2}B_3(\mu) = C_2(\mu). \tag{12c}$$

Further restrictions are not placed on the other coefficients to this order. This is also discussed in Neubert.

2.1 References

- Luke and Manohar,*Reparameterisation Invariance Constraints on Heavy Particle Effective Field Theories.* Phys. Lett. **B286** (1992) 348-354. `arXiv:hep-ph/9205228`

3 Question 3: Non-perturbative Form Factor Corrections

This question concerns problem 3 of Chapter 4 of Manohar and Wise. References are made to problems in Chapter 2 of the same book. I.e. problem 2.6 refers to problem 6 in chapter 2.

In problem 2.6, the matrix F_v^μ is introduces which contains the $D_2^{*\mu\nu}$ and D_2^μ fields and which satisfies various relations listed there. The B fields are contained in the matrix $H_v = H_v^{(b)}$. Both of these transform as

$$F_v^\mu \to D(R)_c F_v^\mu, \qquad H_v \to D(R)_b H_v, \tag{13}$$

under heavy quark spin transformations, so using the usual symmetry arguments whereby this symmetry is fully restored in the currents

$$\bar{c}(\cdots)\Gamma(\cdots)b \tag{14}$$

by having

$$\Gamma \to D(R)_c \Gamma \overline{D}(R)_b \tag{15}$$

we must have the combination $\overline{F}^{\mu}_{v}\Gamma H_v$. We parameterize the remaining structure in matrices $S_{\sigma\lambda}$ as given in the book. Note that these relationships are weak: they only hold between appropriate matrix elements and are not operator relations.

Now we consider the most general structure for $S_{\sigma\lambda}$. From the properties of $\overline{F}^{\mu}_{v}$ and H_v as well as cyclic properties of the trace, we have the following:

$$\overline{F}^{\sigma}_{v}\not{v} = \overline{F}^{\sigma}_{v}, \qquad \not{v}\overline{F}^{\sigma}_{v} = -\overline{F}^{\sigma}_{v}, \tag{16a} \quad \boxed{\texttt{eq:2}}$$

$$H_v\not{v} = -H_v, \qquad \not{v}H_v = H_v, \tag{16b} \quad \boxed{\texttt{eq:3}}$$

$$\gamma_\sigma\overline{F}^{\sigma}_{v} = 0, \qquad v'_\sigma\overline{F}^{\sigma}_{v} = 0, \tag{16c} \quad \boxed{\texttt{eq:4}}$$

$$\not{v}\not{v}' = 2v\cdot v' - \not{v}'\not{v}, \qquad \not{v}\gamma^{\mu} = 2v^{\mu} - \gamma^{\mu}\not{v}, \tag{16d} \quad \boxed{\texttt{eq:5}}$$

Thus, factors of $\not{v}'$ can be absorbed into $\overline{F}^{\mu}_{v'}$ and factors of $\not{v}$ can be absorbed into H_v and factors of γ_σ and v'_σ vanish. The last relations show that we can commute the terms as appropriate to facilitate the cancellations. Thus, we are left constructing terms with v_λ, v_σ, v'_λ, γ_λ and $g_{\lambda\sigma}$. Finally, parity forbids terms with γ_5 and terms with multiple γ_μ can be reduced to terms linear in γ_μ and $g_{\mu\nu}$. From these we can construct the form of $S^{Q}_{\sigma\lambda}$ given which is the most general form with linearly independent terms constructed from these ingredients.

The relevant equations of motion are

$$\bar{c}_{v'}(i\overleftarrow{D}\cdot v') = 0, \qquad (v\cdot D)b = 0. \tag{17}$$

By contracting v^{μ} and v'^{μ} into the equations we have

$$v'^{\lambda}\,\mathrm{Tr}(S^{(c)}_{\sigma\lambda}\overline{F}^{\sigma}_{v'}\Gamma H_v) = 0, \qquad v^{\lambda}\,\mathrm{Tr}(S^{(b)}_{\sigma\lambda}\overline{F}^{\sigma}_{v'}\Gamma H_v) = 0. \tag{18}$$

Expanding

$$\begin{aligned}
v^{\lambda}S^{(b)}_{\sigma\lambda} &= v_\sigma[\tau^{(b)}_1 + \tau^{(b)}_2 w + \tau^{(b)}_3\not{v} + \tau^{(b)}_4], && (19)\\
&\equiv v_\sigma[\tau^{(b)}_1 + \tau^{(b)}_2 w - \tau^{(b)}_3 + \tau^{(b)}_4], && (20)\\
v'^{\lambda}S^{(c)}_{\sigma\lambda} &= v_\sigma[\tau^{(c)}_1 w + \tau^{(c)}_2 + \tau^{(c)}_3\not{v}'] + v'_\sigma\tau^{(c)}_4, && (21)\\
&\equiv v_\sigma[\tau^{(c)}_1 w + \tau^{(b)}_2 w - \tau^{(c)_3}]. && (22)
\end{aligned}$$

Here we have used relations (16a) and (16b) to simplify the coefficient of τ_3, set $w = v\cdot v'$ as well as (16c) to eliminate $\tau^{(c)}_4$ in the full expression. The remaining terms are generally non-zero, and so must be zero for the equations of motion to hold. This gives the desired relations.

