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Global Fit for Branching Fractions and Form Factor Parameters of $B \to D^{(*)}\ell\nu$ Decays

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Abstract

We present details of our Global fit analysis in $B \to D^{(*)}\ell\nu$ decays. We use 86×10^6 $B\bar{B}$ events collected by the Babar detector. The primary target of this analysis is to determine the branching fractions and form factor parameters of $B \to D\ell\nu$ and $B \to D^*\ell\nu$ decays and to extract $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$.

Main changes from previous version

- Final F(1) from unquenched lattice QCD.

Contents

List of Figures

List of Tables

1 Introduction

The Cabbibo-Kobayashi-Maskawa (CKM) Matrix [1][2] plays an important role in quark mixing and CP violation. The matrix elements of it are free parameters of the Standard Model of particle physics. They need to be determined by experiments. As a B-factory, Babar can measure some of the CKM matrix elements, such as $|V_{ub}|$ and $|V_{cb}|$, in a good precision. To extract $|V_{cb}|$, semileptonic $B \to X_c \ell \nu$ decays have been used. Here, X_c denotes a charmed meson or a particle system including a charmed meson, and ℓ an electron or a muon. Especially two exclusive decay modes, $B \to D^* \ell \nu$ and $B \to D \ell \nu$ have been very useful. A study of these decay modes are also important to measure $|V_{ub}|$ because these are the biggest background of $B \to X_u \ell \nu$ decays. However, there are still unresolved problems in these decay modes such as

- $B^0 \to D^{*-} \ell^+ \nu$ decay [1] branching fraction measurements disagreement.
- Discrepancy between inclusive and sum of exclusive branching fractions.
- $B \to D \ell \nu$ decay form factor slope is not well measured.
- HQET inspired model for $B \to D^{**} \ell \nu$ decays has not been used to evaluate the contribution from these decays.

In existing measurements, each decay mode is reconstructed exclusively to measure branching fractions or form factor parameters. However, the above problems are related with each other and cannot be easily solved by looking at a single decay mode. It is better to look at all semileptonic $B \to D^{(*,**)}(\pi) \ell \nu$ decay modes at the same time. We use a global fit method which simultaneously determine contributions from D, D^* and $D^{**}(D^{(*)}\pi)$. One analysis from Babar [3][4] has similar goals, and can also reconstruct the full set of $B \to D^{(*,**)}(\pi) \ell \nu$ decays. The analysis is complementary because the ability to distinguish among the different channels is better than in our work but the statistics are much lower since they use hadronic B-tag. Another important point of our fitting method is an ability to access form factor parameters. We implement this by Monte Carlo re-weighting method as explained later. Since it is not easy to determine a correct contribution from a decay mode without using a sufficient form factor, an ability to simultaneously measure branching fractions and form factor parameters is essential.

1.1 Problems in $B \to D^{(*,**)}(\pi) \ell \nu$ decays

There are still problems in semileptonic B to charmed meson decays even though many measurements have been done. We list important ones below.

- $B^0 \to D^{*-} \ell^+ \nu$ decay branching fraction measurements disagreement
  There ia s disagreement between $B^0 \to D^{*-} \ell^+ \nu$ decay branching fraction measurements. Those are summarized in Table 1. The results vary from 0.0459 (BELLE) to 0.0609 (CLEO), and those measurements are not consistent with each other.

[1] Charge conjugate modes are implied throughout this text

- Inclusive-exclusive disagreement
  There are mainly 10 exclusive decay modes of the type $B \to X_c \ell\nu$ (see Appendix A)

  - $B \to D\ell\nu$
  - $B \to D^*\ell\nu$
  - $B \to D_0^*\ell\nu$
  - $B \to D_1\ell\nu$
  - $B \to D_1'\ell\nu$
  - $B \to D_2^*\ell\nu$
  - $B \to D\pi^+\ell\nu$ (non-resonant)
  - $B \to D\pi^0\ell\nu$ (non-resonant)
  - $B \to D^*\pi^+\ell\nu$ (non-resonant)
  - $B \to D^*\pi^0\ell\nu$ (non-resonant)

  The D_0^*, D_1, D_1' and D_2^* are called D^{**} as a whole (see Appendix B).
  The sum of the branching fractions of these exclusive decays is supposed to be consistent with the inclusive $B \to X_c\ell\nu$ branching fractions (this X_c denotes a particle system including a charm quark such as D^* or $D\pi$ etc.). Those branching fractions are summarized in Table 2. In this table, non-resonant $B \to D^{(*)}\pi\ell\nu$ decays and $B \to D^{**}\ell\nu$ decays are combined (see Appendix D). The sum of exclusive modes is consistent with the inclusive branching fraction in case of B^+ decays. However, there is a disagreement in B^0 decays. If $\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)$ is as large as to satisfy isospin symmetry with $\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$, the inclusive and exclusive branching fractions agree. Note that the decay modes which include more than 2 pions such as $D^*\pi\pi\ell\nu$ mode or include D_s such as $D_sK\ell\nu$ mode are not included in the table; those contributions may fill the inclusive exclusive difference.

- $B \to D\ell\nu$ decay form factor slope and $G(1)|V_{cb}|$
  $B \to D\ell\nu$ decay form factor slope, ρ_D^2 (see Appendix I), has never been well measured. Existing measurements are listed in Table 3. The best measurement from BELLE has a 23 % uncertainty. Consequently, the uncertainty on $G(1)|V_{cb}|$ is much larger than $F(1)|V_{cb}|$. More precise measurement is necessary also to address the above issues.

- HQET inspired model for $B \to D^{**}\ell\nu$ decays.
  HQET inspired model for $B \to D^{**}\ell\nu$ decays (see Appendix I) is to describe these decays better. But, the HQET model has not been used in existing measurements.

In this analysis, we address these problems. We use a global fit technique to determine relative exclusive branching fractions, $\mathcal{B}(B \to D\ell\nu)$, $\mathcal{B}(B \to D^*\ell\nu)$ and

Branching Fraction	Experiment (year)	Decay mode
$0.0459 \pm 0.0023 \pm 0.0040$	BELLE (2002) [5]	$B^0 \to D^{*-}\ell^+\nu$
$0.0470 \pm 0.0013^{+0.0036}_{-0.0031}$	DELPHI (2001) [6]	$B^0 \to D^{*-}\ell^+\nu$
$0.0490 \pm 0.0007^{+0.0036}_{-0.0035}$	BABAR (2005) [7]	$B^0 \to D^{*-}\ell^+\nu$
$0.0526 \pm 0.0020 \pm 0.0046$	OPAL (2000) [8]	$B^0 \to D^{*-}\ell^+\nu$
$0.0553 \pm 0.0026 \pm 0.0052$	ALEPH (1997) [9]	$B^0 \to D^{*-}\ell^+\nu$
$0.0590 \pm 0.0022 \pm 0.0050$	DELPHI (2004) [10]	$B^0 \to D^{*-}\ell^+\nu$
$0.0609 \pm 0.0019 \pm 0.0040$	CLEO (2003) [11]	$B^0(B^+) \to D^{*-}(\bar{D}^{*0})\ell^+\nu$

Table 1: [$B^0 \to D^*\ell\nu$ **BF**] $B^0 \to D^{*-}\ell^+\nu$ branching fractions from different experiments. The first uncertainty is statistical and the second one is systematic. In the most right column, the decay modes measured in each analysis are listed. Only CLEO (2003) measured both $B^0 \to D^{*-}\ell^+\nu$ and $B^+ \to \bar{D}^{*0}\ell^+\nu$ modes.

$\mathcal{B}(B \to D^{**}\ell\nu) + \mathcal{B}(B \to D^{(*)}\pi\ell\nu)$, and form factor parameters of $B \to D\ell\nu$ and $B \to D^*\ell\nu$ decays. Using these results along with $\mathcal{G}(1)$ and $\mathcal{F}(1)$, we can also extract $|V_{cb}|$ as explained in section 20. We do not explicitly reconstruct D^* or D^{**}. Thus, we do not have to rely on slow pion reconstruction. We also do not have to determine the efficiency to reconstruct and detect a specific decay mode, wich is usually done by Monte Carlo simulation (MC). This is a different approach from the existing measurements listed above. We, however, depend on MC to construct the probability density functions (PDF) to be used in the fit. Details of our fit technique is explained in section 3. Since we use a different approach, our results are useful to address the above problems.

Decay mode (B^+)	Branching Fraction (%)	Reference
$B^+ \to \bar{D}^0\ell^+\nu$	2.15 ± 0.22	[12]
$B^+ \to \bar{D}^{*0}\ell^+\nu$	6.5 ± 0.5	[12]
$B^+ \to D^-\pi^+\ell^+\nu$	0.53 ± 0.10	[12]
$B^+ \to \bar{D}^0\pi^0\ell^+\nu$	0.265 ± 0.05	Isospin symmetry
$B^+ \to D^{*-}\pi^+\ell^+\nu$	0.64 ± 0.15	[12]
$B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu$	0.32 ± 0.075	Isospin symmetry
Sum of exclusive modes	10.41 ± 0.61	
Inclusive $B^+ \to X_c\ell^+\nu$	10.8 ± 0.4	[12]
Decay mode (B^0)	Branching Fraction (%)	Reference
$B^0 \to D^-\ell^+\nu$	2.13 ± 0.18	[12]
$B^0 \to D^{*-}\ell^+\nu$	5.20 ± 0.24	[12]
$B^0 \to \bar{D}^0\pi^-\ell^+\nu$	0.32 ± 0.095	[12]
$B^0 \to D^-\pi^0\ell^+\nu$	0.16 ± 0.0475	Isospin symmetry
$B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu$	0.65 ± 0.16	[12]
$B^0 \to D^{*-}\pi^0\ell^+\nu$	0.325 ± 0.08	Isospin symmetry
Sum of exclusive modes	8.79 ± 0.41	
Inclusive $B^0 \to X_c\ell^+\nu$	10.1 ± 0.4	[12]

Table 2: [**Exclusive-Inclusive comparison**] Branching fractions of $B \to X_c\ell\nu$ type of decays. $B \to D^{(*)}\pi\ell\nu$ decays include not only non-resonant $B \to D^{(*)}\pi\ell\nu$ decays but also $B \to D^{**}\ell\nu$ decays.

ρ_D^2	Experiment (year)	Decay mode
$0.97 \pm 0.98 \pm 0.38$	ALEPH (1997) [9]	$B^0 \to D^-\ell^+\nu$
$1.12 \pm 0.22 \pm 0.14$	BELLE (2002) [13]	$B^0 \to D^-\ell^+\nu$
$1.27 \pm 0.25 \pm 0.14$	CLEO (1999) [14]	$B^0 \to D^-\ell^+\nu$ and $B^+ \to \bar{D}^0\ell^+\nu$

Table 3: [$B \to D\ell\nu$ **FF slope**] Form factor slope ρ_D^2 from different experiments. The first uncertainty is statistic and the second one is systematic.

2 Signal and Background Events

2.1 Signal

We are interested in $B \to D^{(*,**)}(\pi)\ell\nu$ decays. We reconstruct either D^0 or D^+ and a lepton (electron or muon) to form a Dl candidate as a candidate of a B meson. We do not explicitly reconstruct D^* or D^{**}. However, we can access those higher charmed meson states because, as described in Appendix A, D^* and D^{**} eventually decay into either D^0 or D^+. We separate decay modes into two categories: the decay modes to produce D^0 and the ones to produce D^+. Contributions of each component can be seen in Fig 1.

2.1.1 D^0 modes

The decay processes to produce D^0 are

- $B^+ \to \bar{D}^0\ell^+\nu$
- $B^+ \to \bar{D}^{*0}\ell^+\nu$ with $\bar{D}^{*0} \to \bar{D}^0\pi^0$ or $\bar{D}^{*0} \to \bar{D}^0\gamma$
- $B^0 \to D^{*-}\ell^+\nu$ with $D^{*-} \to \bar{D}^0\pi^-$
- $B^+ \to \bar{D}^{**0}\ell^+\nu$ with $\bar{D}^{**0} \to D^0\pi^0$
  $B^+ \to \bar{D}^{**0}\ell^+\nu$ with $\bar{D}^{**0} \to D^{*0}\pi^0$ and $D^{*0} \to \bar{D}^0\pi^0$ or $D^{*0} \to \bar{D}^0\gamma$
  $B^+ \to \bar{D}^{**0}\ell^+\nu$ with $\bar{D}^{**0} \to D^{*-}\pi^+$ and $D^{*-} \to \bar{D}^0\pi^-$
- $B^0 \to D^{**-}\ell^+\nu$ with $D^{**-} \to \bar{D}^0\pi^-$
  $B^0 \to D^{**-}\ell^+\nu$ with $D^{**-} \to \bar{D}^{*0}\pi^-$ and $\bar{D}^{*0} \to \bar{D}^0\pi^0$ or $\bar{D}^{*0} \to \bar{D}^0\gamma$
  $B^0 \to D^{**-}\ell^+\nu$ with $D^{**-} \to D^{*-}\pi^0$ and $D^{*-} \to \bar{D}^0\pi^-$
- $B^+ \to \bar{D}^0\pi^0\ell^+\nu$
- $B^0 \to \bar{D}^0\pi^-\ell^+\nu$
- $B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu$ with $\bar{D}^{*0} \to \bar{D}^0\pi^0$ or $\bar{D}^{*0} \to \bar{D}^0\gamma$
  $B^+ \to D^{*-}\pi^+\ell^+\nu$ with $D^{*-} \to \bar{D}^0\pi^-$
- $B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu$ with $\bar{D}^{*0} \to \bar{D}^0\pi^0$ or $\bar{D}^{*0} \to \bar{D}^0\gamma$
  $B^0 \to D^{*-}\pi^0\ell^+\nu$ with $D^{*-} \to \bar{D}^0\pi^-$

2.1.2 D^+ modes

The decay processes to produce D^+ are

- $B^0 \to D^-\ell^+\nu$
- $B^0 \to D^{*-}\ell^+\nu$ with $D^{*-} \to D^-\pi^0$ or $D^{*-} \to D^-\gamma$
- $B^0 \to D^{**-}\ell^+\nu$ with $D^{**-} \to D^-\pi^0$
  $B^0 \to D^{**-}\ell^+\nu$ with $D^{**-} \to D^{*-}\pi^0$ and $D^{*-} \to D^-\pi^0$ or $D^{*-} \to D^-\gamma$

- $B^+ \to \bar{D}^{**0}\ell^+\nu$ with $\bar{D}^{**0} \to D^-\pi^+$
  $B^+ \to \bar{D}^{**0}\ell^+\nu$ with $\bar{D}^{**0} \to D^{*-}\pi^+$ and $D^{*-} \to D^-\pi^0$ or $D^{*-} \to D^-\gamma$
- $B^0 \to D^-\pi^0\ell^+\nu$
- $B^+ \to D^-\pi^+\ell^+\nu$
- $B^0 \to D^{*-}\pi^0\ell^+\nu$ with $D^{*-} \to D^-\pi^0$ or $D^{*-} \to D^-\gamma$
- $B^+ \to D^{*-}\pi^+\ell^+\nu$ with $D^{*-} \to D^-\pi^0$ or $D^{*-} \to D^-\gamma$

2.2 Backgrounds

Backgrounds are categorized as described in the following subsections. Contributions of each component can be seen in Fig 1. Some of backgrounds are non-peaking backgrounds in D mass plots and almost disappear after D mass sideband subtraction. However, some are peaking backgrounds and remain after D mass sideband subtraction as can be seen in Figure 13 - 15 in section 8. D mass sidebands are defined in section 8.1. The composition of major tree backgrounds after sideband subtraction is given in Tables 4 - 6

2.2.1 UncorrDL and UncorrDLMix: peaking background from $B\bar{B}$

D and ℓ come from different B (Uncorrelated), and ℓ come directly from B (DL = Direct Lepton). (Table 4)
One example of this background is the following $B \to D^{(*)}\bar{D}^{(*)}X$ (One of D may be D_s) type :

- $B^+ \to \bar{D}^0\ell^+\nu$
- $B^- \to D^{(*)0}D^{*-}K^0$ with $D^{*-} \to D^-\pi^0$ or $D^{*-} \to D^-\gamma$

These ℓ^+ from B^+ and D^- from B^- form a Dl candidate.
Another example is $B^0\bar{B}^0$ mixing. We separate $B^0\bar{B}^0$ mixing as **UncorrDLMix**

2.2.2 UncorrCL : peaking background from $B\bar{B}$

D and ℓ come from different B (Uncorrelated), and ℓ does not directly come from B (CL = Cascade Lepton). (Table 5)
Main source of this background is semileptonic D decays such as

- $B^+ \to \bar{D}^0X_1$
- $B^- \to D^0X_2$ with $D^0 \to K^-\ell^+\nu$

These $\bar{D}^0$ from B^+ and ℓ^+ from $B^-(D^0)$ form a Dl candidate.

2.2.3 CascL and CascLTau: peaking background from $B\bar{B}$

D and ℓ come from same B (Correlated), and ℓ does not directly come from B (Cascade Lepton). (Table 6)
One example of this background is the following $B \to D^{(*)}\bar{D}^{(*)}X$ (One of D may be D_s) type :

- $B^+ \to \bar{D}^0 D^{*0} K^+$
  with $\bar{D}^0 \to K^+\ell^-\bar{\nu}$ and $D^{*0} \to D^0\pi^0$

These ℓ^- and D^0 form a Dl candidate.
Another example of this background is $B \to X\tau\nu$ type. For example,

- $B^+ \to \bar{D}^0\tau^+\bar{\nu}$
  with $\tau^+ \to \mu^+\bar{\nu}_\mu\nu_\tau$

These $\bar{D}^0$ and μ^+ form a Dl candidate. We separate this type as **CascLTau**

2.2.4 FakeL : peaking background from $B\bar{B}$

The lepton in a Dl candidate is a mis-identified K, π or p.
90 % of mis identification is between μ and π.
We do not float this component in the fit.

2.2.5 CombRL : non-peaking background from $B\bar{B}$

Combinatorial background from $B\bar{B}$ events. This is the case that reconstructed D is not real. Typical mis-reconstructions are that the kaon and the pion do not come from same D or not directly come from D. In our analysis, D^0 has much less combinatorial background than D^+ because D^0 is reconstructed from two particles (one K and one π) whereas D^+ is reconstructed from three particles (one K and two π) (see section 6.1.1).
This component almost disappear after D mass sideband subtraction. Thus, we do not float this component in the fit.

2.2.6 CombFL : non-peaking background from $B\bar{B}$

D is mis-reconstructed (Combinatorial) and ℓ is mis-identified (Fake Lepton).
This component almost disappear after D mass sideband subtraction. Thus, we do not float this component in the fit.

2.2.7 OffPeak : OffPeak data

OffPeak data. The main contribution to OffPeak data is $c\bar{c}$, $s\bar{s}$, $u\bar{u}$, $d\bar{d}$ and $\tau^+\tau^-$ events. This can be separated into two category : continuum background and "$c\bar{c}$ with true D" background. Continuum background can be subtracted out by D mass sideband subtraction. "$c\bar{c}$ with true D" background is a peaking background in the D mass distribution and does not disappear after sideband subtraction.

process	B^+	B^0
$B \to D^{(*)}\bar{D}^{(*)}X$ (One of D may be D_s)	96 %	45 %
$B^0\bar{B}^0$ mixing	0	53 %

Table 4: [**UncorrDL backgroubd processes**] The compositin of UncorrDL backgrounds. Numbers are approximate because compositions are different for D^0 and D^+.

process	B^+	B^0
lepton from γ	4 %	3 %
lepton from τ	5 %	7 %
lepton from uds (K, π, etc.)	3 %	2 %
lepton from $D^{(*)}$ or $D_s^{(*)}$	86 %	87 %

Table 5: [**UncorrCL backgroubd processes**] The compositin of UncorrCL backgrounds. Numbers are approximate because compositions are different for D^0 and D^+.

process	B^+	B^0
lepton from γ	4 %	3 %
lepton from τ	40 %	43 %
lepton from uds (K, π, etc.)	5 %	4 %
lepton from $D^{(*)}$ or $D_s^{(*)}$ ($B \to D^{(*)}\bar{D}^{(*)}X$ (One of D may be D_s))	52 %	49 %

Table 6: [**CascL backgroubd processes**] The compositin of CascL backgrounds. Numbers are approximate because compositions are different for D^0 and D^+.

