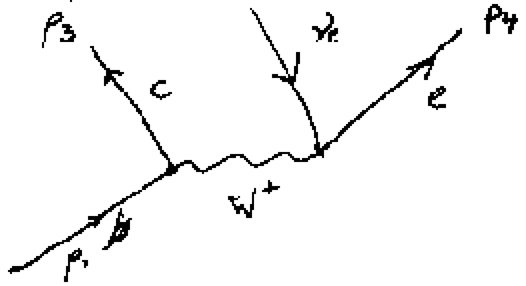
$\boxed{b \to c e \bar{\nu}_e}$



$$\mathcal{M} = \frac{g_w^2}{8(M_W c)^2}\left[\bar{u}(3)\gamma^\mu(1-\gamma^5)u(1)\right]\left[\bar{u}(4)\gamma_\mu(1-\gamma^5)u(2)\right]$$

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_w}{M_W c}\right)^4 (p_1\cdot p_2)(p_3\cdot p_4)$$

In b's rest-frame: $p_1 = (m_b c, 0)$

Therefore, $p_1\cdot p_2 = m_b E_2$

Since $p_1 = p_2 + p_3 + p_4$,

$$\begin{aligned}(p_3+p_4)^2 &= p_3^2 + p_4^2 + 2p_3\cdot p_4\\ &= m_c^2c^2 + m_e^2c^2 + 2p_3\cdot p_4\\ &= (p_1 - p_2)^2\\ &= p_1^2 + p_2^2 - 2p_1\cdot p_2\\ &= m_b^2c^2 - 2p_1\cdot p_2\\ &= m_b^2c^2 - 2m_bE_2\end{aligned}$$

Therefore, $p_3\cdot p_4 = \dfrac{(m_b^2 - m_c^2 - m_e^2)c^2}{2} - m_b E_2$

Set $m_e = 0$.

Then, $\langle|\mathcal{M}|^2\rangle = \left(\dfrac{g_w}{M_W c}\right)^4 m_b E_2\left((m_b^2 - m_c^2)c^2 - 2m_b E_2\right)$

Using Fermi's Golden Rule:

$$d\Gamma = \frac{\langle|\mathcal{M}|^2\rangle}{2\hbar m_b}\left(\frac{c\,d^3\vec{p}_2}{(2\pi)^3 2E_2}\right)\left(\frac{c\,d^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c\,d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)(2\pi)^4\delta^4(p_1-p_2-p_3-p_4)$$

where $E_2 = |\vec{p}_2|c$ and with our approximation $m_e \approx 0$, $E_4 = |\vec{p}_4|c$.

We have: $\delta(p_1 - p_2 - p_3 - p_4) = \delta\left(m_b c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right) \delta^3(\vec{p}_2 + \vec{p}_3 + \vec{p}_4)$

Performing the $\vec{p}_3$ integral gives:

$$d\Gamma = \frac{\langle |\mathcal{M}|^2 \rangle c^3}{16 (2\pi)^5 \hbar m_e} \frac{(d^3\vec{p}_2)(d^3\vec{p}_4)}{E_2 E_3 E_4} \delta\left(m_b c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right)$$

Since the b is at rest, from momentum conservation, we have:

$$\vec{p}_3 = \vec{p}_2 + \vec{p}_4 ;$$

thus, $|\vec{p}_3|^2 = \left(\frac{E_3}{c}\right)^2 - m_e^2 c^2 = |\vec{p}_2 + \vec{p}_4|^2 = \vec{p}_2^{\,2} + p_4^2 + 2\vec{p}_2 \cdot \vec{p}_4$

$$= \frac{1}{c^2}\left(E_2^2 + E_4^2 + 2 E_2 E_4 \cos\theta\right)$$

Also, $d^3\vec{p}_2 = \left(\frac{E_2}{c}\right)^2 \frac{dE_2}{c} \sin\theta \, d\theta \, d\phi$

Let $x \equiv \frac{E_3}{c} = \frac{1}{c}\sqrt{E_2^2 + E_4^2 + 2 E_2 E_4 \cos\theta + m_e^2 c^4}$

Then $dx = -\frac{E_2 E_4 \sin\theta \, d\theta}{c E_3}$,

And carrying out the E_3 integration:

$$\int_0^\pi \frac{\sin\theta \, d\theta}{E_3} \delta\left(m_b c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right)$$

$$= \frac{+c}{E_2 E_4} \int_{x_-}^{x_+} dx \, \delta\left(m_b c - x - \frac{E_2}{c} - \frac{E_4}{c}\right)$$