Using the result of problem 2.9 we have

$$i\partial_\nu(\bar{c}_{v'}\Gamma b_v) = (\overline{\Lambda} v_\nu - \overline{\Lambda}^* v'_\nu)\tau \,\mathrm{Tr}[v_\sigma \overline{F}^\sigma_{v'} x G H^{(b)}_v]. \tag{23}$$

Meanwhile, expanding the derivatives, we have

$$\begin{aligned} i\partial_\nu(\bar{c}_{v'}\Gamma b_v) &= \bar{c}_{v'} i \overleftarrow{\partial}_\nu \Gamma b_v + \bar{c}_{v'}\Gamma i\partial_\nu b_v - \bar{c}_{v'} g A^a_\nu \lambda^a \Gamma b_v + \bar{c}_{v'} g A^a_\nu \lambda^a \Gamma b_v, & (24)\\ &= \bar{c}_{v'} i(\overleftarrow{\partial}_\nu - igA^a_\nu\lambda^a)\Gamma b_v + \bar{c}_{v'}\Gamma i(\overleftarrow{\partial}_\nu + igA^a_\nu\lambda^a) b_v, & (25)\\ &= \bar{c}_{v'} i \overleftarrow{D}_\nu \Gamma b_v + \bar{c}_{v'}\Gamma i D_\nu b_v, & (26)\\ &= \mathrm{Tr}(S^{(c)}_{\sigma\mu}\overline{F}^\sigma_{v'}\Gamma H_v) + \mathrm{Tr}(S^{(b)}_{\sigma\mu}\overline{F}^\sigma_{v'}\Gamma H_v). & (27) \end{aligned}$$

Equating the coefficients one obtains the relations in problem 3.c.

3.1 References

- Manohar and Wise, *Heavy Quark Physics.* Problems 6-9 of Chapter 2, and Problem 3 of Chapter 4.

8.851 Homework 8

Iain Stewart, April 16, 2003 (due Apr.23)

Problem 1) Heavy Meson Chiral Perturbation Theory

Show that we recover our results for the leading order heavy meson chiral Lagrangian gauged under electromagnetism by first using left and right hand currents and then specializing to the $U(1)$. Explain how the Q' term must be treated.

Problem 2) Do Problem 3 in Chapter 5 of the Book

Problem 3) Chiral Lagrangian for Heavy Vector Mesons

Introduce the vector meson fields as a 3×3 octet matrix

$$\mathcal{O}_\mu = \begin{bmatrix} \frac{\rho^0_\mu}{\sqrt{2}} + \frac{\phi^{(8)}_\mu}{\sqrt{6}} & \rho^+_\mu & K^{*+}_\mu \\ \rho^-_\mu & -\frac{\rho^0_\mu}{\sqrt{2}} + \frac{\phi^{(8)}_\mu}{\sqrt{6}} & K^{*0}_\mu \\ K^{*-}_\mu & \overline{K}^{*0}_\mu & -\frac{2\phi^{(8)}_\mu}{\sqrt{6}} \end{bmatrix}, \tag{1}$$

and as a singlet

$$S_\mu = \phi^{(0)}_\mu. \tag{2}$$

(a linear combination of $\phi^{(0,8)}_\mu$ will be the ϕ and ω). Under chiral $SU(3)_L \times SU(3)_R$ let

$$\mathcal{O}_\mu \to U \mathcal{O}_\mu U^\dagger, \qquad S_\mu \to S_\mu. \tag{3}$$

Under charge conjugation,

$$C\mathcal{O}_\mu C^{-1} = -\mathcal{O}^T_\mu,\ CS_\mu C^{-1} = -S_\mu,\ C\xi C^{-1} = \xi^T. \tag{4}$$

Construct the $\mathcal{O}(p)$ chiral $SU(3)$ Lagrangian for these vector mesons treating them as heavy static fields with fixed four-velocity v^μ. Comment on the expansion parameter in this theory. (Bonus: Discuss the terms with the quark mass matrix as well.)

8.851 Solutions 8

Michael McNeil Forbes

June 10, 2003

1 Question 1: Heavy Meson Chiral Perturbation Theory

The tricky part about including chiral symmetry with heavy mesons is that the natural field $\hat{H}_v$ does not have nice chiral transformation properties. This is discussed in Chapter 5 of Manohar and Wise. We follow their conventions and perform a field redefinition H_v which transforms as

$$H = \hat{H}\xi \to HU \tag{1}$$

where

$$\xi = \sqrt{\Sigma}. \tag{2}$$

Note what happens here: first of all, these are all matrix relations. Second, we have the matrix Σ of light mesons which has definite chiral transformation properties:

$$\Sigma \to L\Sigma R^\dagger. \tag{3}$$

The matrix U is defined such that

$$\xi \to L\xi U^\dagger = U\xi R^\dagger. \tag{4}$$

The complication of this is that Σ, ξ and therefore U have spacetime dependence. Thus, derivatives must be constructed in a covariant way to preserve these symmetries. One way to do this is to introduce the combinations

$$\mathbb{V}_\mu = \frac{i}{2}(\xi^\dagger\partial_\mu\xi + \xi\partial_\mu\xi^\dagger), \qquad \mathbb{A}_\mu = \frac{i}{2}(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger). \tag{5}$$