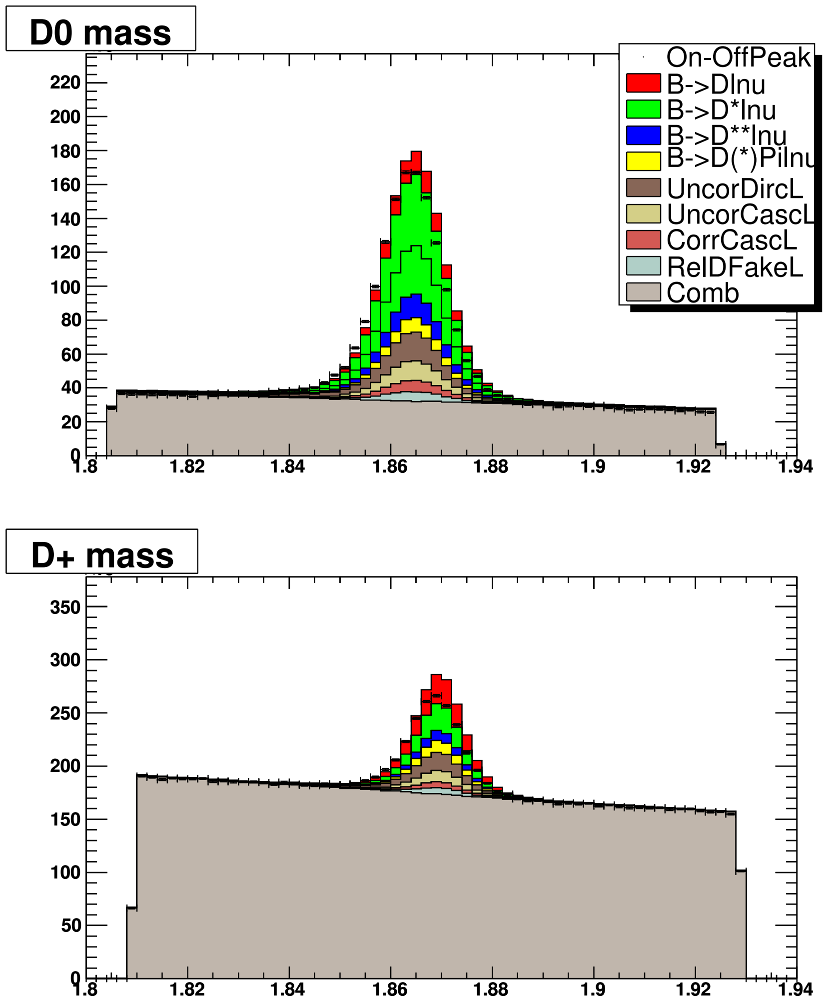


Figure 1: [D **mass plot**] An example of D mass plot showing all components of signal and backgroud. The top is D^0 and the bottom is D^+. Black line with error bar is OnPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, light gray is Comb, and dark gray is OffPeak.

3 Fitting Method

We use binned χ^2 fitting.

3.1 3D binning (i)

We use 3D binning by three variables :

- lepton momentum in the CM frame p_ℓ^*
- D momentum in the CM frame p_D^*
- $\cos\theta_{B-Dl}$ (see below)

The reason of this choice is the following. In the $B \to D\ell\nu$ decays, there are only three independent kinematic variables. For example, in Figure 29, supposing D^* is D and does not decay, independent variables can be

- The magnitude of lepton momentum
- The magnitude of D momentum
- The angle θ_ℓ

because χ and θ_V are irrelevant. The same is true also in other decay modes when we reconstruct only $D\ell$. We chose three variables, p_ℓ^*, p_D^* and $\cos\theta_{B-Dl}$, because this combination is simple, well known and useful to separate each signal decay modes and backgrounds. As you can see from Figures 13-15 in section 8, the shape of the histogram of each MC mode is quite different. We also tried $\cos\theta_{D-\ell}$ (the cosine of the angle between D and ℓ in the CM frame) instead of $\cos\theta_{B-Dl}$ to assure that $\cos\theta_{B-Dl}$ is a better parameter.

3.2 Calculation of $\cos\theta_{B-Dl}$

$\cos\theta_{B-Dl}$ is given by (see Appendix F)

$$\cos\Theta_{B-Dl} = \frac{2E_B E_{Dl} - m_B^2 - m_{Dl}^2}{2|\vec{p}_B||\vec{p}_{Dl}|} \tag{1}$$

where E_B and $p_B = |\vec{p}_B|$ are calculated from beam energy E_{beam} by

$$\begin{aligned} E_B &= E_{beam}/2 \\ p_B &= \sqrt{E_B^2 - m_B^2} \end{aligned} \tag{2}$$

because we don not reconstruct B. We use

$$\begin{aligned} E_{beam} &= E_{\text{data}} \text{ in Table 22} \\ m_B &= m_{B^0} = 5.2795\,\text{GeV} \end{aligned} \tag{3}$$

for both data and MC (both BpBm and B0B0bar MC) because we do not know whether the B in data is B^0 or B^+. Here $E_{\rm data}$ is given in Table 22 in section 9.6. Thus,

$$\cos\Theta_{B-Dl} = \frac{E_{beam}E_{Dl} - m_{B^0}^2 - m_{Dl}^2}{2\sqrt{(E_{beam}/2)^2 - m_{B^0}^2}\,|\vec{p}_{Dl}|} \tag{4}$$

$D\ell$ mass, momentum and energy are reconstructed values.

3.2.1 $\cos\theta_{B-Dl}$ of OffPeak data

OffPeak data is taken at the energy 39MeV below the OnPeak data energy. Thus, the $D\ell$ mass, momentum and energy are different from the continuum background in OnPeak data. In order to get similar $\cos\theta_{B-Dl}$ distribution to OnPeak data, we need to do a modification.
One idea is to scale up $D\ell$ mass, momentum and energy by the ratio of beam energies

$$r_{\rm beam} = \frac{E_{\rm data}^{OnPeak}}{E_{\rm data}^{OffPeak}} = \frac{E_{\rm data}}{E_{\rm data} - 0.039 \text{ GeV}} \tag{5}$$

where $E_{\rm data}$ is given in Table 22 in section 9.6. Thus

$$\cos\Theta_{B-Dl} = \frac{E_{beam}(r_{\rm beam}E_{Dl}) - m_{B^0}^2 - (r_{\rm beam}\, m_{Dl})^2}{2\sqrt{(E_{beam}/2)^2 - m_{B^0}^2}\,|r_{\rm beam}\,\vec{p}_{Dl}|} \tag{6}$$

In reality, $D\ell$ mass, momentum and energy do not have to scale as the same rate to beam energy. Thus, we introduce scale modification factors x_E and x_p for $D\ell$ energy and momentum, respectively. Then, scale factors for $D\ell$ energy is given by

$$r_{\rm beam}^E = \frac{E_{\rm data}}{E_{\rm data} - 0.039x_E} \tag{7}$$

and for $D\ell$ momentum is given by

$$r_{\rm beam}^p = \frac{E_{\rm data}}{E_{\rm data} - 0.039x_p} \tag{8}$$

$D\ell$ mass is calculated from these energy and momentm

$$m_{Dl}^{corrected} = \sqrt{(r_{\rm beam}^E)^2 E_{Dl}^2 - (r_{\rm beam}^p)^2 p_{Dl}^2} \tag{9}$$

Thus, $\cos\theta_{B-Dl}$ is given by

$$\cos\Theta_{B-Dl} = \frac{E_{beam}(r_{\rm beam}^E E_{Dl}) - m_{B^0}^2 - (m_{Dl}^{corrected})^2}{2\sqrt{(E_{beam}/2)^2 - m_{B^0}^2}\,|r_{\rm beam}^p\,\vec{p}_{Dl}|} \tag{10}$$

We use $x_E = 1$ and $x_p = 1$ in our nominal fit and, for systematic study, we use the following three combinations :

- $x_E = 1$ and $x_p = 0$
- $x_E = 0$ and $x_p = 1$
- $x_E = 0$ and $x_p = 0$

3.3 χ^2

The χ^2 is given by

$$\chi^2 = \sum_{i=\text{bin}}^{D^0} \frac{(N_i^{OnPeak} - N_i^{OffPeak} - \sum_j C_j^{MC} N_{ij}^{MC})^2}{(\sigma_i^{OnPeak})^2 + (\sigma_i^{OffPeak})^2 + \sum_j (C_j^{MC} \sigma_{ij}^{MC})^2} + \sum_{i=\text{bin}}^{D^+} \frac{(N_i^{OnPeak} - N_i^{OffPeak} - \sum_j C_j^{MC} N_{ij}^{MC})^2}{(\sigma_i^{OnPeak})^2 + (\sigma_i^{OffPeak})^2 + \sum_j (C_j^{MC} \sigma_{ij}^{MC})^2} \tag{11}$$

where,

- i represents each bin
- j represents each MC mode.

MC modes are listed in section 4 separately for D^0 and D^+ modes. See also Appendix A and section 2.
Quantities in the χ^2 are:

- N_i^{OnPeak}
  The number of OnPeak data candidates in the i-th bin.
- $N_i^{OffPeak}$
  The number of OffPeak data candidates in the i-th bin, luminosity normalized to OnPeak data.
- N_{ij}^{MC}
  The number of j-th MC mode candidates in the i-th bin, luminosity normalized to OnPeak data. Details are in section 3.4. The effect of data MC differences from tracking and PID are corrected (see section 7). MC re-weighting is also done. We can fit for form factor parameters through MC re-weighting (see sections 9 and 10)
- C_j^{MC}
  The coefficient of each MC mode (j). These include branching fractions to be determined in the fit. Details are in section 4.
- σ_i^{OnPeak}
  The statistical uncertainty of N_i^{OnPeak}. $\sigma_i^{OnPeak} = \sqrt{N_i^{OnPeak}}$.
- $\sigma_i^{OffPeak}$
  The statistical uncertainty of $N_i^{OffPeak}$. Details are in section 3.5.
- σ_{ij}^{MC}
  The statistical uncertainty of N_{ij}^{MC}. Details are in section 3.5.

We use bins which have more than 10 expected candidates

$$\sum_j C_j^{MC} N_{ij}^{MC} > 10 \tag{12}$$

and neglect other bins. The number 10 can be changed for systematic study.

3.4 Number of MC Candidates N_{ij}^{MC}

We calculate the number of candidates from MC using MC truth information. We use five generic MC samples: BpBm, B0B0bar, ccbar, uds and tautau. Note that ccbar, uds and tautau MCs are used only for validation of the fit. Tracking efficiency and PID corrections are applied to adjust MC to OnPaek data (see section 7). MCs are also luminosity normalized to OnPeak data.

3.4.1 Luminosity normalization

All MC and OffPeak data are normalized to OnPeak data by luminosity. Supposing OnPeak luminosity is $\mathcal{L}^{OnPeak}$ and MC (mode j) luminosity is $\mathcal{L}_j^{MC}$, the normalization factor r_j^{Lumi} is given by

$$r_j^{Lumi} = \frac{\mathcal{L}^{OnPeak}}{\mathcal{L}_j^{MC}} \tag{13}$$

where the luminosity of $B\bar{B}$ MC is calculated by the generated number of events and B counting number from OnPeak data. For the luminosity of ccbar, uds and tautau MC, their cross sections are used. Details are in Appendix E. Note that depending on the given MC mode j, luminosity is given. For example, when j=1,2,6,8,etc., $\mathcal{L}_j^{MC}$ is the luminosity of BpBm MC and when j=3,5,7,9,etc., $\mathcal{L}_j^{MC}$ is the luminosity of B0B0bar MC (see section 4 for details of MC modes j).
Note that, the luminosity calculation was done based on the assumption that produced number of B^+B^- and $B^0\bar{B}^0$ are equal. The decay rate difference is taken into account in the coefficients (see section 4).
In case of OffPeak data, since OffPeak data is taken at 40 MeV lower energy than OnPeak data, we need to correct for $1/s$ dependence of cross section, where s is the square of CM energy :

$$\begin{aligned} s_{OnPeak} &= 10.58^2 \text{ GeV} \\ s_{OffPeak} &= (10.58 - 0.04)^2 = 10.54^2 \text{ GeV} \end{aligned} \tag{14}$$

Thus, correction factor is

$$r_{OffPeak}^{Lumi} = \frac{\mathcal{L}^{OnPeak} s_{OffPeak}}{\mathcal{L}^{OffPeak} s_{OnPeak}} \tag{15}$$

3.4.2 N_{ij}^{MC}

The number of MC candidates N_{ij}^{MC} is given by

$$N_{ij}^{MC} = r_j^{Lumi} \sum_k r_k^{Trk} r_k^{KPID} r_k^{LPID} W_{jk}^{MC} \tag{16}$$

where

- k represents the k-th candidate in i-th bin and MC mode j.

and

- r_k^{Trk} is the tracking efficiency correction factor (⇒ section 7)
- r_k^{KPID} is the kaon PID correction factor (⇒ section 7)
- r_k^{LPID} is the lepton PID correction factor (⇒ section 7)
- W_{jk}^{MC} is the MC weight (⇒ sections 9 and 10)

Tracking efficiency and PID corrections are described in section 7. MC weights are described in sections 9 and 10.

3.5 Statistical uncertainty $\sigma_i^{OffPeak}$ and σ_{ij}^{MC}

Statistical uncertainty of OnPeak data is given by

$$\sigma_i^{OnPeak} = \sqrt{N_i^{OnPeak}} \tag{17}$$

However, since statistical uncertainty of OffPeak data and MCs are luminosity normalized and re-weighted, statistical uncertainties are not simply given by $\sigma_i = \sqrt{N_i}$. In this section, we explain how the statistical uncertainty of OffPeak data and MCs are calculated.

3.5.1 Luminosity normalization and statistical uncertainty

OffPeak and MCs are luminosity normalized to OnPeak data. We take OffPeak data as an example. If the number of OffPeak candidate in i-th bin is N_i^0, the number of luminosity normalized candidate in the bin is given by

$$N_i^{OffPeak} = r_{OffPeak}^{Lumi} N_i^0 \tag{18}$$

and the statistical uncertainty of the bin is given by

$$\sigma_i^{OffPeak} = r_{OffPeak}^{Lumi} \sqrt{N_i^0} \tag{19}$$

3.5.2 MC re-weighting and statistical uncertainty

Supposing the total weight of the k-th candidate of MC mode j is

$$w_{jk}^T \equiv r_k^{Trk} r_k^{KPID} r_k^{LPID} W_{jk}^{MC} \tag{20}$$

The number of candidate in the i-th bin is given by

$$N_{ij}^{MC} = r_j^{Lumi} \sum_k w_{jk}^T \tag{21}$$

We add statistical uncertainty quadratically. Thus,

$$\sigma_{ij}^{MC} = r_j^{Lumi} \sqrt{\sum_k \left(w_{jk}^T\right)^2} \tag{22}$$

4 Coefficients in the fitting C_j^{MC}

Coefficients consist of branching fractions, some of which are to be determined by the fit. Coefficients are given by taking ratios between new branching fractions and the ones used in SP5/SP6 MC (the ones with superscript 0 in the following subsections). The decays contributing to each MC mode are shown in Appendix A and B. New measurements of branching fractions and constraints that we use in our fit are explained in section 5.

4.1 Number of produced B

Branching fractions of B^+B^- and $B^0\bar{B}^0$ of $\Upsilon(4S)$ resonance are denoted by

$$\begin{aligned} f_{+-} &= \mathcal{B}(\Upsilon(4S) \to B^+B^-) \\ f_{00} &= \mathcal{B}(\Upsilon(4S) \to B^0\bar{B}^0) \end{aligned} \tag{23}$$

We define the ratio of these two branching fractions

$$f_{+0} \equiv \frac{f_{+-}}{f_{00}} \tag{24}$$

In the first approximation, $f_{+-} = f_{00} = 0.5$ and $f_{+0} = 1$. In SP5/SP6 MC

$$f^0_{+0} = 1 \tag{25}$$

is used [2]. The number of produced B is related to this ratio. Supposing $N_{B\bar{B}}$ of $B\bar{B}$-pairs are produced, the number of produced neutral B is

$$N^{prod}_{neut} = 2f_{00}N_{B\bar{B}} \tag{26}$$

and the number of produced charged B is

$$N^{prod}_{chrg} = 2f_{+-}N_{B\bar{B}} = f_{+0}N^{prod}_{neut} \tag{27}$$

Thus,

$$2N_{B\bar{B}} = N^{prod}_{neut} + N^{prod}_{chrg} = N^{prod}_{neut} + f_{+0}N^{prod}_{neut} \tag{28}$$

We get

$$N^{prod}_{neut} = \frac{2}{1+f_{+0}}N_{B\bar{B}} \tag{29}$$

and

$$N^{prod}_{chrg} = \frac{2f_{+0}}{1+f_{+0}}N_{B\bar{B}} \tag{30}$$

This f_{+0} can be floated in the fit.

[2] In this section the superscript "0" is used to denote SP5/SP6 values. For example, the BF that have a superscript "0" after the closing parenthesis are the values from the SP5/SP6 MC

4.2 $B\bar{B}$ luminosity correction

As explained in section 3.4.1, we assumed $f_{+0} = 1$, namely, the same number of B^+B^- and $B^0\bar{B}^0$ to calculate luminosity. We need to correct these. The correction factors are given by

$$\begin{aligned} C_{B^+} &= \frac{2f_{+0}}{1+f_{+0}} \quad \text{for BpBm MC} \\ C_{B^0} &= \frac{2}{1+f_{+0}} \quad \text{for B0B0bar MC} \end{aligned} \tag{31}$$

The coefficients C_j^{MC} include the factor C_{B^+} if those are of BpBm MC and C_{B^0} if those are of B0B0bar MC.

4.3 D^0 MC modes

4.3.1 Coefficients for signals : $j = 1 - 15$

There are 15 signal MC modes.