At $\begin{cases} \theta = \pi, & x \equiv x_- = \frac{1}{c}\sqrt{E_2^2 + E_4^2 - 2E_2 E_4 + m_e^2 c^4} \\ \theta = 0, & x \equiv x_+ = \frac{1}{c}\sqrt{E_2^2 + E_4^2 + 2E_2 E_4 + m_e^2 c^4} \end{cases}$

Therefore,

$$\int_0^\pi \frac{\sin\theta \, d\theta}{E_3} \delta\left(m_b c - \frac{E_2}{c} - \frac{E_3}{c} - \frac{E_4}{c}\right) = \begin{cases} \frac{c}{E_2 E_4}, & \text{for } x_- < m_b c - \frac{E_2}{c} - \frac{E_4}{c} < x_+ \\ 0, & \text{otherwise.} \end{cases}$$

The inequality $x_- < m_b c - \frac{E_2}{c} - \frac{E_4}{c} < x_+$ yields:

$$E_2^2 + E_4^2 - 2E_2E_4 + m_c^2c^4 < \left(m_b^2c^4 + E_2^2 + E_4^2 - 2m_bc^2E_2 - 2m_bc^2E_4 + 2E_2E_4 \right)$$

and

$$E_2^2 + E_4^2 + 2E_2E_4 + m_c^2c^4 > \left(m_b^2c^4 + E_2^2 + E_4^2 - 2m_bc^2E_2 - 2m_bc^2E_4 + 2E_2E_4 \right)$$

Thus, $2m_bc^2(E_2+E_4) > (m_b^2 - m_c^2)c^4$

and $2m_bc^2(E_2+E_4) - 4E_2E_4 < (m_b^2 - m_c^2)c^4$

So, $$E_2 < \frac{m_bc^2}{2} - \left(\frac{m_c^2c^4}{2m_bc^2 - 4E_4} \right)$$

$$= \frac{m_bc^2}{2}\left[1 - \frac{m_c^2c^4}{m_b^2c^4}\left(1 - 2\frac{E_4}{m_bc^2}\right)^{-1} \right]$$

Approx. not allowed because for $b \to c e \nu_e$ $E_\nu \simeq E_e$ in b rest-frame (see next)

$$\simeq \frac{m_bc^2}{2}\cancel{\left[1 - \frac{m_c^2c^4}{m_b^2c^4}\left(1 + 2\frac{E_4}{m_bc^2}\right) \right]}$$

And $$E_2 > \left(\frac{m_bc^2}{2} - \frac{m_c^2c^2}{2m_b} \right) - E_4$$

Having performed the θ and ϕ integrals:

$$d\Gamma = \frac{2\pi \langle |\mathcal{M}|^2 \rangle c^3}{16(2\pi)^5 \hbar m_c} \left(\frac{E_2^2}{c^2} \right) \frac{dE_2 \, d^3\vec{p}_4}{E_2^2 E_4^2}$$

$$= \frac{\langle |\mathcal{M}|^2 \rangle c}{(4\pi)^4 \hbar m_c} \, dE_2 \, \frac{d^3\vec{p}_4}{E_4^2}$$

Carrying the E_2 integral:

$$E_{max} = \frac{m_b c^2}{2} - \left(\frac{m_c^2 c^4}{2m_b c^2 - 4E_4}\right)$$

$$d\Gamma = \left(\frac{g\omega}{4\pi M_w c}\right)^4 \frac{m_b c}{\hbar m_e} \frac{d^3\vec{p}_4}{E_4^2} \int_{min = \left(\frac{m_b c^2}{2} - \frac{m_c^2 c^2}{2m_b}\right) - E_4} E_2\left[(m_b^2 - m_c^2)c^2 - 2m_b E_2\right] dE_2$$

where $\dfrac{d^3\vec{p}_4}{E_4^2} = \dfrac{1}{\cancel{E_4^2}}\left(4\pi\left(\dfrac{E_4}{c}\right)^2 \dfrac{dE_4}{c}\right) = \dfrac{4\pi}{c^3}\, dE_4$

and $\displaystyle\int E_2\left[(m_b^2 - m_c^2)c^2 - 2m_b E_2\right] dE_2$

$$= \left[E_2^2\left(\left(\frac{m_b^2 - m_c^2}{2}\right)c^2 - \frac{2}{3} m_b E_2\right)\right]_{min}^{max}$$

$$E^2_{2\,max} = \frac{m_b^2 c^4}{4} - m_b c^2 \left(\frac{m_c^2 c^4}{2m_b c^2 - 4E_4}\right) + \left(\frac{m_c^2 c^4}{2m_b c^2 - 4E_4}\right)^2$$