Now we construct the covariant derivative

$$D_\mu = \partial_\mu - i\mathbb{V}_\mu. \tag{6}$$

For a file which transforms as $\phi \to U\phi$ such as $H^\dagger$, one can easily verify that this acts covariantly:

$$D_\mu\phi \to UD_\mu\phi. \tag{7}$$

To act on H one must be careful about the indices:

$$D_\mu H \equiv \partial_\mu H + iH\mathbf{V}_\mu. \tag{8}$$

eq:1

This only holds, however, for global transformations L and R independent of x_μ. To promote these to local symmetries, one must introduce the currents l_μ and r_μ which transform as

$$l_\mu \to L l_\mu L^\dagger + i(\partial_\mu L)L^\dagger, \qquad r_\mu \to R l_\mu R^\dagger + i(\partial_\mu R)R^\dagger. \tag{9}$$

Now we can recover covariance by redefining

$$\mathbf{V}_\mu = \frac{i}{2}(\xi^\dagger(\partial_\mu + il_\mu)\xi + \xi(\partial_\mu + ir_\mu)\xi^\dagger), \tag{10}$$

$$\mathbb{A}_\mu = \frac{i}{2}(\xi^\dagger(\partial_\mu + il_\mu)\xi - \xi(\partial_\mu + ir_\mu)\xi^\dagger)\xi^\dagger). \tag{11}$$

The covariant derivative now acts covariantly for local chiral transformations:

$$D_\mu\phi \to UD_\mu\phi. \tag{12}$$

To include the $U(1)_{\text{em}}$ gauge field, one sets

$$l_\mu = r_\mu = eQA_\mu. \tag{13}$$

In addition, one must consider the heavy quarks which have charges Q'. This has a similar structure but acts on the heavy quark indices of $H \sim Qq$. Thus, we have the full covariant derivative

$$\tilde{D}_\mu H = D_\mu H + ieB_\mu Q' H \tag{14}$$

where $D_\mu H$ is defined in (8).

Expanding these out, one obtains the results stated in class. (See the attached notes for some explicit calculations.)

1.1 References

- Manohar and Wise, *Heavy Quark Physics.* Chapter 5.

2 Question 2: Form Factors for $D^+ \to K^-\pi^+\bar{e}\nu_e$.

This is Problem 3 of Chapter 5 in Manohar and Wise. Please see the scanned solution.

2.1 References

- Manohar and Wise, *Heavy Quark Physics.* Problem 3 of Chapter 5.

3 Problem 3: Chiral Lagrangian for Heavy Vector Mesons.

This problem is discussed in detail in the following reference.

3.1 References

- E. Jenkins, A. V. Manohar and M. B. Wise, *Chiral Perturbation Theory for Vector Mesons.* Phys. Rev. Lett. **75** (1995), 2272.

Problem ①

In the case where we consider spacetime independent L, R transformations we have the following expression for the covariant derivative (Which takes care of the fact that $U(L, R, \pi(x))$ has an x dependence)

$$D_\nu = \partial_\nu + \frac{1}{2}\left(\xi^+ \partial_\nu \xi + \xi \partial_\nu \xi^+\right)$$

This covariant derivative has to be applied to fields transforming under U. For example if $\phi \mapsto U\phi$ then (L, R indip of x)

$$D_\nu \phi \mapsto \partial_\nu (U\phi) + \frac{1}{2}\left(U\xi^+ L^+ \partial_\nu (L\xi U^+) + U\xi R^+ \partial_\nu (R\xi^+ U^+)\right) U\phi =$$

$$= U\partial_\nu \phi + (\partial_\nu U)\phi + \frac{1}{2}\left(U\xi^+ \partial_\nu \xi U^+ + U\xi^+ \xi \partial_\nu U^+ + U\xi \partial_\nu \xi^+ U^+ + U\partial_\nu U^+\right) U\phi$$

$$= U\partial_\nu \phi + (\partial_\nu U)\phi + \frac{U}{2}\left(\xi\partial_\nu \xi + \xi \partial_\nu \xi^+\right)\phi - (\partial_\nu U)\phi =$$

$$= U D_\nu \phi \quad \checkmark$$

In the case of L, R dependend of x the above construction will take care of $\partial_\nu U$ terms but not of $\partial_\nu L$ or $\partial_\nu R$ terms.

In order to deal with these we can insert by hand a l_ν, r_ν left handed current and right handed current respectively transforming as:

$$l_\nu \rightarrow L l_\nu L^+ + i(\partial_\nu L)L^+ \qquad r_\nu \rightarrow R r_\nu R^+ + i(\partial_\nu R)R^+$$

and substitute ∂_ν with $\partial_\nu + i l_\nu$ in the first factor and $\partial_\nu + i r_\nu$ in the second in that case

$$D_\nu \phi \mapsto \partial_\nu (U\phi) + \frac{1}{2}\left(U\xi^+ L^+ \left(\partial_\nu + i L l_\nu L^+ - (\partial_\nu L)L^+\right)(L\xi U^+) + \right. \quad \checkmark$$

$$\left. U\xi R^+ \left(\partial_\nu + i R r_\nu R^+ - (\partial_\nu R)R^+\right)(R\xi^+ U^+)\right) U\phi =$$

$$= UD_\nu \phi + \frac{1}{2}\left(U\xi^+ L^+ \partial_\nu L \xi U^+ - U\xi^+ L^+ (\partial_\nu L) L^+ L \xi U^+ + U\xi R^+ \partial_\nu R \xi^+ U^+ + \right.$$

$$\left. - U\xi R^+ \partial_\nu R R^+ R \xi^+ U^+\right) U\phi = U D_\nu \phi \quad \text{as required.} \ \checkmark$$

So we have a good covariant derivative to take care of unitary space dependent transformations on the light quarks.