- $j = 1 :\ B^+ \to \bar{D}^0\ell^+\nu$
  Number of produced events is

$$N = N_{chrg}^{prod}\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)B_{D0} = N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)B_{D0} \tag{32}$$

  As $N_{B\bar{B}}$ is common, it will be canceled by taking a ratio. Thus, the coefficient is given by

$$C_1^{MC} = \frac{C_{B^+}\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)^0\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{33}$$

- $j = 2 :\ B^+ \to \bar{D}^{*0}\ell^+\nu$ with $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  Number of produced events is

$$\begin{aligned} N &= N_{chrg}^{prod}\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\ &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)\mathcal{B}(D^0 \to K^-\pi^+) \end{aligned} \tag{34}$$

  Thus

$$C_2^{MC} = \frac{C_{B^+}\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)^0\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{35}$$

- $j = 3 :\ B^0 \to D^{*-}\ell^+\nu$ with $D^{*-} \to \bar{D}^0\pi^-$
  Number of produced events is

$$N = N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \tag{36}$$

  Thus

$$C_3^{MC} = \frac{C_{B^0}\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{37}$$

- $j=4:\ B^+ \to \bar{D}_0^{*0}\ell^+\nu$ with $\bar{D}_0^{*0} \to \bar{D}^0\pi^0$
  The coefficient is

$$C_4^{MC} = \frac{C_{B^+}\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)\mathcal{B}(D_0^{*0} \to D^0\pi^0)\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)^0\mathcal{B}(D_0^{*0} \to D^0\pi^0)^0\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{38}$$

- $j=5:\ B^0 \to D_0^{*-}\ell^+\nu$ with $D_0^{**-} \to \bar{D}^0\pi^-$
  The coefficient is

$$C_5^{MC} = \frac{C_{B^0}\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)\mathcal{B}(D_0^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)^0\mathcal{B}(D_0^{*+} \to D^0\pi^+)^0\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{39}$$

- $j=6:\ B^+ \to \bar{D}_1^0\ell^+\nu$
  The decays contribute to this mode are
  $B^+ \to \bar{D}_1^0\ell^+\nu$ with $\bar{D}_1^0 \to D^{*-}\pi^+$ and $D^{*-} \to \bar{D}^0\pi^-$
  $B^+ \to \bar{D}_1^0\ell^+\nu$ with $\bar{D}_1^0 \to \bar{D}^{*0}\pi^0$ and $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  Number of produced events is

$$\begin{aligned} N &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\mathcal{B}(D_1^0 \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\ &\quad +N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\mathcal{B}(D_1^0 \to D^{*0}\pi^0) \\ &\qquad \times[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\ &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) \\ &\qquad \times[\mathcal{B}(D_1^0 \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1^0 \to D^{*0}\pi^0)]\mathcal{B}(D^0 \to K^-\pi^+) \end{aligned} \tag{40}$$

  The coefficient is

$$C_6^{MC} = \frac{\begin{array}{l} C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\times \\ {[\mathcal{B}(D_1^0 \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1^0 \to D^{*0}\pi^0)]\mathcal{B}(D^0 \to K^-\pi^+)} \end{array}}{\begin{array}{l} \mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)^0\times \\ {[\mathcal{B}(D_1^0 \to D^{*+}\pi^-)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(D_1^0 \to D^{*0}\pi^0)^0]\mathcal{B}(D^0 \to K^-\pi^+)^0} \end{array}} \tag{41}$$

- $j=7:\ B^0 \to D_1^-\ell^+\nu$
  The decays contribute to this mode are
  $B^0 \to D_1^-\ell^+\nu$ with $D_1^- \to D^{*-}\pi^0$ and $D^{*-} \to \bar{D}^0\pi^-$
  $B^0 \to D_1^-\ell^+\nu$ with $D_1^- \to \bar{D}^{*0}\pi^-$ and $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  Number of produced events is

$$\begin{aligned} N &= N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_1^-\ell^+\nu)\mathcal{B}(D_1^+ \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\ &\quad +N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_1^-\ell^+\nu)\mathcal{B}(D_1^+ \to D^{*0}\pi^+) \\ &\qquad \times[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\ &= N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_1^-\ell^+\nu) \\ &\quad \times[\mathcal{B}(D_1^+ \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1^+ \to D^{*0}\pi^+)]\mathcal{B}(D^0 \to K^-\pi^+) \end{aligned} \tag{42}$$

The coefficient is

$$C_7^{MC} = \frac{\begin{array}{l}C_{B^0}\mathcal{B}(B^0 \to D_1^- \ell^+ \nu) \times \\ {[\mathcal{B}(D_1^+ \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1^+ \to D^{*0}\pi^+)]\mathcal{B}(D^0 \to K^-\pi^+)}\end{array}}{\begin{array}{l}\mathcal{B}(B^0 \to D_1^- \ell^+ \nu)^0 \times \\ {[\mathcal{B}(D_1^+ \to D^{*+}\pi^0)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(D_1^+ \to D^{*0}\pi^+)^0]\mathcal{B}(D^0 \to K^-\pi^+)^0}\end{array}} \tag{43}$$

- $j = 8:\ B^+ \to \bar{D}_1'^0 \ell^+ \nu$
  The decays contribute to this mode are
  $B^+ \to \bar{D}_1'^0 \ell^+ \nu$ with $\bar{D}_1'^0 \to D^{*-}\pi^+$ and $D^{*-} \to \bar{D}^0\pi^-$
  $B^+ \to \bar{D}_1'^0 \ell^+ \nu$ with $\bar{D}_1'^0 \to \bar{D}^{*0}\pi^0$ and $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  Number of produced events is

$$\begin{aligned} N &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu)\mathcal{B}(D_1'^0 \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\ &\quad + N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu)\mathcal{B}(D_1'^0 \to D^{*0}\pi^0) \\ &\qquad \times[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\ &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu) \\ &\quad \times[\mathcal{B}(D_1'^0 \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1'^0 \to D^{*0}\pi^0)]\mathcal{B}(D^0 \to K^-\pi^+) \end{aligned} \tag{44}$$

  The coefficient is

$$C_8^{MC} = \frac{\begin{array}{l}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu) \times \\ {[\mathcal{B}(D_1'^0 \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1'^0 \to D^{*0}\pi^0)]\mathcal{B}(D^0 \to K^-\pi^+)}\end{array}}{\begin{array}{l}\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu)^0 \times \\ {[\mathcal{B}(D_1'^0 \to D^{*+}\pi^-)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(D_1'^0 \to D^{*0}\pi^0)^0]\mathcal{B}(D^0 \to K^-\pi^+)^0}\end{array}} \tag{45}$$

- $j = 9:\ B^0 \to D_1'^- \ell^+ \nu$
  The decays contribute to this mode are
  $B^0 \to D_1'^- \ell^+ \nu$ with $D_1'^- \to D^{*-}\pi^0$ and $D^{*-} \to \bar{D}^0\pi^-$
  $B^0 \to D_1'^- \ell^+ \nu$ with $D_1'^- \to \bar{D}^{*0}\pi^-$ and $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  Number of produced events is

$$\begin{aligned} N &= N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_1'^- \ell^+ \nu)\mathcal{B}(D_1'^+ \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\ &\quad + N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_1'^- \ell^+ \nu)\mathcal{B}(D_1'^+ \to D^{*0}\pi^+) \\ &\qquad \times[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\ &= N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_1'^- \ell^+ \nu) \\ &\quad \times[\mathcal{B}(D_1'^+ \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1'^+ \to D^{*0}\pi^+)]\mathcal{B}(D^0 \to K^-\pi^+) \end{aligned} \tag{46}$$

$$C_9^{MC} = \frac{\begin{array}{l} C_{B^0}\mathcal{B}(B^0 \to D_1'^{-}\ell^+\nu)\times \\ [\mathcal{B}(D_1'^{+} \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_1'^{+} \to D^{*0}\pi^+)]\mathcal{B}(D^0 \to K^-\pi^+) \end{array}}{\begin{array}{l} \mathcal{B}(B^0 \to D_1'^{-}\ell^+\nu)^0\times \\ [\mathcal{B}(D_1'^{+} \to D^{*+}\pi^0)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(D_1'^{+} \to D^{*0}\pi^+)^0]\mathcal{B}(D^0 \to K^-\pi^+)^0 \end{array}} \quad (47)$$

- $j = 10: \; B^+ \to \bar{D}_2^{*0}\ell^+\nu$
  The decays contribute to this mode are
  $B^+ \to \bar{D}_2^{*0}\ell^+\nu$ with $\bar{D}_2^{*0} \to D^{*-}\pi^+$ and $D^{*-} \to \bar{D}^0\pi^-$
  $B^+ \to \bar{D}_2^{*0}\ell^+\nu$ with $\bar{D}_2^{*0} \to \bar{D}^{*0}\pi^0$ and $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  $B^+ \to \bar{D}_2^{*0}\ell^+\nu$ with $\bar{D}_2^{*0} \to \bar{D}^0\pi^0$
  Number of produced events is

$$\begin{aligned} N &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\ &\quad + N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^{*0}\pi^0) \\ &\qquad \times[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\ &\quad + N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^0\pi^0)\mathcal{B}(D^0 \to K^-\pi^+) \\ &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu) \\ &\quad \times[\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_2^{*0} \to D^{*0}\pi^0) + \mathcal{B}(D_2^{*0} \to D^0\pi^0)] \\ &\quad \times\mathcal{B}(D^0 \to K^-\pi^+) \end{aligned} \quad (48)$$

  The coefficient is

$$C_{10}^{MC} = \frac{\begin{array}{l} C_{B^+}\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\times \\ [\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_2^{*0} \to D^{*0}\pi^0) + \mathcal{B}(D_2^{*0} \to D^0\pi^0)] \\ \times\mathcal{B}(D^0 \to K^-\pi^+) \end{array}}{\begin{array}{l} \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)^0\times \\ [\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(D_2^{*0} \to D^{*0}\pi^0)^0 + \mathcal{B}(D_2^{*0} \to D^0\pi^0)^0] \\ \times\mathcal{B}(D^0 \to K^-\pi^+)^0 \end{array}} \quad (49)$$

- $j = 11: \; B^0 \to D_2^{*-}\ell^+\nu$
  The decays contribute to this mode are
  $B^0 \to D_2^{*-}\ell^+\nu$ with $D_2^{*-} \to D^{*-}\pi^0$ and $D^{*-} \to \bar{D}^0\pi^-$
  $B^0 \to D_2^{*-}\ell^+\nu$ with $D_2^{*-} \to \bar{D}^{*0}\pi^-$ and $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  $B^0 \to D_2^{*-}\ell^+\nu$ with $D_2^{*-} \to \bar{D}^0\pi^-$

Number of produced events is

$$\begin{aligned}
N &= N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\
&\quad +N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) \\
&\qquad \times[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\
&\quad +N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\
&= N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu) \\
&\quad \times[\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) + \mathcal{B}(D_2^{*+} \to D^0\pi^+)] \\
&\quad \times\mathcal{B}(D^0 \to K^-\pi^+)
\end{aligned} \tag{50}$$

The coefficient is

$$C_{11}^{MC} = \frac{\begin{array}{l} C_{B^0}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\times \\ {[\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) + \mathcal{B}(D_2^{*+} \to D^0\pi^+)]} \\ \times\mathcal{B}(D^0 \to K^-\pi^+) \end{array}}{\begin{array}{l} \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)^0\times \\ {[\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(D_2^{*+} \to D^{*0}\pi^+)^0 + \mathcal{B}(D_2^{*+} \to D^0\pi^+)^0)} \\ \times\mathcal{B}(D^0 \to K^-\pi^+)^0 \end{array}} \tag{51}$$

- $j = 12:\ B^+ \to \bar{D}^0\pi^0\ell^+\nu$
  The coefficient is
  $$C_{12}^{MC} = \frac{C_{B^+}\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)^0\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{52}$$

- $j = 13:\ B^0 \to \bar{D}^0\pi^-\ell^+\nu$
  The coefficient is
  $$C_{13}^{MC} = \frac{C_{B^0}\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)^0\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{53}$$

- $j = 14:\ B^+ \to D^*\pi\ell^+\nu$
  The decays contribute to this mode are
  $B^+ \to D^{*-}\pi^+\ell^+\nu$ with $D^{*-} \to \bar{D}^0 + \pi^-$
  $B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu$ with $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  Number of produced events is

$$\begin{aligned}
N &= N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\
&\quad +N_{B\bar{B}}C_{B^+}\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\
&= N_{B\bar{B}}C_{B^+}[\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)] \\
&\quad \times\mathcal{B}(D^0 \to K^-\pi^+)
\end{aligned} \tag{54}$$

The coefficient is

$$C_{14}^{MC} = \frac{C_{B^+}[\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)] \times\mathcal{B}(D^0 \to K^-\pi^+)}{[\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)^0] \times\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{55}$$

- $j = 15:\ B^0 \to D^*\pi\ell^+\nu$
  The decays contribute to this mode are
  $B^0 \to D^{*-}\pi^0\ell^+\nu$ with $D^{*-} \to \bar{D}^0 + \pi^-$
  $B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu$ with $\bar{D}^{*0} \to \bar{D}^0 + \gamma/\pi^0$
  Number of produced events is

$$\begin{aligned} N &= N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+)\mathcal{B}(D^0 \to K^-\pi^+) \\ &\quad +N_{B\bar{B}}C_{B^0}\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)[\mathcal{B}(D^{*0} \to D^0\pi^0) + \mathcal{B}(D^{*0} \to D^0\gamma)]\mathcal{B}(D^0 \to K^-\pi^+) \\ &= N_{B\bar{B}}C_{B^0}[\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)] \\ &\quad \times\mathcal{B}(D^0 \to K^-\pi^+) \end{aligned} \tag{56}$$

The coefficient is

$$C_{15}^{MC} = \frac{C_{B^0}[\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)\mathcal{B}(D^{*+} \to D^0\pi^+) + \mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)] \times\mathcal{B}(D^0 \to K^-\pi^+)}{[\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)^0\mathcal{B}(D^{*+} \to D^0\pi^+)^0 + \mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)^0] \times\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{57}$$

4.3.2 Coefficients for backgrounds : $j = 16 - 27$

There are 12 background MC modes. The coefficient of each mode is

- $j = 16$, UncorrDL from B^+B^-

$$C_{16}^{MC} = C_{UncorDL}^{D^0}\frac{C_{B^+}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{58}$$

- $j = 17$, UncorrDL from $B^0\bar{B}^0$

$$C_{17}^{MC} = C_{UncorDL}^{D^0}\frac{C_{B^0}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{59}$$

- $j = 18$, UncorrCL from B^+B^-

$$C_{18}^{MC} = C_{UncorCL}^{D^0}\frac{C_{B^+}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{60}$$

- $j = 19$, UncorrCL from $B^0\bar{B}^0$

$$C_{19}^{MC} = C_{UncorCL}^{D^0} \frac{C_{B^0}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{61}$$

- $j = 20$, CascL from B^+B^-

$$C_{20}^{MC} = C_{CascL}^{D^0} \frac{C_{B^+}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{62}$$

- $j = 21$, CascL from $B^0\bar{B}^0$

$$C_{21}^{MC} = C_{CascL}^{D^0} \frac{C_{B^0}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{63}$$

- $j = 22$, FakeL from B^+B^-

$$C_{22}^{MC} = C_{FakeL}^{D^0} \frac{C_{B^+}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{64}$$

- $j = 23$, FakeL from $B^0\bar{B}^0$

$$C_{23}^{MC} = C_{FakeL}^{D^0} \frac{C_{B^0}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{65}$$

- $j = 24$, CombRL from B^+B^-

$$C_{24}^{MC} = \frac{C_{B^+}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{66}$$

- $j = 25$, CombRL from $B^0\bar{B}^0$

$$C_{25}^{MC} = \frac{C_{B^0}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{67}$$

- $j = 26$, CombFL from B^+B^-

$$C_{26}^{MC} = \frac{C_{B^+}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{68}$$

- $j = 27$, CombFL from $B^0\bar{B}^0$

$$C_{27}^{MC} = \frac{C_{B^0}\mathcal{B}(D^0 \to K^-\pi^+)}{\mathcal{B}(D^0 \to K^-\pi^+)^0} \tag{69}$$

The factors such as $C_{UncorDL}^{D^0}$ are free parameters to be floated with constraints in the fit. See section 5.3 for details of constraints.

4.4 D^+ MC modes

4.4.1 Coefficients for signals : $j = 1 - 14$

There are 14 signal MC modes.

- $j = 1$: $B^0 \to D^- \ell^+ \nu$
  The coefficient is

$$C_1^{MC} = \frac{C_{B^0} \mathcal{B}(B^0 \to D^- \ell^+ \nu) \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)}{\mathcal{B}(B^0 \to D^- \ell^+ \nu)^0 \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)^0} \tag{70}$$

- $j = 2$: $B^0 \to D^{*-} \ell^+ \nu$ with $D^{*-} \to D^- + \gamma/\pi^0$
  The coefficient is

$$C_2^{MC} = \frac{\begin{array}{l} C_{B^0} \mathcal{B}(B^0 \to D^{*-} \ell^+ \nu) \times \\ {[\mathcal{B}(D^{*+} \to D^+ \pi^0) + \mathcal{B}(D^{*+} \to D^+ \gamma)] \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)} \end{array}}{\begin{array}{l} \mathcal{B}(B^0 \to D^{*-} \ell^+ \nu)^0 \times \\ {[\mathcal{B}(D^{*+} \to D^+ \pi^0)^0 + \mathcal{B}(D^{*+} \to D^+ \gamma)^0] \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)^0} \end{array}} \tag{71}$$

- $j = 3$: $B^+ \to \bar{D}_0^{*0} \ell^+ \nu$ with $\bar{D}_0^{*0} \to D^- + \pi^+$
  The coefficient is

$$C_3^{MC} = \frac{C_{B^+} \mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+ \nu) \mathcal{B}(D_0^{*0} \to D^+ \pi^-) \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)}{\mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+ \nu)^0 \mathcal{B}(D_0^{*0} \to D^+ \pi^-)^0 \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)^0} \tag{72}$$

- $j = 4$: $B^0 \to D_0^{*-} \ell^+ \nu$ with $D_0^{*-} \to D^- \pi^0$
  The coefficient is

$$C_4^{MC} = \frac{C_{B^0} \mathcal{B}(B^0 \to D_0^{*-} \ell^+ \nu) \mathcal{B}(D_0^{*+} \to D^+ \pi^0) \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)}{\mathcal{B}(B^0 \to D_0^{*-} \ell^+ \nu)^0 \mathcal{B}(D_0^{*+} \to D^+ \pi^0)^0 \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)^0} \tag{73}$$

- $j = 5$: $B^+ \to \bar{D}_1^0 \ell^+ \nu$ with $\bar{D}_1^0 \to D^{*-} + \pi^+$ and $D^{*-} \to D^- + \pi^0/\gamma$
  The coefficient is

$$C_5^{MC} = \frac{\begin{array}{l} C_{B^+} \mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu) \mathcal{B}(D_1^0 \to D^{*+} \pi^-) \times \\ {[\mathcal{B}(D^{*+} \to D^+ \pi^0) + \mathcal{B}(D^{*+} \to D^+ \gamma)] \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)} \end{array}}{\begin{array}{l} \mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu)^0 \mathcal{B}(D_1^0 \to D^{*+} \pi^-)^0 \times \\ {[\mathcal{B}(D^{*+} \to D^+ \pi^0)^0 + \mathcal{B}(D^{*+} \to D^+ \gamma)^0] \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)^0} \end{array}} \tag{74}$$

- $j = 6$: $B^0 \to D_1^- \ell^+ \nu$ with $D_1^- \to D^{*-} \pi^0$ and $D^{*-} \to D^- + \pi^0/\gamma$
  The coefficient is

$$C_6^{MC} = \frac{\begin{array}{l} C_{B^0} \mathcal{B}(B^0 \to D_1^- \ell^+ \nu) \mathcal{B}(D_1^+ \to D^{*+} \pi^0) \times \\ {[\mathcal{B}(D^{*+} \to D^+ \pi^0) + \mathcal{B}(D^{*+} \to D^+ \gamma)] \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)} \end{array}}{\begin{array}{l} \mathcal{B}(B^0 \to D_1^- \ell^+ \nu)^0 \mathcal{B}(D_1^+ \to D^{*+} \pi^0)^0 \times \\ {[\mathcal{B}(D^{*+} \to D^+ \pi^0)^0 + \mathcal{B}(D^{*+} \to D^+ \gamma)^0] \mathcal{B}(D^+ \to K^- \pi^+ \pi^+)^0} \end{array}} \tag{75}$$

- $j = 7:\ B^+ \to \bar{D}_1'^0 \ell^+ \nu$ with $\bar{D}_1'^0 \to D^{*-} + \pi^+$ and $D^{*-} \to D^- + \pi^0/\gamma$
  The coefficient is

$$C_7^{MC} = \frac{\begin{array}{l} C_{B^+}\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu)\mathcal{B}(D_1'^0 \to D^{*+}\pi^-)\times \\ {[\mathcal{B}(D^{*+} \to D^+\pi^0) + \mathcal{B}(D^{*+} \to D^+\gamma)]\mathcal{B}(D^+ \to K^-\pi^+\pi^+)} \end{array}}{\begin{array}{l} \mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu)^0\mathcal{B}(D_1'^0 \to D^{*+}\pi^-)^0\times \\ {[\mathcal{B}(D^{*+} \to D^+\pi^0)^0 + \mathcal{B}(D^{*+} \to D^+\gamma)^0]\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \end{array}} \quad (76)$$