$$\left(\frac{m_b^2 - m_c^2}{2}\right)c^2 - \frac{2}{3} m_b E_{2\,max} = \left(\frac{m_b^2 - m_c^2}{2}\right)c^2 - \frac{1}{3} m_b^2 c^2 + \frac{2}{3}\left(\frac{m_b m_c^2 c^4}{2m_b c^2 - 4E_4}\right)$$

$$\boxed{\text{let } \chi = 2m_b c^2 - 4E_4 \quad \Rightarrow \quad dE_4 = -\frac{1}{4}\, d\chi}$$

$$E_{2\,min} = \left(\frac{m_b c^2}{2} - \frac{m_c^2 c^2}{2m_b}\right) - E_4 = \frac{1}{4}\left(2m_b c^2 - 4E_4\right) - \frac{m_c^2 c^2}{2m_b}$$

$$= \frac{1}{4}\chi - \frac{m_c^2 c^2}{2m_b}$$

$$E^2_{2\,min} = \frac{1}{4}\left(\frac{1}{4}\chi^2 - \frac{m_c^2 c^2}{m_b}\chi + \frac{m_c^4 c^4}{m_b^2}\right)$$

$$\left(\frac{m_b^2 - m_c^2}{2}\right)c^2 - \frac{2}{3} m_b E_{2\,min} = \left(\frac{m_b^2 - m_c^2}{2}\right)c^2 - \frac{2}{3} m_b\left(\frac{1}{4}\chi - \frac{m_c^2 c^2}{2m_b}\right)$$

$$\Big[\ \Big]_{min}^{max} \rightarrow \left(\frac{m_b^2 c^4}{4} - \frac{m_b^2 c^2 m_c^2 c^4}{\chi} + \frac{m_c^4 c^8}{\chi^2}\right)\left(\left(\frac{2m_b^2 - 3m_c^2}{6}\right)c^2 + \frac{2}{3}\frac{m_b m_c^2 c^4}{\chi}\right)$$

$$- \frac{1}{4}\left(\frac{1}{4}\chi^2 - \frac{m_c^2 c^2}{m_b}\chi + \frac{m_c^4 c^4}{m_b^2}\right)\left(\left(\frac{m_b^2 - m_c^2}{2}\right)c^2 - \frac{2}{3} m_b\left(\frac{1}{4}\chi - \frac{m_c^2 c^2}{2m_b}\right)\right)$$

$$= \frac{1}{\chi^3}\left(\frac{2}{3} m_b m_c^6 c^{12}\right) + \frac{1}{\chi^2}\left(-\frac{2m_b^2 m_c^4 - 3m_c^6}{6}\right)c^{10}$$

$$+ \frac{1}{\chi}\left(\frac{-m_b^3 m_c^2 c^8 + 3 m_b m_c^4 c^8}{6}\right) + \frac{\chi}{4}\left(\frac{m_c^2 c^2}{m_b}\left(\frac{3m_b^2 - 2m_c^2}{6}\right)c^2\right)$$

$$- \frac{1}{16}\chi^2\left(\frac{3m_b^2 + 2m_c^2}{6}\right)c^2 + \frac{\chi^3}{96} - \frac{m_c^4 c^4}{4m_b^2}\left(\frac{m_b^2 + m_c^2}{6}\right)c^2$$

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$$\Rightarrow \int_{\chi_{min}} \left[\quad\right] d\chi$$

$$= \left[-\frac{1}{\cancel{4}\chi^2}\left(\frac{2}{3} m_b m_c^6 c^{12}\right) + \frac{1}{\chi}\frac{m_c^4 c^6}{3}\left(\frac{2m_b^2+3m_c^2}{2}\right)c^2 \right.$$

$$+ (\ln\chi)\,\frac{m_c^2 c^2}{6}\left(3m_b m_c^2 c^6 - m_b^3 c^6\right)$$

$$+ \frac{1}{8}\chi^2\left(\frac{m_c^2 c^2}{6 m_b}\left(3m_b^2 - 2m_c^2\right)c^2\right) - \frac{1}{16}\frac{\chi^3}{3}\left(\frac{3m_b^2+2m_c^2}{6}\right)c^2$$

$$\left. + \frac{1}{96}\frac{\chi^4}{4} - \frac{m_c^4 c^4}{4 m_b^2}\left(\frac{m_b^2+m_c^2}{6}\right)c^2\chi \right]_{\chi_{min}}^{\chi_{max}}$$

Recall: $\chi = 2m_b c^2 - 4E_4$

so $\chi_{max} = 2m_b c^2$

and $\chi_{min} = 0$.

$$\Gamma = -\frac{1}{4}\left(\frac{1}{4\pi c^2}\right)^3\left(\frac{g_w}{M_W}\right)^4 \frac{m_b}{m_e\hbar}\Big[\quad\Big]_{min}^{max}.$$