Solution to Problem Ch 1 #2

A $N \times N$ unitary matrix can be written in the following form

$$U_{ij} = a_{ij} + ib_{ij}$$

So, there are N^2 of a's and N^2 of b's, and $2N^2$ of parameters altogether. Unitary condition is given by

$$\begin{aligned}(U^\dagger U)_{ij} &= U^\dagger_{ik} U_{kj} \\ &= (a_{ki} - ib_{ki})(a_{kj} + ib_{kj}) \\ &= a_{ki}a_{kj} + b_{ki}b_{kj} + i(a_{ki}b_{kj} - a_{kj}b_{ki}) = \delta_{ij}\end{aligned}$$

Hence

$$a_{ki}a_{kj} + b_{ki}b_{kj} = \delta_{ij} \tag{1}$$

$$a_{ki}b_{kj} - a_{kj}b_{ki} = 0 \tag{2}$$

Equation (1) is symmetric under i and j excahnge, so this equation actually shows $N+(N^2-N)/2$ independent conditions. Equation (2) is antisymmetric under i and j exchange, so this shows $(N^2-N)/2$ independent conditions. Hence, tere are

$$N + \frac{N^2 - N}{2} + \frac{N^2 - N}{2} = N^2$$

independent conditions in this unitary condition. Therefore, number of independent parameters in a $N \times N$ unitary matrix is

$$2N^2 - N^2 = N^2$$

Let us put a colum qurk matrix of up type quark and down type quarks as

$$u = \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_N \end{pmatrix}$$

and

$$d = \begin{pmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_N \end{pmatrix}$$

The charged current J_C^μ is given in this form

$$J_C^\mu = A\bar{u}_L \gamma^\mu V d_L$$

where A is a constant and V is the CKM matrix. We know from quantum mechanics that the phase of a wavefunction in not a measurable quantity. This situation is also true in quantum field theory. So, transformation

$$u_{1L} \to e^{i\phi_{u1}} u_{1L}$$

does not change observavles. In other words, this phase ϕ_{u1} is not a measurable quantity. There are N such quantities for up type quarks and also N for down type quarks, and $2N$ altogether. Defining matrices $M(u)$ and $M(d)$

$$M(u)_{ij} = e^{i\phi_{ui}} \delta_{ij}$$

$$M(d)_{ij} = e^{i\phi_{di}} \delta_{ij}$$

because the charged current should be invariant under this phase transformation, the CKM matrix transforms as follows

$$V \to M(u)^\dagger V M(d)$$

Hence

$$V_{ij} \to \exp[i(\phi_{ui} - \phi_{di})] V_{ij}$$

So, change of phase $(\phi_{ui} - \phi_{di})$ in the CKM matrix is unmeasurable. There are $2N - 1$ "phase differences" because there are $2N$ phases. Hence, actual number of free parameters in the CKM matrix is

$$N^2 - (2N - 1) = (N - 1)^2$$