Solution to Problem Ch 1 #5

Donoghue, Golowich and Holstein "Dynamics of the Standard Model" (Chaper I and Cahpter IV-1) is a good reference.
Effective Lagrangian of $SU(3)_L \times SU(3)_R$ chiral symmetry is given by (1.98)

$$\mathcal{L}_{eff} = \frac{f^2}{8}\mathrm{Tr}\partial^\mu\Sigma\partial_\mu\Sigma^+ + \text{higher derivative terms}$$

where (1.99)

$$\Sigma = \exp\left(\frac{2iM}{f}\right),$$

M represents meson fields (1.100) and f is a constant with dimensions of mass. By expanding $_sigma$, leading order of interacction terms is given by (1.107)

$$\begin{aligned} & \tfrac{1}{6f^2}\mathrm{Tr}[M,\partial_\mu M][M,\partial^\mu M] \\ = & \tfrac{1}{6f^2}\mathrm{Tr}[M(\partial_\mu M)M(\partial^\mu M) - M(\partial_\mu M)(\partial^\mu M)M \\ & -(\partial_\mu M)MM(\partial^\mu M) + (\partial_\mu M)M(\partial^\mu M)M] \end{aligned}$$

These terms show four meson interactions with vertex factor $\sim 1/f^2$, and two derivatives bring down two momentums. Hence, an amplitude of pi-pi scatteringsis given by (1.108)

$$\mathcal{M} = \frac{iq^2}{f^2}$$

where $q = (p_\pi^{final} - p_\pi^{initial})$.
In this problem, we set $m_\pi^+ = m_\pi^- = m$. In CM frame, initial 4-momentums are given by

$$\pi^+ : p_1 = (E, \vec{p}), \quad \pi^- : p_2 = (E, -\vec{p})$$

where $p = |\vec{p}| = \sqrt{E^2 - m^2}$. We set final 4-momentums

$$\pi^+ : p_3 = (E_3, \vec{p}_3), \quad \pi^- : p_4 = (E_4, \vec{p}_4)$$

We use Fermi's Golden Rule

$$d\sigma = |\mathcal{M}|^2 \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - (m_1 m_2)^2}} \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}(2\pi)^4\delta(p_1+p_2-p_3-p_4)$$

where $|\mathcal{M}|^2$ is given by

$$|\mathcal{M}|^2 = \frac{q^4}{f^4} = \frac{(p_3 - p_1)^4}{f^4} = \frac{(0 - |\vec{p}_3 - \vec{p}|)^4}{f^4} = \frac{|\vec{p}_3 - \vec{p}|^4}{f^4}$$

and

$$\begin{aligned} 4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2} &= \sqrt{(E^2 + p^2)^2 - m^4} = \sqrt{(2E^2 - m^2)^2 - m^4} \\ &= 2E\sqrt{E^2 - m^2} \end{aligned}$$

These has nothing to do with p_3 and p_4 integration. Delta function may be splitted as

$$\delta(p_1 + p_2 - p_3 - p_4) = \delta(2E - E_3 - E_4)\delta(0 - \vec{p}_3 - \vec{p}_4)$$

Therfore,

$$\begin{aligned} d\sigma &= \tfrac{|\vec{p}_3 - \vec{p}|^4}{f^4} \tfrac{1}{8E\sqrt{E^2 - m^2}} \tfrac{1}{4(2\pi)^2} \tfrac{d^3\vec{p}_3}{\sqrt{p_3^2 + m^2}} \tfrac{d^3\vec{p}_4}{\sqrt{p_4^2 + m^2}} \\ &\quad \times \delta\left(2E - \sqrt{p_3^2 + m^2} - \sqrt{p_4^2 + m^2}\right) \delta(0 - \vec{p}_3 - \vec{p}_4) \end{aligned}$$

Integration by $\vec{p}_4$ just replaces $\vec{p}_4$ by $\vec{p}_3$ and gives

$$d\sigma = \frac{|\vec{p}_3 - \vec{p}|^4}{128\pi^2 f^4 E\sqrt{E^2 - m^2}} \frac{d^3\vec{p}_3}{\sqrt{p_3^2 + m^2}} \frac{\delta\left(2E - 2\sqrt{p_3^2 + m^2}\right)}{\sqrt{p_3^2 + m^2}}$$

Using following relations

$$d^3\vec{p}_3 = p_3^2 dp_3 d\Omega$$

$$\delta\left(2E - 2\sqrt{p_3^2 + m^2}\right) = \frac{1}{2}\delta\left(E - \sqrt{p_3^2 + m^2}\right)$$

and

$$E_3 = \sqrt{p_3^2 + m^2} \Rightarrow dE_3 = \frac{p_3 dp_3}{\sqrt{p_3^2 + m^2}}$$

$d\sigma$ is given by

$$d\sigma = \frac{|\vec{p}_3 - \vec{p}|^4}{256\pi^2 f^4 E\sqrt{E^2 - m^2}} \frac{\delta(E - E_3)\sqrt{E_3^2 - m^2}}{E_3} dE_3 d\Omega$$

Intrgration by E_3 just replaces E_3 by E, and

$$E = \sqrt{p^2 - m^2} = E_3 = \sqrt{p_3^2 - m^2}$$
$$\Rightarrow |\vec{p}_3| = |\vec{p}| = p$$

then

$$|\vec{p}_3 - \vec{p}|^2 = p^2 + p^2 - 2p^2 \cos\theta = 2(E^2 - m^2)(1 - \cos\theta)$$

wehre θ is the angle between initial and final π^+. Hence

$$\frac{d\sigma}{d\Omega} = \frac{(E^2 - m^2)^2(1 - \cos\theta)^2}{64\pi^2 f^4 E\sqrt{E^2 - m^2}} \frac{\sqrt{E^2 - m^2}}{E}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{(E^2 - m^2)^2(1 - \cos\theta)^2}{64\pi^2 f^4 E^2}$$

Integration by ϕ gives 2π, and because

$$(1-\cos\theta)^2 = 1-2\cos\theta+\cos^2\theta = 1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta) = \frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta$$

integration by θ gives $3\pi/2$. Finally,

$$\sigma = \frac{(E^2 - m^2)^2}{64\pi^2 f^4 E^2} \cdot 2\pi \cdot \frac{3\pi}{2} = \frac{3(E^2 - m^2)^2}{64 f^4 E^2}$$