- $j = 8:\ B^0 \to D_1'^- \ell^+ \nu$ with $D_1'^- \to D^{*-}\pi^0$ and $D^{*-} \to D^- + \pi^0/\gamma$
  The coefficient is

$$C_8^{MC} = \frac{\begin{array}{l} C_{B^0}\mathcal{B}(B^0 \to D_1'^- \ell^+ \nu)\mathcal{B}(D_1'^+ \to D^{*+}\pi^0)\times \\ {[\mathcal{B}(D^{*+} \to D^+\pi^0) + \mathcal{B}(D^{*+} \to D^+\gamma)]\mathcal{B}(D^+ \to K^-\pi^+\pi^+)} \end{array}}{\begin{array}{l} \mathcal{B}(B^0 \to D_1'^- \ell^+ \nu)^0\mathcal{B}(D_1'^+ \to D^{*+}\pi^0)^0\times \\ {[\mathcal{B}(D^{*+} \to D^+\pi^0)^0 + \mathcal{B}(D^{*+} \to D^+\gamma)^0]\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \end{array}} \quad (77)$$

- $j = 9:\ B^+ \to \bar{D}_2^{*0} \ell^+ \nu$
  The decays contribute to this mode are
  $B^+ \to \bar{D}_2^{*0} \ell^+ \nu$ with $\bar{D}_2^{*0} \to D^{*-} + \pi^+$ and $D^{*-} \to D^- + \pi^0/\gamma$
  $B^+ \to \bar{D}_2^{*0} \ell^+ \nu$ with $\bar{D}_2^{*0} \to D^- + \pi^+$
  The coefficient is

$$C_9^{MC} = \frac{\begin{array}{l} C_{B^+}\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu)\times \\ \{\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)[\mathcal{B}(D^{*+} \to D^+\pi^0) + \mathcal{B}(D^{*+} \to D^+\gamma)] + \mathcal{B}(D_2^{*0} \to D^+\pi^-)\} \\ \times\mathcal{B}(D^+ \to K^-\pi^+\pi^+) \end{array}}{\begin{array}{l} \mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu)^0\times \\ \{\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)^0[\mathcal{B}(D^{*+} \to D^+\pi^0)^0 + \mathcal{B}(D^{*+} \to D^+\gamma)^0] + \mathcal{B}(D_2^{*0} \to D^+\pi^-)^0)\} \\ \times\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0 \end{array}} \quad (78)$$

- $j = 10:\ B^0 \to D_2^{*-} \ell^+ \nu$
  The decays contribute to this mode are
  $B^0 \to D_2^{*-} \ell^+ \nu$ with $D_2^{*-} \to D^{*-}\pi^0$ and $D^{*-} \to D^- + \pi^0/\gamma$
  $B^0 \to D_2^{*-} \ell^+ \nu$ with $D_2^{*-} \to D^-\pi^0$

The coefficient is

$$C_{10}^{MC} = \frac{\begin{array}{l} C_{B^0}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\times \\ \{\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0)[\mathcal{B}(D^{*+} \to D^+\pi^0) + \mathcal{B}(D^{*+} \to D^+\gamma)] + \mathcal{B}(D_2^{*+} \to D^+\pi^0)\} \\ \times\mathcal{B}(D^+ \to K^-\pi^+\pi^+) \end{array}}{\begin{array}{l} \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)^0\times \\ \{\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0)^0[\mathcal{B}(D^{*+} \to D^+\pi^0)^0 + \mathcal{B}(D^{*+} \to D^+\gamma)^0] + \mathcal{B}(D_2^{*+} \to D^+\pi^0)^0)\} \\ \times\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0 \end{array}} \tag{79}$$

- $j = 11 :\ B^+ \to D^-\pi^+\ell^+\nu$
  The coefficient is

$$C_{11}^{MC} = \frac{C_{B^+}\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)^0\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{80}$$

- $j = 12 :\ B^0 \to D^-\pi^0\ell^+\nu$
  The coefficient is

$$C_{12}^{MC} = \frac{C_{B^0}\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)^0\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{81}$$

- $j = 13 :\ B^+ \to D^{*-}\pi^+\ell^+\nu$ with $D^{*-} \to D^- + \gamma/\pi^0$
  The coefficient is

$$C_{13}^{MC} = \frac{\begin{array}{l} C_{B^+}\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)\times \\ [\mathcal{B}(D^{*+} \to D^+\pi^0) + \mathcal{B}(D^{*+} \to D^+\gamma)]\mathcal{B}(D^+ \to K^-\pi^+\pi^+) \end{array}}{\begin{array}{l} \mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)^0\times \\ [\mathcal{B}(D^{*+} \to D^+\pi^0)^0 + \mathcal{B}(D^{*+} \to D^+\gamma)^0]\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0 \end{array}} \tag{82}$$

- $j = 14 :\ B^0 \to D^{*-}\pi^0\ell^+\nu$ with $D^{*-} \to D^- + \gamma/\pi^0$
  The coefficient is

$$C_{14}^{MC} = \frac{\begin{array}{l} C_{B^0}\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)\times \\ [\mathcal{B}(D^{*+} \to D^+\pi^0) + \mathcal{B}(D^{*+} \to D^+\gamma)]\mathcal{B}(D^+ \to K^-\pi^+\pi^+) \end{array}}{\begin{array}{l} \mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)^0\times \\ [\mathcal{B}(D^{*+} \to D^+\pi^0)^0 + \mathcal{B}(D^{*+} \to D^+\gamma)^0]\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0 \end{array}} \tag{83}$$

4.4.2 Coefficients for backgrounds : $j = 15 - 26$

There are 16 background MC modes (j). The coefficient of each mode is

- $j = 15$, UncorrDL B^+B^-

$$C_{15}^{MC} = C_{UncorDL}^{D^+} \frac{C_{B^+}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{84}$$

- $j = 16$, UncorrDL from $B^0\bar{B}^0$

$$C_{16}^{MC} = C_{UncorDL}^{D^+} \frac{C_{B^0}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{85}$$

- $j = 17$, UncorrCL from B^+B^-

$$C_{17}^{MC} = C_{UncorCL}^{D^+} \frac{C_{B^+}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{86}$$

- $j = 18$, UncorrCL from $B^0\bar{B}^0$

$$C_{18}^{MC} = C_{UncorCL}^{D^+} \frac{C_{B^0}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{87}$$

- $j = 19$, CascL from B^+B^-

$$C_{19}^{MC} = C_{CascL}^{D^+} \frac{C_{B^+}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{88}$$

- $j = 20$, CascL from $B^0\bar{B}^0$

$$C_{20}^{MC} = C_{CascL}^{D^+} \frac{C_{B^0}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{89}$$

- $j = 21$, FakeL from B^+B^-

$$C_{21}^{MC} = C_{FakeL}^{D^+} \frac{C_{B^+}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{90}$$

- $j = 22$, FakeL from $B^0\bar{B}^0$

$$C_{22}^{MC} = C_{FakeL}^{D^+} \frac{C_{B^0}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{91}$$

- $j = 23$, CombRL from B^+B^-

$$C_{23}^{MC} = \frac{C_{B^+}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{92}$$

- $j = 24$, CombRL from $B^0\bar{B}^0$

$$C_{24}^{MC} = \frac{C_{B^0}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{93}$$

- $j = 25$, CombFL from B^+B^-

$$C_{25}^{MC} = \frac{C_{B^+}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{94}$$

- $j = 26$, CombFL from $B^0\bar{B}^0$

$$C_{26}^{MC} = \frac{C_{B^0}\mathcal{B}(D^+ \to K^-\pi^+\pi^+)}{\mathcal{B}(D^+ \to K^-\pi^+\pi^+)^0} \tag{95}$$

5 Constraints on variables in the fit

5.1 $B \to D^{**}\ell\nu$ and $B \to D^{(*)}\pi\ell\nu$ decay branching fractions

[Note] The calculated results in this subsection are not used in our fit as explained later. We keep this section for the following purposes :

- **to show the origin of our parametrization.**
- **to keep the original numbers for a reference.**

Our fitting method does not have the sensitivity to determine all 16 branching fractions of $B \to D^{**}\ell\nu$ and $B \to D^{(*)}\pi\ell\nu$ (non-resonant) decays. Thus, we need constraints to reduce free parameters. Available measurements of these decay modes are very limited. Currently we have only four usable results. The first is from PDG [12]

$$\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) = 0.0056 \pm 0.0013 \pm 0.0009 = 0.0056 \pm 0.00158 \tag{96}$$

The next is from the most recent Babar measurement [15]

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) &= 0.00448 \pm 0.00026 \pm 0.00035 = 0.00448 \pm 0.000436 \\
\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu) &= 0.00354 \pm 0.00032 \pm 0.00054 = 0.00354 \pm 0.000627 \\
\mathcal{B}(B^0 \to D_1^-\ell^+\nu) &= 0.00364 \pm 0.00032 \pm 0.00049 = 0.00364 \pm 0.000585 \\
\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu) &= 0.00270 \pm 0.00035 \pm 0.00043 = 0.00270 \pm 0.000554
\end{aligned} \tag{97}$$

The third is from the most recent DELPHI measurement [16]

$$\begin{aligned}
&\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) = 0.0042 \pm 0.0033 \pm 0.0022 = 0.0042 \pm 0.0040 \\
&\mathcal{B}(B^0 \to D_1'^-\ell^+\nu) = 0.0124 \pm 0.0025 \pm 0.0027 = 0.0124 \pm 0.0037 \\
&\mathcal{BT}(B^0 \to D^{**-}\ell^+\nu) \\
&\equiv \mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) + \mathcal{B}(B^0 \to D_1'^-\ell^+\nu) + \mathcal{B}(B^0 \to D_1^-\ell^+\nu) + \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu) \\
&= 0.027 \pm 0.007 \pm 0.002 = 0.027 \pm 0.073
\end{aligned} \tag{98}$$

The last is also from PDG [12]

$$\begin{aligned}
\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) &= 0.0053 \pm 0.0009 \pm 0.0005 = 0.0053 \pm 0.0010 \\
\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu) &= 0.0064 \pm 0.0015 \pm 0.0002 = 0.0064 \pm 0.0015 \\
\mathcal{BT}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) &= 0.0032 \pm 0.0009 \pm 0.0003 = 0.0032 \pm 0.00095 \\
\mathcal{BT}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) &= 0.0065 \pm 0.0015 \pm 0.0005 = 0.0065 \pm 0.0016
\end{aligned} \tag{99}$$

These values are based on a BELLE measurement [17] and $B \to D^{**}\ell\nu$ decays are included in these branching fractions.

As we show in the following sections, it is convenient to divide $B \to D^{**}\ell\nu$ and $B \to D^{(*)}\pi\ell\nu$ (non-resonant) decay branching fractions into three categories : "$B \to D_1\ell\nu$ and $B \to D_2^*\ell\nu$", "$B \to D_0^*\ell\nu$ and $B \to D\pi\ell\nu$", and "$B \to D_1'\ell\nu$ and $B \to D^*\pi\ell\nu$".

5.1.1 $B \to D_1 \ell\nu$ and $B \to D_2^* \ell\nu$

There are 4 branching fractions in these decay modes : $\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu)$, $\mathcal{B}(B^0 \to D_1^- \ell^+\nu)$, $\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu)$ and $\mathcal{B}(B^0 \to D_2^{*-} \ell^+\nu)$. We define two parameters, one total branching fraction and one branching fraction ratio, and relate the 4 branching fractions to the parameters. Based on isospin symmetry

$$\begin{aligned}\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) &= t_{+0}\mathcal{B}(B^0 \to D_1^- \ell^+\nu) \\ \mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu) &= t_{+0}\mathcal{B}(B^0 \to D_2^{*-} \ell^+\nu)\end{aligned} \tag{100}$$

The Babar results (Equation (97)) show that isospin symmetry holds. Thus, we take averages. As we do not know the correlation between these branching fractions, we take a conservative side and assume the maximum correlation

$$\begin{aligned}\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) &= \tfrac{1}{2}\left[(0.00448 \pm 0.000436) + (0.00364 \pm 0.000585) \times (1.071 \pm 0.009)\right] \\ &= \tfrac{1}{2}\left[(0.00448 \pm 0.000436) + (0.003898 \pm 0.0006274)\right] \\ &= 0.00419 \pm 0.000532\end{aligned} \tag{101}$$

and

$$\begin{aligned}\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu) &= \tfrac{1}{2}\left[(0.00354 \pm 0.000627) + (0.00270 \pm 0.000554) \times (1.071 \pm 0.009)\right] \\ &= \tfrac{1}{2}\left[(0.00354 \pm 0.000627) + (0.002892 \pm 0.0005938)\right] \\ &= 0.003215 \pm 0.0006104\end{aligned} \tag{102}$$

Thus, if we define a total branching fraction

$$\begin{aligned}\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0} \ell^+\nu) &\equiv \mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu) \\ &= (0.00419 \pm 0.000532) + (0.003215 \pm 0.0006104) \\ &= 0.007405 \pm 0.0011874\end{aligned} \tag{103}$$

and a branching fraction ratio

$$\begin{aligned}f_{D_2^*/D_1} \equiv \frac{\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu)} &= \frac{0.003215 \pm 0.0006104}{0.00419 \pm 0.000532} \\ &= 0.767 \pm 0.243\end{aligned} \tag{104}$$

We can relate 4 branching fractions to $f_{D_2^*/D_1}$ and $\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0} \ell^+\nu)$

$$\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) = \frac{1}{1+f_{D_2^*/D_1}}\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0} \ell^+\nu) \tag{105}$$

$$\begin{aligned}\mathcal{B}(B^0 \to D_1^- \ell^+\nu) &= \frac{\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu)}{t_{+0}} \\ &= \frac{1}{t_{+0}(1+f_{D_2^*/D_1})}\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0} \ell^+\nu)\end{aligned} \tag{106}$$

$$\begin{aligned}\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu) &= f_{D_2^*/D_1}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) \\ &= \frac{f_{D_2^*/D_1}}{1+f_{D_2^*/D_1}}\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)\end{aligned} \qquad (107)$$

$$\begin{aligned}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)}{t_{+0}} \\ &= \frac{\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)}{t_{+0}(1+f_{D_2^*/D_1})}\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)\end{aligned} \qquad (108)$$

5.1.2 $B \to D_0^*\ell\nu$ and non-resonant $B \to D\pi\ell\nu$

There are 6 branching fractions in this category : $\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)$, $\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)$, $\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)$, $\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)$, $\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)$ and $\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)$. As in the previous section, we define two parameters, one total branching fraction and one branching fraction ratio, and relate the 6 branching fractions to the parameters. Isospin symmetry gives

$$\begin{aligned}&\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) \\ &\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu) \\ &\quad = t_{+0}\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu) \\ &\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) = t_{+0}\mathcal{BT}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)\end{aligned} \qquad (109)$$

Note that $\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu)$ is what is given in Equation (99). This includes not only non-resonant $D^-\pi^+$ mode but also relevant D^{**} modes. Thus,

$$\begin{aligned}&\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) \\ &= \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)\mathcal{B}(\bar{D}_0^{*0} \to D^-\pi^+) + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(\bar{D}_2^{*0} \to D^-\pi^+) \\ &\quad + \mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)\end{aligned} \qquad (110)$$

Using the above isospin condition, we can take an average of Equation (99). Here, we also take the maximum correlation

$$\begin{aligned}&\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) \\ &= \tfrac{1}{2}\left[(0.0053 \pm 0.0010) + (0.00323 \pm 0.00095) \times (1.071 \pm 0.009)\right] \\ &= \tfrac{1}{2}\left[(0.0053 \pm 0.0010) + (0.00346 \pm 0.001018)\right] \\ &= 0.00438 \pm 0.00101\end{aligned} \qquad (111)$$

From Equation (110),

$$\begin{aligned}&\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) \\ &= \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) \times \tfrac{2}{3} + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(\bar{D}_2^{*0} \to D^-\pi^+) \\ &\quad + \mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)\end{aligned} \qquad (112)$$

Using equations (197) (in section 9) and (102),

$$\begin{aligned}
&\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) + \tfrac{2}{3}\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) \\
&= \mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) - \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(\bar{D}_2^{*0} \to D^-\pi^+) \\
&= (0.00438 \pm 0.00101) - (0.003215 \pm 0.0006104) \times (0.452 \pm 0.167) \\
&= (0.00438 \pm 0.00101) - (0.001453 \pm 0.0006036) \\
&= 0.00293 \pm 0.00118
\end{aligned} \tag{113}$$

Thus, if we define a total branching fraction

$$\begin{aligned}
&\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu) \\
&\equiv \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) + \mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu) \\
&= \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) + \tfrac{3}{2}\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) \\
&= \tfrac{3}{2}(0.00293 \pm 0.00118) = 0.0044 \pm 0.0018
\end{aligned} \tag{114}$$

and a branching fraction ratio

$$f_{D\pi/D_0^*} \equiv \frac{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)} \tag{115}$$

we can relate 6 branching fractions to $\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)$ and $f_{D\pi/D_0^*}$.

$$\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) = \frac{2}{2+3f_{D\pi/D_0^*}}\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu) \tag{116}$$

$$\begin{aligned}
\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) &= f_{D\pi/D_0^*}\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) \\
&= \frac{2f_{D\pi/D_0^*}}{2+3f_{D\pi/D_0^*}}\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)
\end{aligned} \tag{117}$$

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)}{2} \\
&= \frac{f_{D\pi/D_0^*}}{2+3f_{D\pi/D_0^*}}\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)
\end{aligned} \tag{118}$$

$$\begin{aligned}
\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)}{t_{+0}} \\
&= \frac{2}{t_{+0}(2+3f_{D\pi/D_0^*})}\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)
\end{aligned} \tag{119}$$

$$\begin{aligned}
\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)}{t_{+0}} \\
&= \frac{2f_{D\pi/D_0^*}}{t_{+0}(2+3f_{D\pi/D_0^*})}\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)
\end{aligned} \tag{120}$$

$$\begin{aligned}
\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)}{t_{+0}} \\
&= \frac{f_{D\pi/D_0^*}}{t_{+0}(2+3f_{D\pi/D_0^*})}\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)
\end{aligned} \tag{121}$$

Now, let us calculate $f_{D\pi/D_0^*}$. Using Equation (98)

$$\begin{aligned}&\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)\\&= (1.071 \pm 0.009)(0.0042 \pm 0.0040) = 0.0045 \pm 0.0043\end{aligned} \tag{122}$$

Thus, from Equation (113),

$$\begin{aligned}\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) &= (0.00293 \pm 0.00118) - \tfrac{2}{3}(0.0045 \pm 0.0043)\\&= -0.00007 \pm 0.00310\end{aligned} \tag{123}$$

Then, $f_{D\pi/D_0^*}$ is given by

$$f_{D\pi/D_0^*} = \frac{-0.00007 \pm 0.00310}{0.0045 \pm 0.0043} = -0.016 \pm 0.689 \tag{124}$$

The ratio $f_{D\pi/D_0^*}$ cannot be negative. Thus, we take a positive number as described below.

5.1.3 $B \to D_1'\ell\nu$ and non-resonant $B \to D^*\pi\ell\nu$.