The same holds also for the structure
$A_\nu = \frac{i}{2}\left[\xi^+ \partial_\nu \xi - \xi \partial_\nu \xi^+ \right]$ because each term will be separately covariant as before. ✓

Now we can specialize to a vectorial abelian group with charge matrix on the light quarks eQ $l_\nu = r_\nu = eQ$
(Notice that by now we are considering the heavy quarks unchanged)
Then we will have the following covariant derivative on the heavy meson H

$$\partial_\nu H + i H V_\nu =$$

$$= \partial_\nu H + \frac{1}{2} H \left(\xi^+ (\partial_\nu + i l_\nu) \xi + \xi (\partial_\nu + i r_\nu) \xi^+ \right) =$$

$$= \partial_\nu H - \frac{i}{2} H e B_\nu \left(\xi^+ Q \xi^{\checkmark} + \xi Q \xi^+ \right) - \frac{1}{2} H^{\checkmark} \left(\xi^+ \partial_\nu \xi + \xi \partial_\nu \xi^+ \right)$$

Instead $A_\nu = \frac{i}{2}\left(\xi^+ (\partial_\nu + i l_\nu) \xi - \xi (\partial_\nu + i r_\nu) \xi^+ \right) =$

$$= \frac{i}{2}\left(\xi^+ \partial_\nu \xi - \xi \partial_\nu \xi^+ \right) - \frac{eB_\nu}{2}\left(\xi^+ Q \xi - \xi Q \xi^+ \right)$$

Now we can remember that also the heavy quarks are charged and modify the covariant derivative ~~on~~ H accordingly:
$D_\nu H \hookrightarrow D_\nu H + i e B_\nu Q' H$ where Q' is the charge matrix ✓
for the heavy quarks. (Or we could have inserted currents for heavy quarks flavour unitary transformations specializing to the abelian case later)

This is the same as done in class:

$$D_\nu \xi = \partial_\nu \xi + ie\, B_\nu [Q, \xi]$$

$$V_\nu = \frac{i}{2}\left[\xi^+ D_\nu \xi + \xi (D_\nu \xi)^+\right] = \mathbb{V}_\nu - e B_\nu Q_\xi^+ + e B_\nu Q$$

$$A_\nu = \frac{i}{2}\left[\xi^+ D_\nu \xi - \xi (D_\nu \xi)^+\right] = \mathbb{A}_\nu - e B_\nu Q_\xi^-$$

where $Q_\xi^{\pm} = \frac{1}{2}\left(\xi^+ Q \xi \pm \xi Q \xi^+\right)$

And

$$D_\nu H = \partial_\nu H + ieB_\nu Q' H - ieB_\nu H Q + i \cancel{H} V_\nu =$$

$$= \partial_\nu H + ie B_\nu Q' H - ie \cancel{B_\nu} H Q - \frac{H}{2}\left[\xi \partial_\nu \xi + \xi \partial_\nu \xi^+\right]$$

$$- ie B_\nu \frac{H}{2}\left(\xi^+ Q \xi \pm \xi Q \xi^+\right) + ie \cancel{B_\nu} H Q =$$

$$= \partial_\nu H + ie B_\nu Q' H - \frac{H}{2}\left[\xi^+ \partial_\nu \xi + \xi \partial_\nu \xi^+\right] - ie B_\nu \frac{H}{2}\left(\xi^+ Q \xi + \xi Q \xi^+\right)$$

$$A_\nu = \frac{i}{2}\left[\xi^+ \partial_\nu \xi - \xi(\partial_\nu \xi)^+\right] - \frac{e B_\nu}{2}\left(\xi^+ Q \xi - \xi Q \xi^+\right)$$

which are exactly our results for the same quantities.

The Lagrangian is therefore:

$$\mathcal{L} = -i \,\mathrm{Tr}\, \bar{H}\, v_\nu D^\nu H + g_\pi \,\mathrm{Tr}\, \bar{H}_a H_b \gamma_\nu \gamma_5 A^\nu_{ba}$$

Problem 2 : form factors for $D^+ \to K^- \pi^+ \bar{e} \nu_e$

For $\underset{c\bar{d}}{\overset{\uparrow}{D^+}} \to \underset{s\bar{d}}{\overset{\uparrow}{K^-}} \pi^+ \bar{e}^- \nu_e$, the form factors are defined by

$$\langle \pi(p_\pi) K(p_K) | \bar{s} \gamma_\mu P_L c | D(p_D) \rangle = i w_+ \underset{(p_K+p_\pi)_\mu}{\underset{\uparrow}{P_\mu}} + i w_- \underset{(p_K-p_\pi)_\mu}{\underset{\uparrow}{Q_\mu}} + i r (p_D - P)_\mu + h \epsilon_{\mu\alpha\beta\gamma} p_D^\alpha P^\beta Q^\gamma \quad \text{--- ①}$$

Note that there'll be 2 factor when we ~~use our results in ① from $P_L = \frac{1}{2}(1-\gamma_5)$~~

So, consider the QCD operator $\bar{q}_a \gamma_\mu (1-\gamma_5) Q_v$, which transforms as $(\bar{3}_L, 1_R)$ under $SU(3)_L \times SU(3)_R$ chiral symmetry. In the chiral Lagrangian at zeroth order, this takes the form

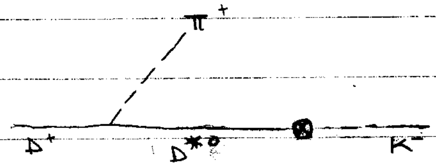
$$\bar{q}_a \gamma_\mu (1-\gamma_5) Q_v = \frac{a}{2} \mathrm{Tr}\, \gamma_\mu (1-\gamma_5) H_b \xi^{+}_{ba} \quad \text{--- ②}$$

(since $H \xi^+ \xrightarrow{SU(3)_L \times SU(3)_R} H U^+ U \xi^+ L^+ = (H\xi^+) L^+$.)

Now, $\xi = \sqrt{\Sigma} = \exp(iM/f) = 1 + iM/f - M^2/2f^2 + \ldots$ --- ③

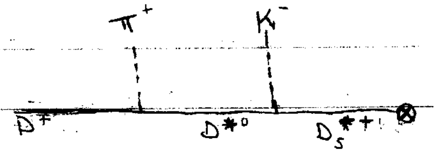
∴ the Feynman diagrams are :

a)

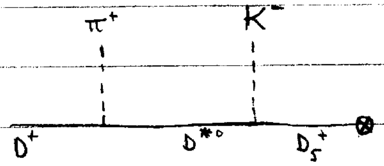


⊗ = current insertion

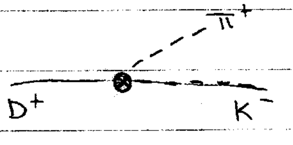
b)



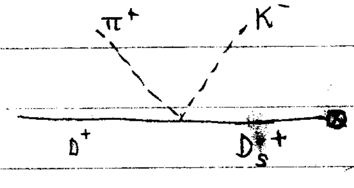
c)



d)



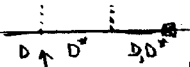
e)



b), c) & e) correspond to the 1 term in the expansion of ξ^+;
a) corresponds to the $-iM/f$ term & d) corresponds to the $-M^2/f^2$ term.

Note that the $PP^*\pi$ coupling comes from the g_π term in the chiral Lagrangian, i.e. expanding A_ν gave

$$\mathcal{L}_{int} = \left(\frac{2ig_\pi}{f} P_a^{*\nu\dagger} P_b \partial_\nu M_{ba} + h.c.\right) - \frac{2ig_\pi}{f} P_a^{*\alpha\dagger} P_b^{*\beta} \partial^\nu M_{ba} \varepsilon_{\alpha\lambda\beta\nu} v^\lambda \quad \ldots ④$$

Furthermore, the PPM coupling vanishes by parity:
hence the absence of  vertices, etc.
Other diagrams vanish due to charge conservation & strangeness conservation
e.g  $=0$ (violates charge cons.) &  $=0$
(violates strangeness cons.)

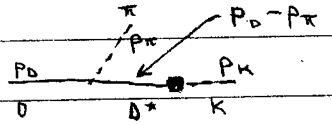
Consider the R.H.S. of ②

At $\mu = m_Q$, $\quad a = \sqrt{m_{P(Q)}}\, f_{P(Q)}$

Also, $\frac{a}{2}\mathrm{Tr}[\Gamma^\mu H] = a \times \begin{cases} -iv^\mu P_v^{(Q)} & \text{if } \Gamma^\mu = \gamma^\mu\gamma_5 \\ P_v^{*(Q)\mu} & \text{if } \Gamma^\mu = \gamma^\mu \end{cases} \quad \ldots ⑤$

by explicit evaluation of the trace. (i.e. $\frac{a}{2}\mathrm{Tr}[\gamma_\mu(1-\gamma_5)H] = a\left(P^{*(Q)}_{v\mu} + iv_\mu P_v^{(Q)}\right)$)

$\therefore$ we can evaluate the Feynman diagrams:

a) 

$$v\cdot p_D = -\frac{3}{4}\Delta \qquad \therefore \frac{1}{v\cdot p_D - v\cdot p_\pi - \frac{1}{4}\Delta} = -\frac{1}{v\cdot p_\pi + \Delta}$$

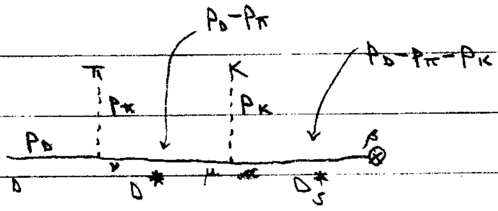
$\mathcal{O}(M)$ term of $J^\dagger$

$$= -\frac{\cancel{2}g_\pi}{f}(+ip_\pi^\nu)\frac{(+i\delta_{ab})(g_{\mu\nu} - v_\mu v_\nu)}{\cancel{2}(v\cdot p_\pi + \Delta)}\sqrt{m_D}\, f_D \sqrt{m_D}\left(\frac{-i}{f}\right)$$