There are also 6 branching fractions in this category : $\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu)$, $\mathcal{B}(B^0 \to D_1'^-\ell^+\nu)$, $\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)$, $\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)$, $\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)$ and $\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)$. Again, we define two parameters, one total branching fraction and one branching fraction ratio, and relate the 6 branching fractions to the parameters. From isospin symmetry

$$\begin{aligned}&\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_1'^-\ell^+\nu)\\&\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)\\&\quad = t_{+0}\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)\\&\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu) = t_{+0}\mathcal{BT}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)\end{aligned} \tag{125}$$

Note that $\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu)$ is what is given in Equation (99). This includes not only non-resonant $D^{*-}\pi^+$ mode but also relevant D^{**} modes. Thus,

$$\begin{aligned}&\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu)\\&= \mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\mathcal{B}(D_1^0 \to D^{*+}\pi^-) + \mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu)\mathcal{B}(D_1'^0 \to D^{*+}\pi^-)\\&\quad +\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-) + \mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)\end{aligned} \tag{126}$$

Using the above isospin condition, we can take an average of Equation (99). Again, assuming the maximum correlation,

$$\begin{aligned}&\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu)\\&= \tfrac{1}{2}\left[(0.0064 \pm 0.0015) + (0.0065 \pm 0.0016) \times (1.071 \pm 0.009)\right]\\&= \tfrac{1}{2}\left[(0.0064 \pm 0.0015) + (0.00696 \pm 0.00171)\right]\\&= 0.00668 \pm 0.00161\end{aligned} \tag{127}$$

From Equation (126),

$$\begin{aligned}
&\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu) \\
&= \tfrac{2}{3}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) + \tfrac{2}{3}\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) \\
&\quad +\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) + \mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)
\end{aligned} \tag{128}$$

Using equations (101), (199) (in section 9) and (102),

$$\begin{aligned}
&\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) + \tfrac{2}{3}\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) \\
&= \mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu) - \tfrac{2}{3}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) - \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) \\
&= (0.00668 \pm 0.00161) - \tfrac{2}{3} \times (0.00419 \pm 0.000532) \\
&\qquad -(0.003215 \pm 0.0006104) \times (0.215 \pm 0.045) \\
&= (0.00668 \pm 0.00161) - (0.00279 \pm 0.0003547) - (0.000691 \pm 0.0001953) \\
&= 0.00319 \pm 0.00166
\end{aligned} \tag{129}$$

Thus, if we define a total branching fraction

$$\begin{aligned}
&\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu) \\
&\equiv \mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) + \mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) \\
&= \mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) + \tfrac{3}{2}\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) \\
&= \tfrac{3}{2}(0.00319 \pm 0.00166) = 0.0048 \pm 0.0025
\end{aligned} \tag{130}$$

and a branching fraction ratio

$$f_{D^*\pi/D_1'} \equiv \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu)} \tag{131}$$

We can relate the 6 branching fractions to $\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)$ and $f_{D^*\pi/D_1'}$

$$\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) = \frac{2}{2+3f_{D^*\pi/D_1'}}\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu) \tag{132}$$

$$\begin{aligned}
\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) &= f_{D^*\pi/D_1'}\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) \\
&= \frac{2f_{D^*\pi/D_1'}}{2+3f_{D^*\pi/D_1'}}\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)
\end{aligned} \tag{133}$$

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{2} \\
&= \frac{f_{D^*\pi/D_1'}}{2+3f_{D^*\pi/D_1'}}\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)
\end{aligned} \tag{134}$$

$$\begin{aligned}
\mathcal{B}(B^0 \to D_1'^-\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu)}{t_{+0}} \\
&= \frac{2}{t_{+0}(2+3f_{D^*\pi/D_1'})}\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)
\end{aligned} \tag{135}$$

$$\begin{aligned}\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{t_{+0}} \\ &= \frac{2f_{D^*\pi/D_1'}}{t_{+0}(2+3f_{D^*\pi/D_1'})}\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu) \end{aligned} \tag{136}$$

$$\begin{aligned}\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu) &= \frac{\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)}{t_{+0}} \\ &= \frac{f_{D^*\pi/D_1'}}{t_{+0}(2+3f_{D^*\pi/D_1'})}\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu) \end{aligned} \tag{137}$$

Now, let us calculate $f_{D^*\pi/D_1'}$. Using Equation (98)

$$\begin{aligned}&\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_1'^-\ell^+\nu) \\ &= (1.071 \pm 0.009)(0.0124 \pm 0.0037) = 0.0133 \pm 000725\end{aligned} \tag{138}$$

Thus, by Equation (129),

$$\begin{aligned}&\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) = (0.00319 \pm 0.00166) - \tfrac{2}{3}(0.0133 \pm 0.00725) \\ &\quad = -0.005677 \pm 0.00511\end{aligned} \tag{139}$$

Then, $f_{D\pi/D_0^*}$ is

$$f_{D^*\pi/D_1'} = \frac{-0.005677 \pm 0.00511}{0.0133 \pm 0.00725} = -0.427 \pm 0.449 \tag{140}$$

The ratio $f_{D^*\pi/D_1'}$ cannot be negative. Thus, we take a positive number as described below.

5.1.4 $B \to D^{**}\ell\nu$ and non-resonant $B \to D^{(*)}\pi\ell\nu$

We define a total branching fraction

$$\begin{aligned}&\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu) \\ &\equiv \mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu) + \mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu) + \mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)\end{aligned} \tag{141}$$

We also define two ratios

$$\begin{aligned}f_{D_0^*D\pi T/D_1D_2^*T} &\equiv \frac{\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = \frac{3}{2}\frac{0.00293 \pm 0.00118}{0.007405 \pm 0.0011874} \\ &= 0.59 \pm 0.26\end{aligned} \tag{142}$$

and

$$\begin{aligned}f_{D_1'D^*\pi T/D_1D_2^*T} &\equiv \frac{\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = \frac{3}{2}\frac{0.00319 \pm 0.00166}{0.007405 \pm 0.0011874} \\ &= 0.65 \pm 0.35\end{aligned} \tag{143}$$

Then, we can relate all 16 branching fractions of $B \to D^{**}\ell\nu$ and non-resonant $B \to D^{(*)}\pi\ell\nu$ decays to $\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)$ and five ratios. First, we have

$$\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu) = \frac{\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)}{1 + f_{D_0^* D\pi T/D_1 D_2^* T} + f_{D_1' D^*\pi T/D_1 D_2^* T}} \tag{144}$$

$$\begin{aligned}\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu) &= f_{D_0^* D\pi T/D_1 D_2^* T}\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu) \\ &= \frac{f_{D_0^* D\pi T/D_1 D_2^* T}\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)}{1 + f_{D_0^* D\pi T/D_1 D_2^* T} + f_{D_1' D^*\pi T/D_1 D_2^* T}}\end{aligned} \tag{145}$$

$$\begin{aligned}\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu) &= f_{D_1' D^*\pi T/D_1 D_2^* T}\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu) \\ &= \frac{f_{D_1' D^*\pi T/D_1 D_2^* T}\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)}{1 + f_{D_0^* D\pi T/D_1 D_2^* T} + f_{D_1' D^*\pi T/D_1 D_2^* T}}\end{aligned} \tag{146}$$

Substituting these into equations (105)-(108), (116)-(121) and (132)-(137),

$$\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) = \frac{\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)}{(1 + f_{D_2^*/D_1})(1 + f_{D_0^* D\pi T/D_1 D_2^* T} + f_{D_1' D^*\pi T/D_1 D_2^* T})} \tag{147}$$

$$\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu) = f_{D_2^*/D_1}\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) \tag{148}$$

$$\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) = \frac{2f_{D_0^* D\pi T/D_1 D_2^* T}\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)}{(2 + 3f_{D\pi/D_0^*})(1 + f_{D_0^* D\pi T/D_1 D_2^* T} + f_{D_1' D^*\pi T/D_1 D_2^* T})} \tag{149}$$

$$\mathcal{B}(B^+ \to \bar{D}_1^{\prime 0}\ell^+\nu) = \frac{2f_{D_1' D^*\pi T/D_1 D_2^* T}\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)}{(2 + 3f_{D^*\pi/D_1'})(1 + f_{D_0^* D\pi T/D_1 D_2^* T} + f_{D_1' D^*\pi T/D_1 D_2^* T})} \tag{150}$$

$$\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) = f_{D\pi/D_0^*}\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) \tag{151}$$

$$\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu) = \frac{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)}{2} \tag{152}$$

$$\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) = f_{D^*\pi/D_1'}\mathcal{B}(B^+ \to \bar{D}_1^{\prime 0}\ell^+\nu) \tag{153}$$

$$\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) = \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{2} \tag{154}$$

and

$$\mathcal{B}(B^0 \to D_1^-\ell^+\nu) = \frac{\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)}{t_{+0}} \tag{155}$$

$$\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu) = \frac{\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)}{t_{+0}} \tag{156}$$

$$\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) = \frac{\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)}{t_{+0}} \tag{157}$$

$$\mathcal{B}(B^0 \to D_1^{\prime -}\ell^+\nu) = \frac{\mathcal{B}(B^+ \to \bar{D}_1^{\prime 0}\ell^+\nu)}{t_{+0}} \tag{158}$$

$$\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) = \frac{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)}{t_{+0}} \tag{159}$$

$$\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu) = \frac{\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)}{t_{+0}} \tag{160}$$

$$\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) = \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{t_{+0}} \tag{161}$$

$$\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu) = \frac{\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)}{t_{+0}} \tag{162}$$

Thus, we have only one branching fraction, $\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)$, to be determined by the fit. We also have five ratios. Three of them are determined from measurements as described above (Equations (104), (142) and (143)).

$$f_{D_2^*/D_1} = 0.767 \pm 0.243 \tag{163}$$

$$f_{D_0^* D\pi T/D_1 D_2^* T} = 0.59 \pm 0.26 \tag{164}$$

$$f_{D_1' D^*\pi T/D_1 D_2^* T} = 0.65 \pm 0.35 \tag{165}$$

and the other two have negative central numbers (Equations (124) and (140))

$$f_{D\pi/D_0^*} = -0.016 \pm 0.689 \tag{166}$$

$$f_{D^*\pi/D_1'} = -0.427 \pm 0.449 \tag{167}$$

In our fit, we set

$$f_{D\pi/D_0^*} = 0.2 \tag{168}$$

$$f_{D^*\pi/D_1'} = 0.1 \tag{169}$$

and we vary these values from 0 to 2 to get the systematic uncertainty from these ratios.
It is worth noting that our fit does not have the sensitivity to determine these ratios. We tried to float these ratios in our fit, but the results give large statistical uncertainties, for example, values larger than 100 %. Thus, we decided to fix these ratios in our fit.

5.2 New standard branching fractions from Semileotonic AWG

The following are the new branching fraction values recommended by the Semileptonic AWG [18]. We use these values in our fit, instead of the values shown above.

- B^+ decays
  - $\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) = 0.0045 \pm 0.0009$
  - $\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) = 0.0054 \pm 0.0006$
  - $\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) = 0.0085 \pm 0.0020$
  - $\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu) = 0.0042 \pm 0.0008$
  - $\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) = 0.0019 \pm 0.0012$
  - $\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu) = 0.0010 \pm 0.0006$

  - $\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) = 0.0006 \pm 0.0004$
  - $\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) = 0.0003 \pm 0.0002$

- B^0 decays
  - $\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) = 0.0043 \pm 0.0009$
  - $\mathcal{B}(B^0 \to D_1^{-}\ell^+\nu) = 0.0050 \pm 0.0008$
  - $\mathcal{B}(B^0 \to D_1'^{-}\ell^+\nu) = 0.0080 \pm 0.0020$
  - $\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu) = 0.0039 \pm 0.0007$
  - $\mathcal{B}(B^0 \to \bar{D}^{0}\pi^-\ell^+\nu) = 0.0019 \pm 0.0012$
  - $\mathcal{B}(B^0 \to D^{-}\pi^0\ell^+\nu) = 0.0009 \pm 0.0006$
  - $\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) = 0.0006 \pm 0.0004$
  - $\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu) = 0.0003 \pm 0.0004$

Here, isospin symmetry is assumed. We calculate the branching fraction ratios by assuming the above branching fractions are independent.

$$\begin{aligned}
&f_{D_2^*/D_1} \equiv \frac{\mathcal{B}(B^+\to\bar{D}_2^{*0}\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^{0}\ell^+\nu)} = 0.78 \pm 0.17 \\
&f_{D_0^*D\pi T/D_1D_2^*T} \equiv \frac{\mathcal{BT}(B^+\to\bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^{0}/\bar{D}_2^{*0}\ell^+\nu)} = 0.77 \pm 0.19 \\
&f_{D_1'D^*\pi T/D_1D_2^*T} \equiv \frac{\mathcal{BT}(B^+\to\bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^{0}/\bar{D}_2^{*0}\ell^+\nu)} = 0.98 \pm 0.24 \\
&f_{D\pi/D_0^*} \equiv \frac{\mathcal{B}(B^+\to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_0^{*0}\ell^+\nu)} = 0.42 \pm 0.28 \\
&f_{D^*\pi/D_1'} \equiv \frac{\mathcal{B}(B^+\to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^{*0}\ell^+\nu)} = 0.071 \pm 0.050
\end{aligned} \tag{170}$$

5.3 Backgrounds

In our fit there are 8 background coefficients for peaking backgrounds: $C^{D^0}_{UncorDL}$, $C^{D^0}_{UncorCL}$, $C^{D^0}_{CascL}$, $C^{D^0}_{FakeL}$, $C^{D^+}_{UncorDL}$, $C^{D^+}_{UncorCL}$, $C^{D^+}_{CascL}$ and $C^{D^+}_{FakeL}$. We may float some of these and fix the other depending on the fit. Note that combinatorial backgrounds are always fixed.

5.4 Summary of fitting parameters and constraints

As a summary, we are trying to fit for three branching fractions and Υ(4S) decay BF ratio

- $\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$
- $\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)$
- f_{+0}

with isospin constraints

- $\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^-\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}_1^{0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_1^{-}\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}_1^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_1^{\prime -}\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)$
- $\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)$
  $= t_{+0}\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)$
- $\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)$
  $= t_{+0}\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)$

and constraints on relative branching fractions

- $f_{D_2^*/D_1} \equiv \frac{\mathcal{B}(B^+\to\bar{D}_2^{*0}\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^{0}\ell^+\nu)} = 0.78 \pm 0.17$
- $f_{D_0^* D\pi T/D_1 D_2^* T} \equiv \frac{\mathcal{BT}(B^+\to\bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^{0}/\bar{D}_2^{*0}\ell^+\nu)} = 0.77 \pm 0.19$
- $f_{D_1' D^*\pi T/D_1 D_2^* T} \equiv \frac{\mathcal{BT}(B^+\to\bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^{0}/\bar{D}_2^{*0}\ell^+\nu)} = 0.98 \pm 0.24$
- $f_{D\pi/D_0^*} \equiv \frac{\mathcal{B}(B^+\to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_0^{*0}\ell^+\nu)} = 0.42 \pm 0.28$
- $f_{D^*\pi/D_1'} \equiv \frac{\mathcal{B}(B^+\to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^{*0}\ell^+\nu)} = 0.071 \pm 0.050$

and floating or fixing background components

- $C^{D^0}_{UncorDL}$
- $C^{D^0}_{UncorCL}$
- $C^{D^0}_{CascL}$
- $C^{D^0}_{FakeL}$
- $C^{D^+}_{UncorDL}$
- $C^{D^+}_{UncorCL}$
- $C^{D^+}_{CascL}$
- $C^{D^+}_{FakeL}$

We also fit for form factor parameters of $B \to D^{(*)}\ell\nu$ decays. See details in section 10.

6 Event Selection

We use the data sample of approximately 230 million $B\bar{B}$-pairs, collected by the BaBar detector. This corresponds to the integrated luminosity of 207 fb^{-1}. A description of the BaBar detector can be found elsewhere [19]. The $B\bar{B}$-pairs are produced from $\Upsilon(4S)$ resonance created by the asymmetric e^+e^- collider, PEP-II [20], at the Stanford Linear Accelerator Center (SLAC). We also use 21.5 fb^{-1} of off-resonance data collected 40 MeV below the resonance. This can be used to subtract non-$B\bar{B}$ background. The package EvtGen [21] is used to generate events and to model decays, and the GEANT4 software [22] is used to simulate the Babar detector in our Monte Carlo simulation. We use analysis-26 (release 16.1.5) to analyze Run1-4 data and SP5/SP6 MC.

We consider the uncertainty on signal to optimize our cuts. We use generic MC (BpBm, B0B0bar, ccbar, uds and tautau) to model OnPeak data. Since we apply D mass sideband subtraction to reduce combinatorial and continuum backgrounds, we need to consider signal yields and background yields in the D mass peak and the D mass sideband regions. We denote those as N^{Sig}_{peak}, N^{Bkg}_{peak}, N^{Sig}_{side} and N^{Bkg}_{side}. We can determine these numbers using MC truth information and luminosity normalization to OnPeak data. The definition of sidebands are given in section 8.1. Since sidebands are larger than peak region, the number of candidates in the sidebands is scaled down to the peak region. The scale factor is

$$\begin{aligned} r^{D^0}_{side} &= \tfrac{1.888-1.840}{(1.840-1.806)+(1.922-1.888)} = 0.7059 \\ r^{D^+}_{side} &= \tfrac{1.890-1.848}{(1.848-1.810)+(1.928-1.890)} = 0.5526 \end{aligned} \tag{171}$$

Expected OnPeak yields, which are determined by MC, are given by

$$\begin{aligned} N^{OnPeak}_{peak} &= N^{Sig}_{peak} + N^{Bkg}_{peak} \\ N^{OnPeak}_{side} &= r_{side}(N^{Sig}_{side} + N^{Bkg}_{side}) \end{aligned} \tag{172}$$

In actual fit, we are interested in (OnPeak - OffPeak) data. The number of events, $N^{OffPeak}$, and uncertainty, $\sigma^{OffPeak}$, of OffPeak data after luminosity normalization to OnPeak data are given by

$$\begin{aligned} N^{OffPeak}_{peak} &= r^{Lumi}_{OffPeak} N^{OffOrg}_{peak} \\ \sigma^{OffPeak}_{peak} &= r^{Lumi}_{OffPeak} \sqrt{N^{OffOrg}_{peak}} \\ N^{OffPeak}_{side} &= r^{Lumi}_{OffPeak} r_{side} N^{OffOrg}_{side} \\ \sigma^{OffPeak}_{side} &= r^{Lumi}_{OffPeak} r_{side} \sqrt{N^{OffOrg}_{side}} \end{aligned} \tag{173}$$

where N^{OffOrg} is the actual yield of OffPeak data before normalization and $r^{Lumi}_{OffPeak}$ is the normalization factor (see section 3.4.1). We do not need to model OffPeak data by MC. This is because we do not need MC truth information to separate OffPaek candidates into signal and background. The signal yield, S, after sideband subtraction (and OffPeak subtraction) is given by

$$S = N^{Sig}_{peak} - r_{side} N^{Sig}_{side} \tag{174}$$

The variance by performing D mass sideband subtraction and OffPeak subtraction is expected to be

$$\begin{aligned}\sigma_S^2 &= (N_{peak}^{OnPeak} + N_{side}^{OnPeak}) + [(\sigma_{peak}^{OffPeak})^2 + (\sigma_{side}^{OffPeak})^2] \\ &= N_{peak}^{Sig} + N_{peak}^{Bkg} + r_{side}^2(N_{side}^{Sig} + N_{side}^{Bkg}) + \left(r_{OffPeak}^{Lumi}\right)^2 (N_{peak}^{OffOrg} + r_{side}^2 N_{side}^{OffOrg})\end{aligned} \tag{175}$$

Therefore, the uncertainty on signal is given by

$$\frac{\sigma_S}{S} = \frac{\sqrt{N_{peak}^{Sig} + N_{peak}^{Bkg} + r_{side}^2(N_{side}^{Sig} + N_{side}^{Bkg}) + \left(r_{OffPeak}^{Lumi}\right)^2 (N_{peak}^{OffOrg} + r_{side}^2 N_{side}^{OffOrg})}}{N_{peak}^{Sig} - r_{side} N_{side}^{Sig}} \tag{176}$$

6.1 BToDlnu skim

First, the collected data is precessed to select only $B \to D^{(*)}\ell\nu$ candidates. This process is done by BaBar collaboration and the produced data set is called BToDlnu skim. In this subsection, we summarize the selection criteria of BToDlnu skim. See also Appendix E.