(remembering that $|P^{(Q)}(p_P)\rangle = \sqrt{m_{P(Q)}}\left[|P^{(Q)}(v)\rangle + \mathcal{O}\left(\frac{1}{m_Q}\right)\right]$)

$$= -i\frac{f_D}{f^2}\frac{g_\pi m_D}{[v\cdot p_\pi + \Delta]} p_{\pi\mu} + i\frac{f_D}{f^2}\frac{g_\pi v\cdot p_\pi}{[v\cdot p_\pi + \Delta]} p_{D\mu} \quad \ldots ⑥$$

b)



$$= \frac{-\cancel{2}g_\pi}{f}\left(+ip_{\pi\nu}\right)\frac{(+i\delta_{ab})(g^{\nu\mu}-v^\nu v^\mu)}{\cancel{2}(v\cdot p_\pi+\Delta)}\left(\frac{+\cancel{2}g_\pi}{f}\right)\left(+ip_K^\gamma\,\varepsilon_{\alpha\lambda\mu\gamma}\,v^\lambda\right)$$

$$\times\frac{i\delta_{cd}(g^{\alpha\beta}-v^\alpha v^\beta)}{\cancel{2}[v\cdot(p_\pi+p_K)+\mu_s]}\sqrt{m_D}\,f_D\sqrt{m_D}$$

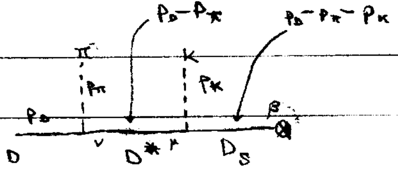
antisym.

0 due to $\varepsilon_{\alpha\lambda\mu\gamma}v^\lambda v^\mu$

$$= -\frac{f_D}{f^2}\frac{g_\pi^2\,m_D}{[v\,p_\pi+\Delta][v\cdot(p_\pi+p_K)+\mu_s]}\left(p_\pi^\mu-\cancel{v\cdot p_\pi v^\mu}\right)p_K^\gamma\,\varepsilon_{\alpha\lambda\mu\gamma}\,v^\lambda\left(g^{\alpha\beta}-\overset{0}{\cancel{v^\alpha v^\beta}}\right)$$

$$= -\frac{f_D}{f^2}\frac{g_\pi^2\,m_D}{[v\cdot p_\pi+\Delta][v\cdot(p_\pi+p_K)+\mu_s]}\,\varepsilon_{\alpha\lambda\mu\gamma}\,p_\pi^\mu\,p_K^\gamma\,v^\lambda\,g^{\alpha\beta} \qquad ⑦$$

c)

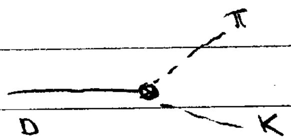


$$= \frac{-\cancel{2}g_\pi}{f}\left(+ip_{\pi\nu}\right)\frac{i(g^{\nu\mu}-v^\nu v^\mu)}{\cancel{2}(v\cdot p_\pi+\Delta)}$$

$$\times\left(\frac{-\cancel{2}g_\pi}{f}\right)\left(+ip_{K\mu}\right)\frac{i\quad -i}{\cancel{2}[v\cdot(p_\pi+p_K)\,-]}\sqrt{m_D}\,f_D\sqrt{m_D}\,(+iv_\beta)$$

$$= \frac{+if_D}{f^2}\frac{g_\pi^2\,m_D}{[v\cdot p_\pi+\Delta][v\cdot(p_\pi+p_K)\,-]}\left(p_\pi^\mu-v\cdot p_\pi v^\mu\right)\cdot p_{K\mu}\;v_\beta \qquad ⑧$$

d)



Here, we need the $-M^2/2f^2$ term of $\xi^\dagger$:

$$M^2 = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}$$

Since $P_b = [D^0 \; D^+ \; D_s^+]$, we want the $b=2$ components in

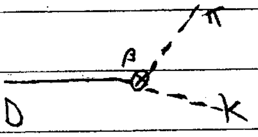
$$H_b \xi^+_{ba} = \dots + H_b \left(-\frac{M^2}{f^2}\right)_{ba} + \dots$$

i.e. the second row of M^2.

This is: $M^2_{2a} = \left((\dots) \quad (\dots) \quad (\pi^- K^+ + \dots) \right)$

(We're only interested in finding $(\pi^+)^\dagger (K^-)^\dagger$ terms, i.e. $\pi^- K^+$ terms)

So

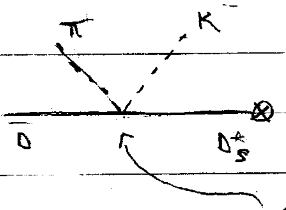


$$= \sqrt{m_D}\, f_D \sqrt{m_D} \left(+iv_\beta\right)\left(-\frac{1}{2f^2}\right)$$

$$\underline{\underline{= -\frac{i f_D}{2f^2} m_D v_\beta}} \qquad \dots ⑨$$

e)

$$\xi = 1 + iM/f - \frac{1}{2}M^2/f^2 + \ldots$$
$$\xi^\dagger = 1 - iM/f - \frac{1}{2}M^2/f^2 + \ldots$$



Comes from expanding $\mathbb{V}^\mu$ to second order in M

$$\mathbb{V}_\mu = \frac{i}{2}\left(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger\right)$$

$$\underset{\substack{2^{\text{nd}}\text{ order}\\\text{terms}}}{=} \frac{i}{2}\left(-\frac{iM}{f}\partial_\mu\left(\frac{iM}{f}\right) + 1\,\partial_\mu\left(-M^2/2f^2\right) + \frac{iM}{f}\partial_\mu\left(\frac{-iM}{f}\right) + 1\,\partial_\mu\left(-M^2/2f^2\right)\right)$$

$$= \frac{i}{f^2}\left(M\partial_\mu M - \frac{1}{2}\left(\partial_\mu M\, M + M\partial_\mu M\right)\right)$$

$$= \frac{i}{2f^2}\left(M\partial_\mu M - (\partial_\mu M) M\right) \quad \ldots\to ⑲$$

$$\text{So}\quad \mathcal{L} = -i\,\mathrm{Tr}\,\bar{H}_a v_\mu\left(\partial^\mu \delta_{ab} + i\mathbb{V}^\mu_{ba}\right) H_b + \ldots$$
$$= \ldots \cancel{-i}\,\mathrm{Tr}\,\bar{H}_a v^\mu \cancel{i}\, \frac{i}{2f^2}\left(M\partial_\mu M - (\partial_\mu M)M\right)_{ba} H_b + \ldots$$

$$\mathrm{Tr}\,\bar{H}H = \mathrm{Tr}\left(\left[P^{*(Q)\dagger}_{v\mu}\gamma^\mu + iP^{(Q)\dagger}_v\gamma_5\right]\underbrace{\frac{1+\not{v}}{2}\,\frac{1+\not{v}}{2}}_{\frac{1+\not{v}}{2}}\left[P^{*(Q)}_{v\nu}\gamma^\nu + iP^{(Q)}_v\gamma_5\right]\right)$$

$$= \frac{1}{2}\mathrm{Tr}\Big(iP^{(Q)\dagger}_v \cancelto{0}{\gamma_5 \not{v}\, iP^{(Q)}_v\gamma_5} + \cancel{iP^{(Q)\dagger}_v \gamma_5 \not{v}\, P^{*(Q)}_{v\nu}\gamma^\nu}$$
$$+ iP^{(Q)\dagger}_v\gamma_5\, iP^{(Q)}_v\gamma_5 + \cancel{iP^{(Q)\dagger}_v \gamma_5\, P^{*(Q)}_{v\nu}\gamma^\nu}$$
$$+ \cancel{P^{*(Q)\dagger}_{v\mu}\gamma^\mu \not{v}\, iP^{(Q)}_v\gamma_5} + \cancel{P^{*(Q)\dagger}_{v\mu}\gamma^\mu \not{v}\, P^{*(Q)}_{v\nu}\gamma^\nu}$$
$$+ \cancel{P^{*(Q)\dagger}_{v\mu}\gamma^\mu\, iP^{(Q)}_v\gamma_5} + P^{*(Q)\dagger}_{v\mu}\gamma^\mu\, P^{*(Q)}_{v\nu}\gamma^\nu \Big)$$

$$= \frac{1}{2}\mathrm{Tr}\left(-P^{(Q)\dagger}_v P^{(Q)}_v \mathbb{1} + P^{*(Q)\dagger}_{v\mu} P^{*(Q)}_{v\nu}\gamma^\mu\gamma^\nu\right)$$
$$= -2P^{(Q)\dagger}_v P^{(Q)}_v + 2P^{*(Q)\dagger}_{v\mu}\cdot P^{*(Q)\mu}_v$$

$$\therefore \mathcal{L} = \frac{2i}{f^2} P^{\dagger}_{a} P_{b} \left(M v\cdot\partial M - (v\cdot\partial M) M\right)_{ba} + \frac{2i}{f^2} P^{*\dagger\mu}_{a} P^{*\nu}_{b} g_{\mu\nu} \left(M v\cdot\partial M - (v\cdot\partial M)M\right)_{ba} \quad \text{---} ⑪$$

We are interested in $b=2$ (incoming D^+), $a=3$ (outgoing D_s)

$$(M v\cdot\partial M) = \Rightarrow \begin{pmatrix} \frac{\pi^0}{\sqrt{2}}+\frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}}+\frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & \frac{-2\eta}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \cdots & \cdots & \overset{\downarrow}{i v\cdot p_K K^+} \\ \cdots & \cdots & i v\cdot p_K K^0 \\ \cdots & \cdots & -\frac{2 i v\cdot p_\eta \eta}{\sqrt{6}} \end{pmatrix}$$

so

$$\begin{aligned}(M v\cdot\partial M)_{23} &= i v\cdot p_K \pi^- K^+ + \cdots \\ &= i v\cdot p_K (\pi^+)^\dagger (K^-)^\dagger + \cdots \end{aligned}$$