6.1.1 Reconstruction of D^0 and D^+

D^0 is reconstructed by the following 4 modes:

- $D^0 \to K^-\pi^+$
- $D^0 \to K^-\pi^+\pi^+\pi^-$
- $D^0 \to K^-\pi^+\pi^0$
- $D^0 \to \bar{K}^0\pi^+\pi^-$

D^+ is reconstructed by the following 2 modes:

- $D^+ \to K^-\pi^+\pi^+$
- $D^+ \to \bar{K}^0\pi^+$

Figure 2 is the D^0 mass plots corresponding to four D^0 reconstruction modes. From top to bottom, $D^0 \to K^-\pi^+$, $D^0 \to K^-\pi^+\pi^+\pi^-$, $D^0 \to K^-\pi^+\pi^0$, $D^0 \to \bar{K}^0\pi^+\pi^-$. Figure 3 is the D^+ mass plots corresponding to two D^+ reconstruction modes. From top to bottom, $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to \bar{K}^0\pi^+$. In these plots, star is the On Peak data, green is the B^+B^- MC, yellow is the $B^0\bar{B}^0$ MC, blue is the Off Peak data, magenta is the $c\bar{c}$ MC, cyan is the uds MC, and gray is the $\tau^+\tau^-$ MC. From these plots, we can observe clear peaks only in the top plots. Thus,

- We use only $D^0 \to K^-\pi^+$ and $D^+ \to K^-\pi^+\pi^+$ modes.

For K^- tracks in the skim,

- GoodTracksLoose (GTL) is used to reconstruct D^0
- KLHNotPion selector is used to reconstruct D^+

We apply the same KLHNotPion to D^0 as described in subsection 6.2.
In BToDlnu skim, D^* is also reconstructed, but we do not use them in our analysis. This avoids the problems in the reconstruction of soft pions from D^* decays.

6.1.2 Reconstruction of Dl candidates

Electrons and muons are are selected by PID selectors.

- PidLHElectron selector for electrons
- muNNTight selector for muons

are used in the BToDlnu skims. These leptons are combined with D^0 or D^+ to form Dl candidates. Since neutrinos cannot be detected, we cannot reconstruct B mesons. Thus, these Dl candidates are the candidates of B mesons. Only one true Dl-pair can be produced from one B meson. However, tens of Dl candidates are usually reconstructed in a event. Most of them are combinatorial backgrounds with a misreconstructed D.
The following cuts are also applied

- Lepton momentum cut
  $0.8 < p^* < 3.0$ GeV, where p^* is the lepton momentum in the $\Upsilon(4S)$ rest frame.
- Lepton and kaon charge correlation cut
  The kaon and lepton from the same B meson must have the same sign of charge. See Appendix A.

6.2 Kaon Selection

In BToDlnu skim, for K^-, GoodTracksLoose (GTL) is used to reconstruct D^0, but KLHNotPion is used to reconstruct D^+. We apply KLHNotPion to D^0 reconstruction as well to reduce kaon misidentification. The uncertainty on signal before and after KLHNotPion are given in Table 7. We can reduce the uncertainty on signal by 43 %. There is no cahnge in D^+ mode because KLHNotPion selector was already applied in the BToDlnu skims.

6.3 Vertexing

No vertexing is done in BToDlnu skim, hence, we apply TreeFitter [23] to D vertex and B vertex. By B vertex, we mean the vertex of Dl combination. We cut on the probability value, $P(\chi^2, dof)$, calculated by the χ^2 and the number of degrees of

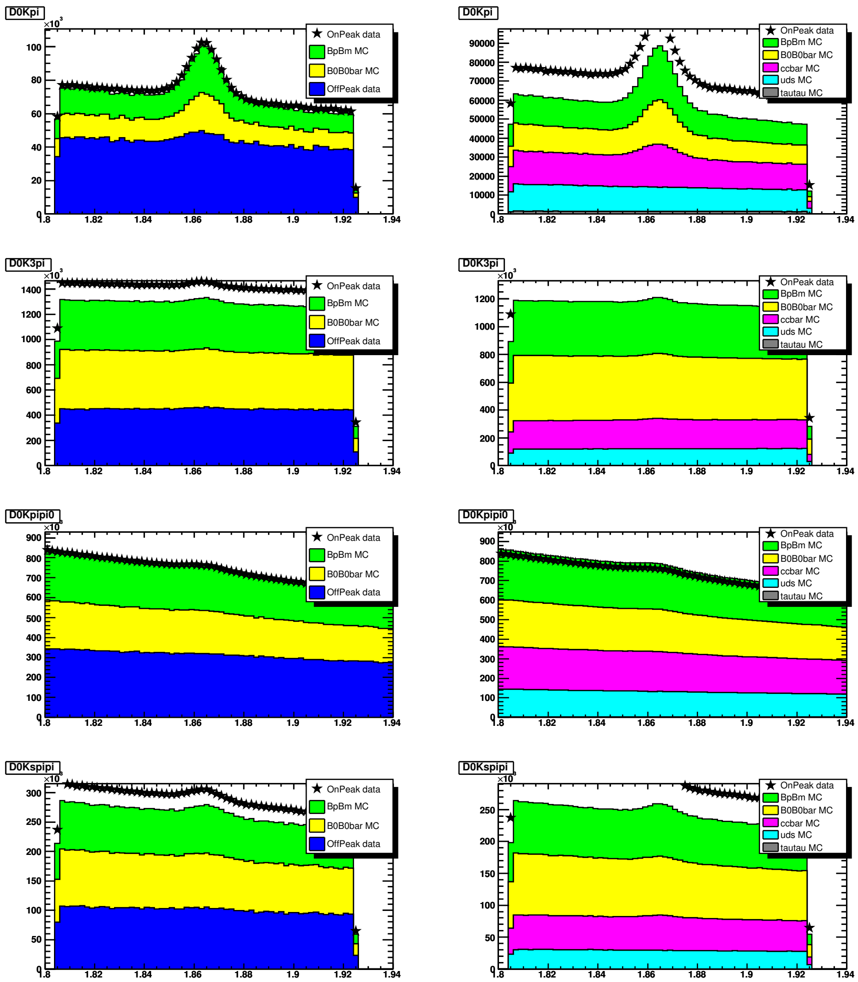


Figure 2: [D^0 **mass plots with 4 reconstruction modes**] D^0 mass plots. From top to bottom, $D^0 \to K^-\pi^+$, $D^0 \to K^-\pi^+\pi^+\pi^-$, $D^0 \to K^-\pi^+\pi^0$, $D^0 \to \bar{K}^0\pi^+$. Left-hand side is the comparison of OnPeak data with $B\bar{B}$ MC + OffPeak data. Right-hand side is of the OnPeak data with $B\bar{B}$ MC + (ccbar, uds and tautau MC). Star is the On Peak data, green is the B^+B^- MC, yellow is the $B^0\bar{B}^0$ MC, blue is the Off Peak data, magenta is the $c\bar{c}$ MC, cyan is the uds MC, and gray is the $\tau^+\tau^-$ MC.

	Before KLHNotPion	After KLHNotPion
D^0 $\frac{\sigma_S}{S}$	0.01152	0.00657
D^+ $\frac{\sigma_S}{S}$	0.01538	0.01538

Table 7: [**KLHNotPion selection**] The uncertainty on signal Signal (σ_S/S) before and after KLHNotPion

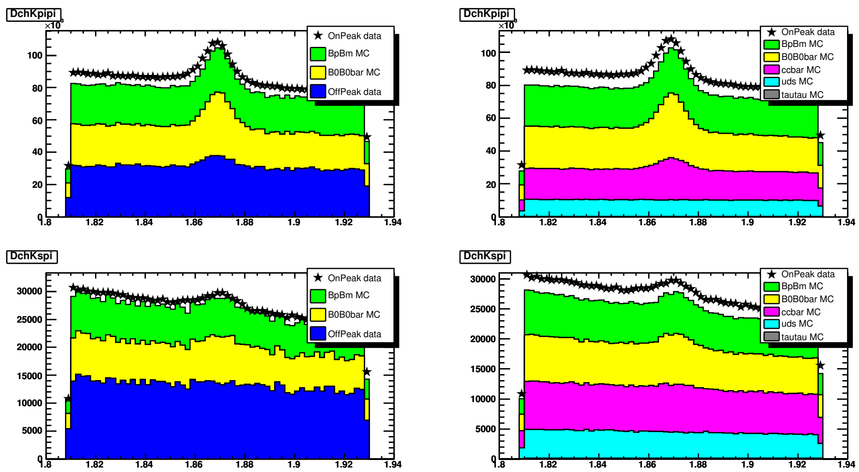


Figure 3: [D^+ **mass plots with 2 reconstruction modes**] D^+ mass plots. From top to bottom, $D^+ \to K^-\pi^+\pi^+$ and $D^+ \to \bar{K}^0\pi^+$. Left-hand side is the comparison of OnPeak data with $B\bar{B}$ MC + OffPeak data. Right-hand side is of the OnPeak data with $B\bar{B}$ MC + (ccbar, uds and tautau MC). Star is the On Peak data, green is the B^+B^- MC, yellow is the $B^0\bar{B}^0$ MC, blue is the Off Peak data, magenta is the $c\bar{c}$ MC, cyan is the uds MC, and gray is the $\tau^+\tau^-$ MC.

freedom (dof) of the vertex fitting. Table 8 shows the uncertainty on signal with variety of $P(\chi^2, dof)$ cuts of D and B vertexing. Tighter cuts give better uncertainties. However, since there is no big difference on uncertainties, we chose looser cuts :

$$P^D(\chi^2, dof) > 0.001, \quad P^B(\chi^2, dof) > 0.01, \tag{177}$$

These cuts improve the uncertainty by 20 % for D^0 mode and 24 % for D^+ mode.

6.4 Thrust cut

$B\bar{B}$ events are isotropic and other $q\bar{q}$ events tend to go back to back in the $\Upsilon(4S)$ rest frame. This is because the mass difference between $\Upsilon(4S)$ and, for example, $D\bar{D}$-pair is big and produced D mesons carry rather large momentum. Thus, we can separate $q\bar{q}$ events from $B\bar{B}$ events using thrust axis. We calculate thrust axis of Dl candidate and "non-Dl" particles in each event. Here, "non-Dl" particles mean all charged tracks and calorimeter energy deposits other than the ones used to reconstruct the Dl candidate. Dl is supposed to be one B in the event; thus, non-Dl should be the other B in the event. However, since we do not reconstruct D^* or D^{**}, soft pions from Dl side are included in non-Dl side. Nevertheless we can achieve a good separation between $B\bar{B}$ events and $q\bar{q}$ events as you can see in the following. We then calculate $|\cos\theta_{Dl-nonDl}|$, which is the cosine of the angle between thrust axis of Dl candidate and non-Dl particles. Those are shown in Figure 4. It is almost flat for signal events but peaked at $|\cos\theta_{Dl-nonDl}| = 1$ for $q\bar{q}$ events as expected.
To optimize the cut, we calculate the uncertainty on signal by changing cuts. Table 9

D^0	no cut	$P^D(\chi^2, dof) > 0.0001$	0.001	0.01	0.1
no cut	0.00657	0.00595	0.00589	0.00581	0.00589
$P^B > 0.0001$	0.00549	0.00548	0.00546	0.00542	0.00554
0.001	0.00538	0.00538	0.00537	0.00535	0.00548
0.01	0.00525	0.00525	**0.00525**	0.00525	0.00539
0.1	0.00520	0.00520	0.00520	0.00520	0.00531
D^+	no cut	$P^D(\chi^2, dof) > 0.0001$	0.001	0.01	0.1
no cut	0.01539	0.01283	0.01270	0.01256	0.01274
$P^B > 0.0001$	0.01221	0.01220	0.01212	0.01203	0.01223
0.001	0.01200	0.01200	0.01199	0.01192	0.01214
0.01	0.01174	0.01174	**0.01174**	0.01174	0.01200
0.1	0.01167	0.01167	0.01167	0.01167	0.01181

Table 8: [**Vertex cuts**] The change in the uncertainty on signal, σ_S/S, with variety of $P(\chi^2, dof)$ cuts of D and B vertexing. D^0 , D^+ are shown separately.

show the result. The lowest uncertainty is at $|\cos\theta_{Dl-nonDl}| < 0.80$. However, since no big difference between 0.80 and 0.92, we choose looser cut

$$\cos\theta_{Dl-nonDl} < 0.92 \tag{178}$$

With this cut, we can reduce the uncertainty by 18 % for D^0 and 16 % for D^+.

6.5 Summary of event selection cuts

The cuts we applied, the uncertainty on signal and the number of selected signal candidates in the D mass peak region are summarized in Table 10. We reduced uncertainty by 63 % for D^0 and 36 % for D^+ in total. We also lose 17 % and 18 % of signal candidates by these selection cuts. Tables 11 and 12 are cut-flow tables for D^0 and D^+ respectively. (OnPeak - OffPeak) data and $B\bar{B}$ MC are compared in these tables. Efficiencies show a good agreement. We apply futher kinematic cuts of $p^*_\ell > 1.2$ GeV, $p^*_D > 0.8$ GeV and $-2 < \cos\theta_{B-Dl} < 1.1$ (see section 13). The numbers after these cuts are also listed in the table.
After event selection cuts, number of candidates per event are

- Signal
  - $B^+ \to D^0/\bar{D}^0 X \ell^+ \nu$ mode : 1.0015
  - $B^+ \to D^+/D^- X \ell^+ \nu$ mode : 1.0012
  - $B^0 \to D^0/\bar{D}^0 X \ell^+ \nu$ mode : 1.0010

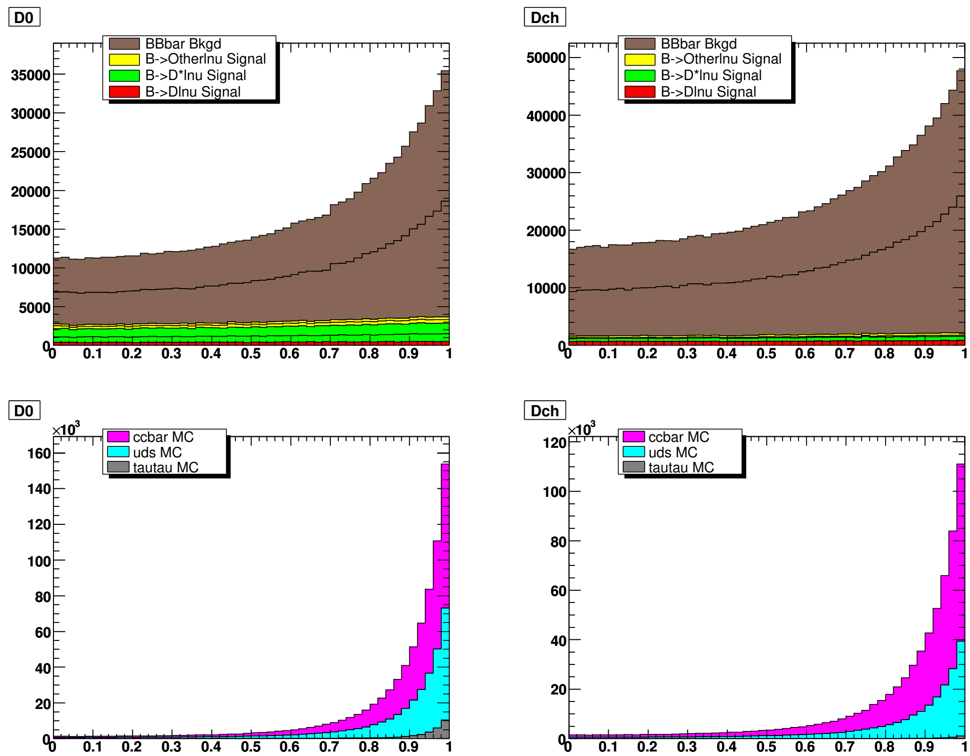


Figure 4: [$|\cos\theta_{Dl-nonDl}|$] $|\cos\theta_{Dl-nonDl}|$ plot for D^0 (lesf) and D^+ (right). In the top plot, red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, yellow is $B \to D^{**}/(D^{(*)}\pi)\ell\nu$ and brown is $B\bar{B}$ backgrounds. In the bottom plot, magenta is $c\bar{c}$, light blue is uds and gray is $\tau^+\tau^-$.

cuts	D^0	D^+
no cut	0.00525	0.01174
$\lvert\cos\theta_{Dl-nonDl}\rvert < 0.96$	0.00463	0.01063
0.92	**0.00428**	**0.00992**
0.88	0.00405	0.00944
0.84	0.00389	0.00911
0.80	0.00378	0.00890

Table 9: [**Thrust cuts**] The change in the uncertainty on signal, $\frac{\sigma_S}{S}$, with variety of $|\cos\theta_{Dl-nonDl}|$ cuts.

	D^0		D^+	
cuts	$\frac{\sigma_S}{S}$	N^{Sig}_{peak}	$\frac{\sigma_S}{S}$	N^{Sig}_{peak}
BToDlnu skim	0.01152	1.0639×10^6	0.01538	628185
KLHNotPion selector	0.00657	1.0349×10^6	0.01538	628170
$P^D(\chi^2, dof) > 0.001$ $P^B(\chi^2, dof) > 0.01$	0.00525	974632	0.01174	568799
$\lvert\cos\theta_{Dl-nonDl}\rvert < 0.92$	0.00428	882038	0.00992	515792
Kinematic cuts	0.00264	492955	0.00660	279686

Table 10: [**Event selection cuts : summary**] The summary of event selection cuts.

  - $B^0 \to D^+/D^- X\ell^+\nu$ mode : 1.0020

- Backgrounds
  - $B^0 \to D^+/D^- X\ell^+\nu$ Uncorrelated Direct Lepton : 1.0038
  - $B^+ \to D^0/\bar{D}^0 X\ell^+\nu$ Uncorrelated Cascade Lepton : 1.0038
  - $B^0 \to D^0/\bar{D}^0 X\ell^+\nu$ Correlated Cascade Lepton : 1.0038

To understand these numbers, we try to estimate the rate to reconstruct $D\ell$ candidate in the $B^+ \to D^0/\bar{D}^0$ case.

$$\begin{aligned}
&\mathcal{B}(B^+ \to X_c\ell^+\nu)\mathcal{B}(X_c \to D^0/\bar{D}^0 X)\mathcal{B}(D^0 \to K^+\pi^-) \times (\text{acceptance} \times \text{efficiency}) \\
&\simeq 0.1 \times 1 \times 0.04 \times 0.4 = 0.0016
\end{aligned} \tag{179}$$

where we assumed $\mathcal{B}(X_c \to D^0/\bar{D}^0 X) = 1$, but this number can be lower than 1. Thus, the above number 1.0015 shows that we just reconstructed true $D\ell$ candidate from both B^+ and B^-.
Figure 5 shows the D mass plot before and after selection cuts are applied. At the bottom of each D^0 and D^+ plot, (number of OnPaek - OffPeak) - (number of $B\bar{B}$ MC) plots are given. Figures 6 shows the D mass plot after all cuts with electrons only (left) and muons only (right). There is a significant data-MC disagreement especially in the D^+ case. This is mailnly because D^+ require more chaged tracks to reconstruct than D^0. If you look at Figures 2 and 3, $D^0 \to K^-\pi^+\pi^+\pi^+$ mode shows more data-MC disagreement than $D^+ \to K^-\pi^+\pi^+$ mode. It is understandable that combinatrial bcakground is less well modeled in MC when it requires more tracks. However, since the data-MC difference for combinatrial bckground is flat throughout the D mass region, this difference disappear after D mass sideband subtraction.

OnPeak - OffPeak	Yield	Efficiency	Cumulative efficiency
BToDlnu skim	1.3388×10^6		
KLHNotPion selector	1.3410×10^6	1.00	1.00
$P^D(\chi^2, dof) > 0.001$ $P^B(\chi^2, dof) > 0.01$	1.1961×10^6	0.89	0.89
$\lvert\cos\theta_{Dl-nonDl}\rvert < 0.92$	1.0651×10^6	0.89	0.78
Kinematic cuts	479296	0.45	0.36
$B\bar{B}$ MC	Yield	Efficiency	Cumulative efficiency
BToDlnu skim	1.4679×10^6		
KLHNotPion selector	1.4449×10^6	0.99	0.99
$P^D(\chi^2, dof) > 0.001$ $P^B(\chi^2, dof) > 0.01$	1.3204×10^6	0.92	0.90
$\lvert\cos\theta_{Dl-nonDl}\rvert < 0.92$	1.1789×10^6	0.89	0.80
Kinematic cuts	502995	0.43	0.34

Table 11: [**Cut-flow Table,** D^0] The cut-flow table for D^0. Yield = number of candidates after D mass sideband subtraction.

OnPeak - OffPeak	Yield	Efficiency	Cumulative efficiency
BToDlnu skim	825616		
KLHNotPion selector	825625	1.00	1.00
$P^D(\chi^2, dof) > 0.001$ $P^B(\chi^2, dof) > 0.01$	700745	0.85	0.85
$\lvert\cos\theta_{Dl-nonDl}\rvert < 0.92$	621453	0.89	0.75
Kinematic cuts	266127	0.43	0.32
$B\bar{B}$ MC	Yield	Efficiency	Cumulative efficiency
BToDlnu skim	1.0061×10^6		
KLHNotPion selector	1.0061×10^6	1.00	1.00
$P^D(\chi^2, dof) > 0.001$ $P^B(\chi^2, dof) > 0.01$	870188	0.85	0.85
$\lvert\cos\theta_{Dl-nonDl}\rvert < 0.92$	766995	0.88	0.76
Kinematic cuts	294623	0.38	0.29

Table 12: [**Cut-flow Table,** D^+] The cut-flow table for D^+. Yield = number of candidates after D mass sideband subtraction.

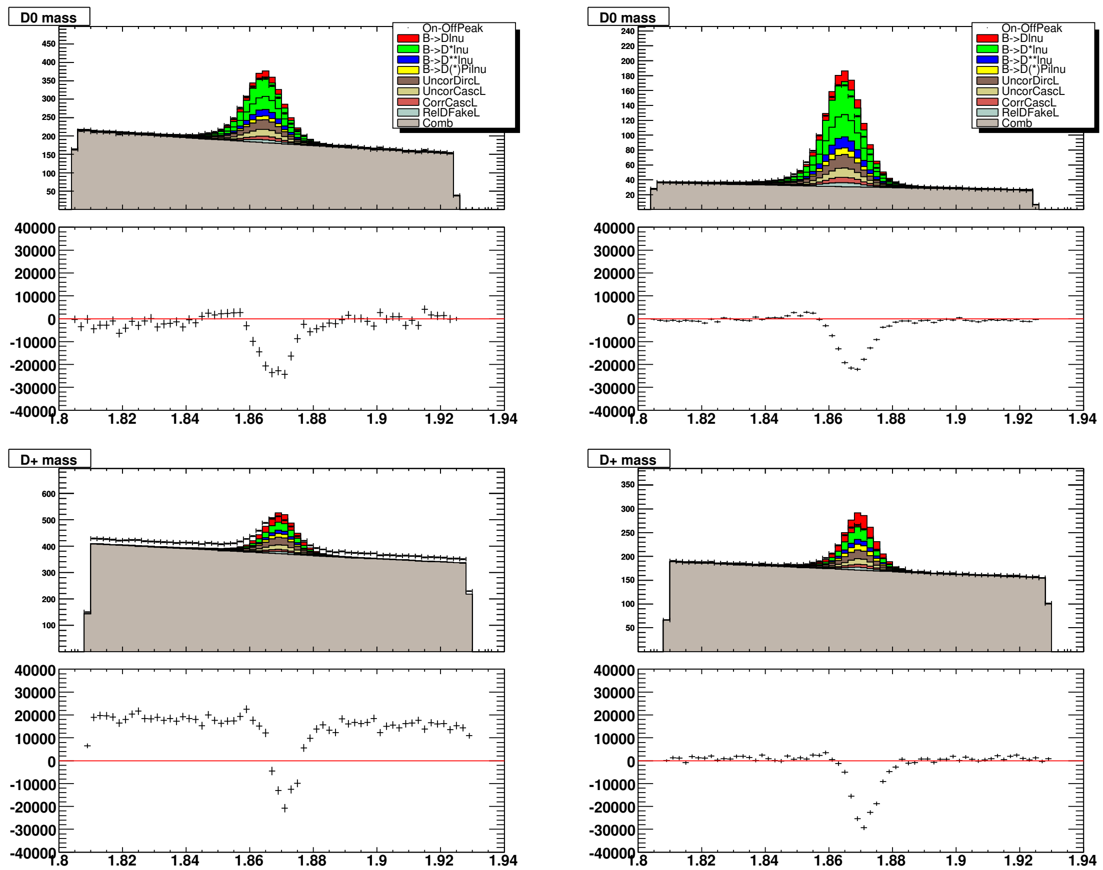


Figure 5: [D **mass before and after event selection**] D mass plot before (left) and after (right) applying selection cuts. The top is D^0 and the bottom is D^+. Black line with error bar is (OnPeak - OffPeak) data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, light gray is Comb. Data - MC difference is also plotted.

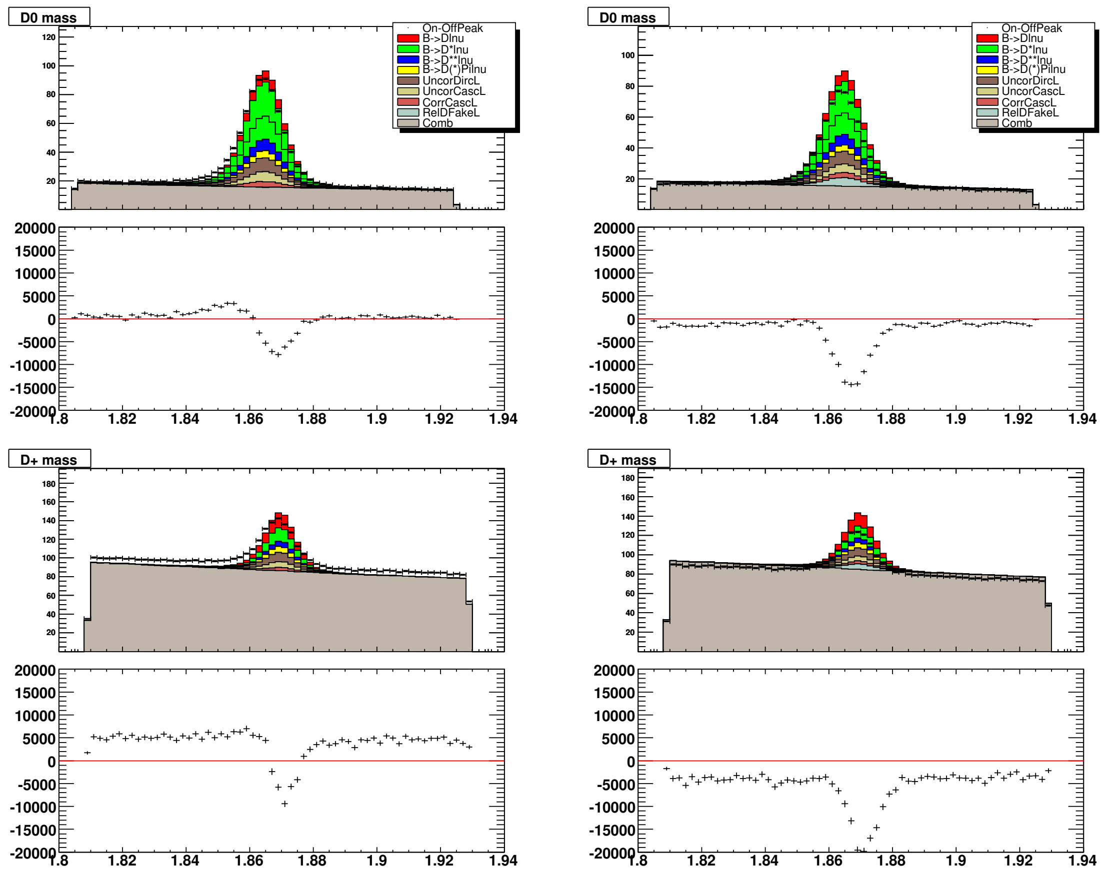


Figure 6: [D **mass after event selection, electron vs muon**] D mass plot after all cuts. **Electrons** (left) and **muons** (right) are compared. The top is D^0 and the bottom is D^+. Black line with error bar is (OnPeak - OffPeak) data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, light gray is Comb. Data - MC difference is also plotted.

7 Tracking Efficiency and PID correction

Tracking efficiency and PID efficiency are not completely equal in both data and MC. There is always a difference no matter how sophisticated the codes to reconstruct charged tracks and to identify particles are. Tracking efficiency and PID corrections are the corrections to account for the residual difference between data and MC. Both corrections are done in similar way. The correction factors are read out from correction tables stored in the condition database. We use R16 correction tables since we use R16b skims.

7.1 Tracking efficiency correction : r_k^{Trk}

Tracking efficiency correction factors are given for each charged track, depending on its transverse momentum, direction in the LAB frame and the multiplicity of the event. There are 3 charged tracks to reconstruct a $D^0(K\pi)l$ candidate, and 4 charged tracks to reconstruct a $D^+(K\pi\pi)l$ candidate. We multiply those correction factors of charged tracks to get a correction factor r_k^{Trk} for each candidate (k). The distribution of r_k^{Trk} for Run2 BpBm MC is shown in Figure 7 as an example.

7.2 PID correction : r_k^{LPID} and r_k^{KPID}

PID correction factors (called weights) are given one for each PID selector. Thus, we have one factor r_k^{LPID} for lepton (PidLHElectron for electrons and muNNTight for muons) and another factor r_k^{KPID} for kaon (KLHNotPion) for each $D\ell$ candidate (k). When the identified particles are real, r_k^{LPID} and r_k^{KPID} are the factor to correct identification efficiency difference. When the identified particles are fake, r_k^{LPID} and r_k^{KPID} give the factor to correct fake rate difference. The distribution of electron r_k^{LPID} for Run2 BpBm MC, as an example, is shown in Figure 8 and muon r_k^{LPID} in Figure 9. r_k^{KPID} is shown in Figure 10.

7.3 Effect of efficiency correction

Figure 11 show the D mass plots after tracking efficiency and PID corrections. At the bottom of each D^0 and D^+ plot, (number of OnPaek - OffPeak) - (number of $B\bar{B}$ MC) plots are given. These Figures need to be compared with Figures 6. The data-MC agreement of D^+ mode is significantly improved.

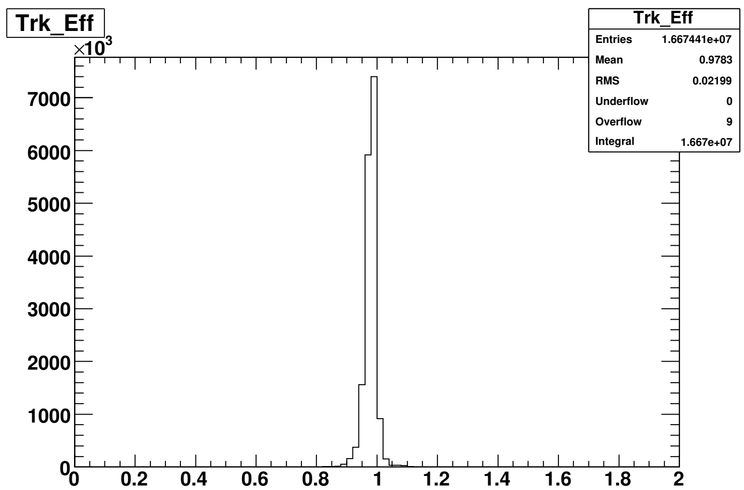


Figure 7: [**TrkEff correction factor**] Tracking efficiency correction factor for each candidate. BpBm MC is used to make this plot.

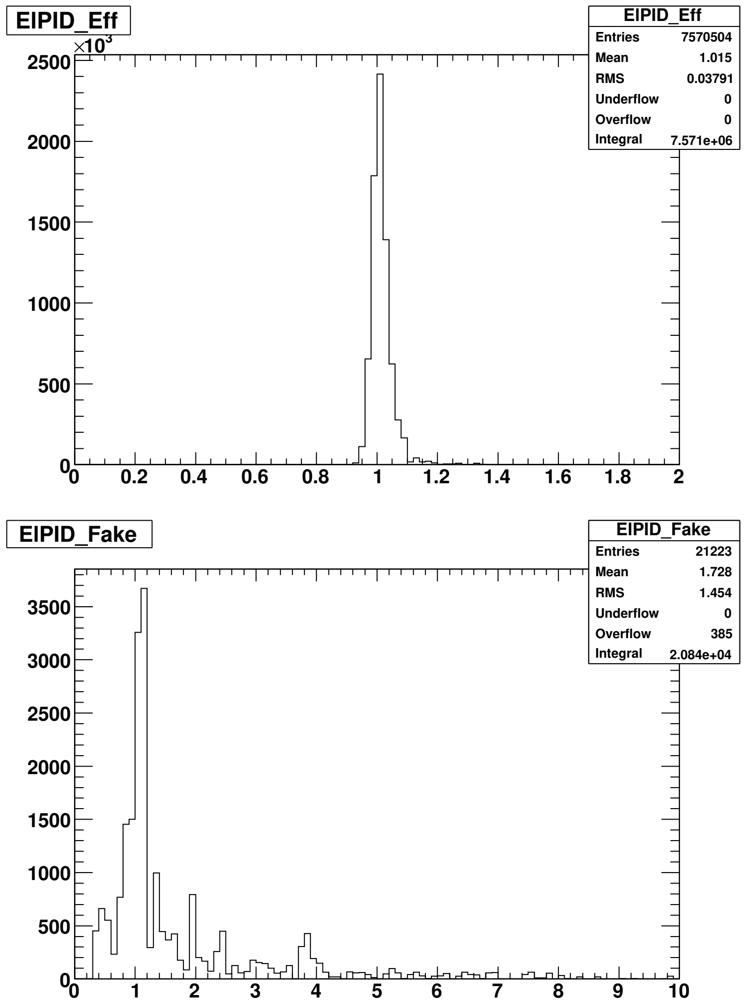


Figure 8: [**Electron PID weight**] Electron PID correction factor for each candidate. BpBm MC is used to make this plot. The top is the efficiency correction factors and the bottom is the fake rate correction factors.

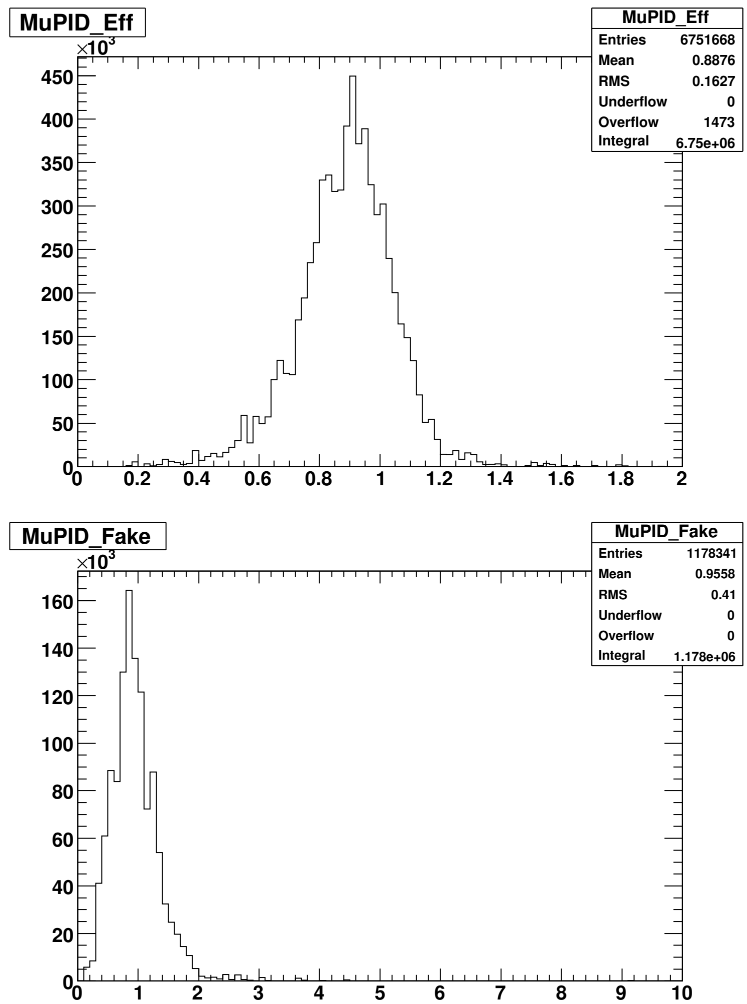


Figure 9: [**Muon PID weight**] Muon PID correction factor for each candidate. BpBm MC is used to make this plot. The top is the efficiency correction factors and the bottom is the fake rate correction factors.

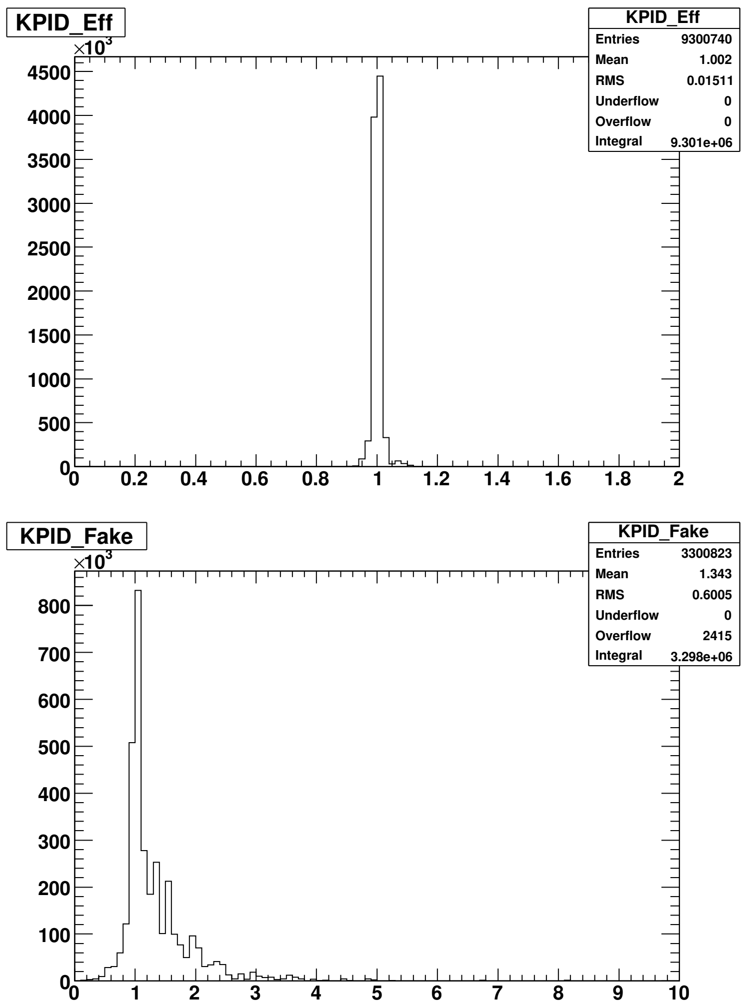


Figure 10: [**Kaon PID weight**] Kaon PID correction factor for each candidate. BpBm MC is used to make this plot. The top is the efficiency correction factors and the bottom is the fake rate correction factors.