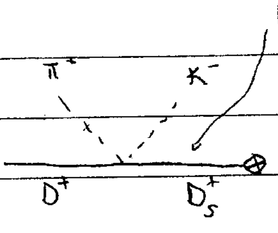
$\leftarrow$ outgoing π^+ & K^-, i.e. what we're interested in

$$\text{Similarly } (v\cdot\partial M) M = \Rightarrow \begin{pmatrix} \cdots & \cdots & \cdots \\ i v\cdot p_\pi \pi^- & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \cdots & \cdots & K^+ \\ \cdots & \cdots & K^0 \\ \cdots & \cdots & \underset{\uparrow}{\frac{-2\eta}{\sqrt{6}}} \end{pmatrix}$$

so

$$\begin{aligned}((v\cdot\partial M) M)_{23} &= i v\cdot p_\pi \pi^- K^+ + \cdots \\ &= i v\cdot p_\pi (\pi^+)^\dagger (K^-)^\dagger + \cdots \end{aligned}$$

So



$$= + \frac{i v\cdot}{f^2} (p_K - p_\pi) \sqrt{m_D} f_D \sqrt{m_D} \frac{-i}{2[v\cdot(p_\pi + p_K)]} (+i v^\mu)$$

$$= +i \frac{f_D}{2f^2} m_D \frac{v\cdot(p_K - p_\pi)}{v\cdot(p_\pi + p_K)} v^\mu \quad \text{---} ⑫$$

So putting all the pieces ⑥-⑨, ⑫ together:

$$\langle \pi^+ K^- | \bar{s}\gamma_\mu P_L c | D \rangle = -i \frac{f_D}{2f^2} \frac{g_\pi m_D}{[v\cdot p_\pi + \Delta]} p_{\pi\mu} + i\frac{f_D}{2f^2} \frac{g_\pi \, v\cdot p_\pi}{[v\cdot p_\pi + \Delta]} p_{D\mu}$$

$$- \frac{f_D}{2f^2} \frac{g_\pi^2}{[v\cdot p_\pi + \Delta][v\cdot(p_\pi + p_K) + \mu_s]} \varepsilon_{\mu\lambda\delta\gamma} p_\pi^\delta p_K^\gamma p_D^\lambda$$

$$+ \frac{i f_D}{2f^2} \frac{g_\pi^2}{[v\cdot p_\pi + \Delta][v\cdot(p_\pi + p_K)]} (p_\pi^\delta - v\cdot p_\pi v^\delta) p_{K\delta} p_{D\mu}$$

$$- \frac{i f_D}{4f^2} p_{D\mu}$$

$$+ \frac{i f_D}{4 f^2} \frac{v\cdot(p_K - p_\pi)}{v\cdot(p_\pi + p_K)} p_{D\mu} \qquad (13)$$

(using $p_D^\mu = m_D v^\mu$)

Then manipulate this so that we can compare it to ① & read off the form factors.

In ①,
$$h\,\epsilon_{\mu\alpha\beta\gamma} p_D^\alpha P^\beta Q^\gamma = h\,\epsilon_{\mu\alpha\beta\gamma} p_D^\alpha (p_K + p_\pi)^\beta (p_K - p_\pi)^\gamma$$
$$= h\,\epsilon_{\mu\alpha\beta\gamma} \left(\cancel{p_D^\alpha p_K^\beta p_K^\gamma} + p_D^\alpha p_\pi^\beta p_K^\gamma - p_D^\alpha p_K^\beta p_\pi^\gamma - \cancel{p_D^\alpha p_\pi^\beta p_\pi^\gamma} \right)$$
$$= 2h\,\epsilon_{\mu\alpha\beta\gamma} p_D^\alpha p_\pi^\beta p_K^\gamma$$

So
$$h = -\frac{f_D}{4f^2} \frac{g_\pi^2}{[v\cdot p_\pi + \Delta][v\cdot(p_\pi + p_K) + \mu_s]} \qquad (14)$$

Looking at what remains, coeff of $p_{D\mu}$ is ir

$$\Rightarrow \quad r = \frac{f_D}{2f^2} \frac{g_\pi \, v\cdot p_\pi}{[v\cdot p_\pi + \Delta]} + \frac{f_D}{2f^2} \frac{g_\pi^2}{[v\cdot p_\pi + \Delta][v\cdot(p_\pi + p_K)]} (p_\pi\cdot p_K - v\cdot p_\pi \, v\cdot p_K)$$
$$- \frac{f_D}{4f^2} + \frac{f_D}{4f^2} \frac{v\cdot(p_K - p_\pi)}{v\cdot(p_K + p_\pi)}$$
$$= \frac{f_D}{2f^2} \left[\frac{g_\pi}{[v\cdot p_\pi + \Delta]} \left(v\cdot p_\pi + \frac{g_\pi (p_\pi\cdot p_K - v\cdot p_\pi \, v\cdot p_K)}{[v\cdot(p_\pi + p_K)]} \right) + \frac{1}{2} \left(\frac{v\cdot(p_K - p_\pi)}{v\cdot(p_K + p_\pi)} - 1 \right) \right] \qquad (15)$$