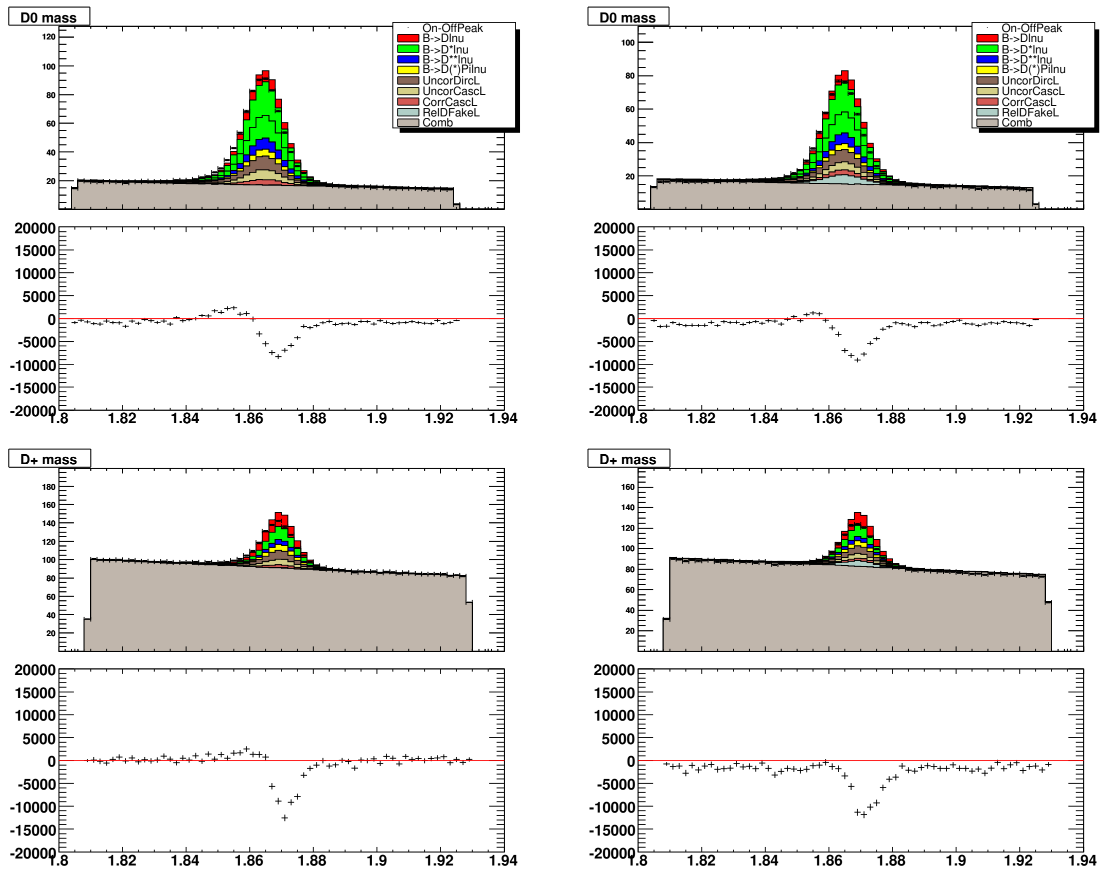


Figure 11: [D **mass after Trk PID correction, Electron vs muon**] D mass plot after tracking efficiency and PID corrections in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black line with error bar is (OnPeak - OffPeak) data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, light gray is Comb. Data - MC difference is also plotted.

8 D mass sideband subtraction

We assume D mass sideband is flat when we do sideand subtraction. Here we check this assumption.

8.1 Definition of D mass sideband

From D mass plots (Figure 11), we decided the range of D mass peak and sidebands as follows

- D^0 mass
  Peak : 1.840 - 1.888 GeV
  Side bands : 1.806 - 1.840 and 1.888 - 1.922 GeV

- D^+ mass
  Peak : 1.848 - 1.890 GeV
  Side bands : 1.810 - 1.848 and 1.890 - 1.928 GeV

Approximately 4.6 % of signal candidates goes into sideband.

8.2 Linearlity of D mass sideband

8.2.1 D mass fit

We try to fit OnPeak D mass plot with the following two combinations :

- 2 Gaussian + a linear function

- 2 Gaussina + a quadratic function

We use two Gaussinas with different width for the peak : one for signal and the other for peaking backgroud. We compare the yeild of sideband from linear and quadratic functions to make sure we get the similar results in both cases. Fitted plots are shown in Figure 12. Yeldsare given in Table 13. Yeilds difference is very small.

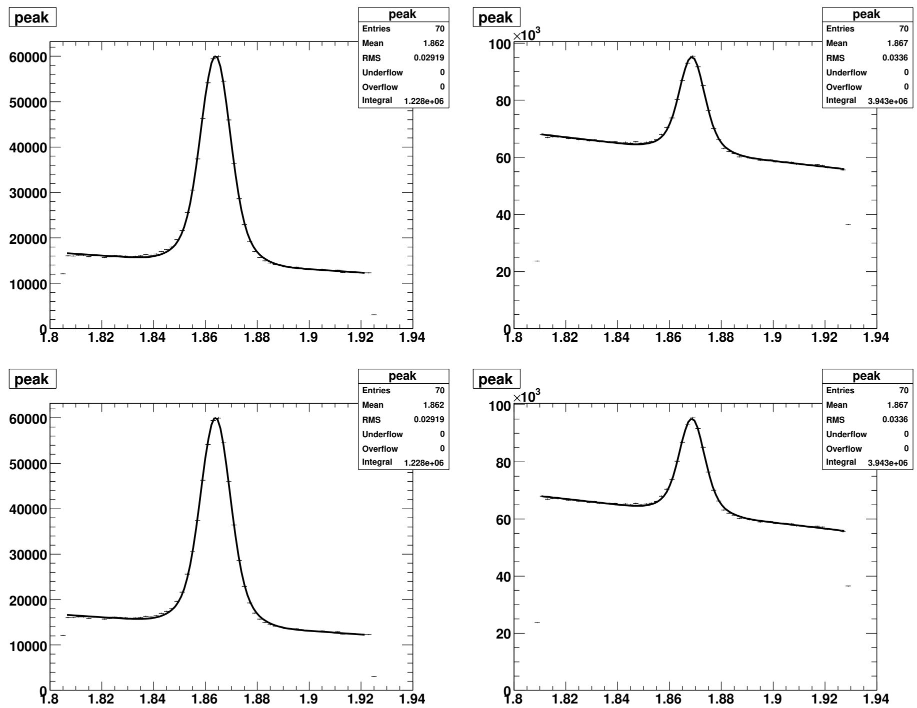


Figure 12: [D^0 **mass fit**] D^0 mass (left) and D^+ masss (right) fit with linear function (top) and quadratic function (bottom).

D^0	Linear	Quadratic
Peak yield	719.9	719.1
Combinatrial yield	693.9	694.7
D^+	Linear	Quadratic
Peak yield	450.3	448.6
Combinatrial yield	2602.7	2604.7

Table 13: [D **mass fit**] D mass fit results.

8.3 Lepton momentum, D momentum and $\cos\theta_{B-Dl}$

Figures 13 - 15 show the lepton momentum, D momentum and $\cos\theta_{B-Dl}$ after tracking efficiency and PID correction with electrons only and muons only. These are the projections of 3D histograms.

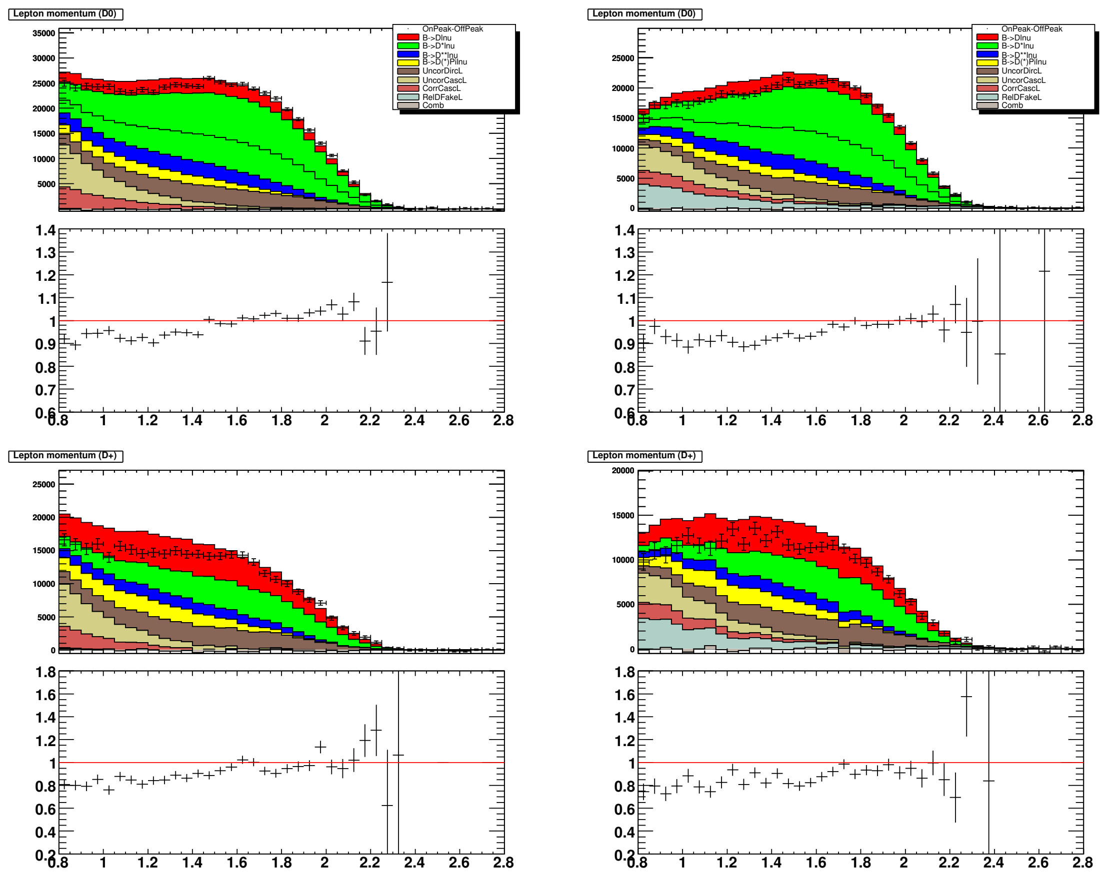


Figure 13: [p_ℓ **after Trk PID, Electron vs muon**] Lepton momentum plot after PID and tracking corrected and D mass sideband subtracted in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

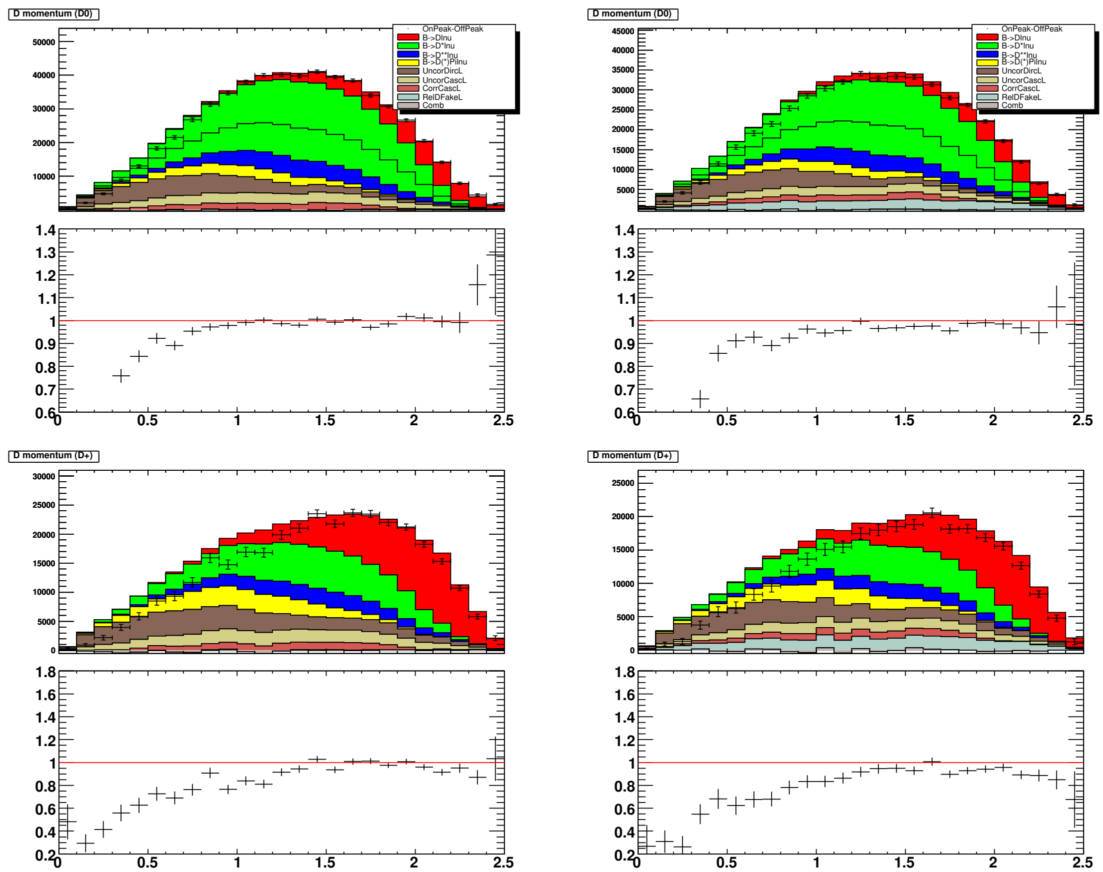


Figure 14: [p_D **after Trk PID, Electron vs muon**] D momentum plot after PID and tracking corrected and D mass sideband subtracted in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

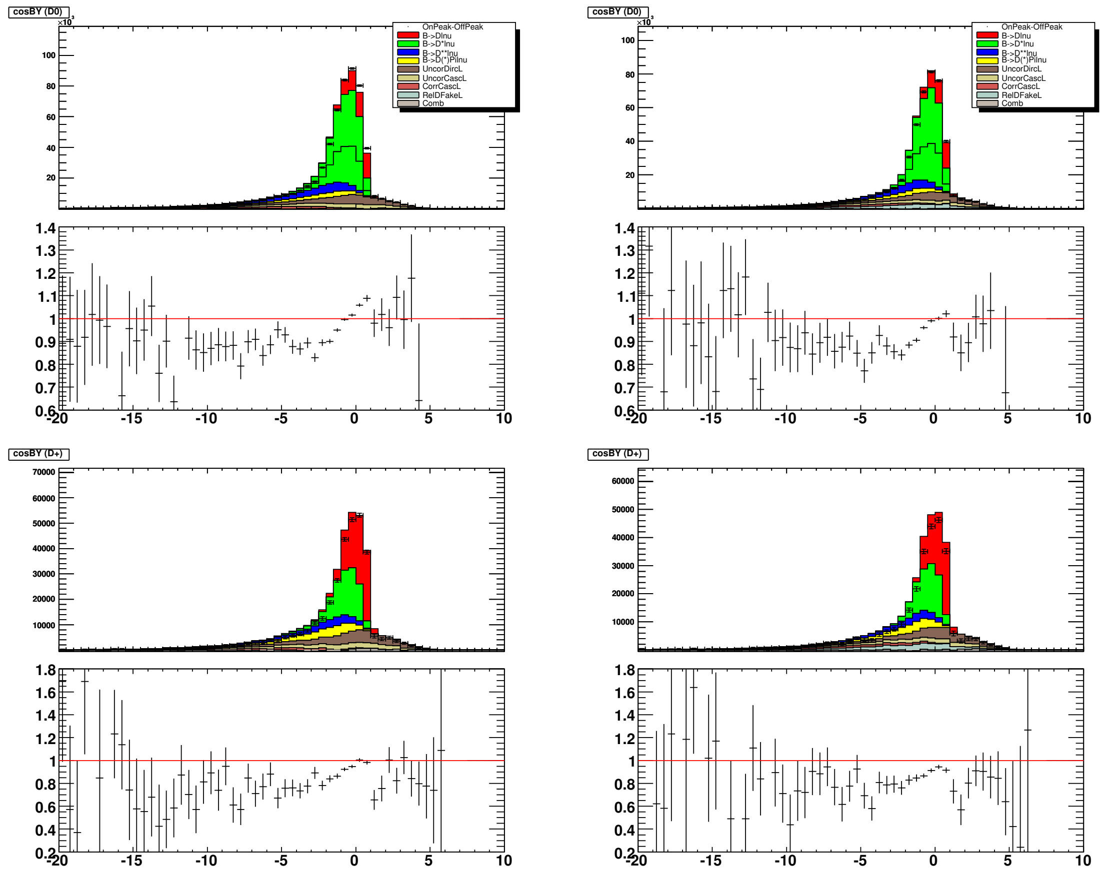


Figure 15: [$\cos\theta_{B-Dl}$ **after Trk PID, Electron vs muon**] $\cos\theta_{B-Dl}$ plot after PID and tracking corrected and D mass sideband subtracted in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

9 Monte Carlo Re-weighting (W^{MC}_{jk})

Because of theoretical and experimental improvements, some of numbers and formulas used in our MC are obsolete. In order to take new results into account, we need to re-wight MC. In this analysis, we re-weight decay branching fractions and form factors. In this section, details of branching fraction re-weighting is presented. Form factor re-weighting is described in section 10

9.1 Background components

As described in section 2, a few specific decay modes dominate the backgrounds. We need to update some of decay branching fraction to get better description of data.

9.1.1 Semileptonic $D(D_s)$ decay branching fractions

In case of CascL, UncrorrCL and CombRL backgrounds, the lepton we reconstruc is mainly come from semileptonic $D(D_s)$ decays. Thus, we correct the branching fractions of semileptonic $D(D_s)$ decays. The values used in SP5/SP6 MC and in PDG [12] are listed in Table 14 along with the weights. The weight is given by the ratio of new and old branching fractions. See section 9.4 for an example.

9.1.2 Inclusive $B \to X\ell\nu$ decay branching fractions

In case of UncorrDL and CombRL backgrounds, we correct the branching fraction of inclusive $B \to X\ell\nu$ decays. In SP5/SP6 MC, the following values are used :

$$\begin{aligned}
\mathcal{B}(B^+ \to X_c\ell\nu) &= \mathcal{B}(B^0 \to X_c\ell\nu) = 0.104 \\
\mathcal{B}(B^+ \to X_u\ell\nu) &= 0.002088 \\
\mathcal{B}(B^0 \to X_u\ell\nu) &= 0.002100
\end{aligned} \tag{180}$$

Thus

$$\begin{aligned}
\mathcal{B}(B^+ \to X\ell\nu) &= 0.106088 \\
\mathcal{B}(B^0 \to X\ell\nu) &= 0.106100
\end{aligned} \tag{181}$$

From PDG [12] and HFAG [24]

$$\mathcal{B}(B \to X\ell\nu) = 0.1075 \pm 0.0016 \tag{182}$$

We impose isospin constrain, then

$$\begin{aligned}
\mathcal{B}(B^+ \to X\ell\nu) &= t_{+0}\mathcal{B}(B^0 \to X\ell\nu) \\
\mathcal{B}(B^+ \to X\ell\nu) + \mathcal{B}(B^0 \to X\ell\nu) &= \mathcal{B}(B \to X\ell\nu)
\end{aligned} \tag{183}$$

Thus, we get

$$\mathcal{B}(B^+ \to X\ell\nu) = \frac{2t_{+0}}{1+t_{+0}}\mathcal{B}(B \to X\ell\nu) = 0.1112 \pm 0.0019 \tag{184}$$

Decay mode	SP5/SP6	PDG [12]	Weight
$D^0 \to K^{*-}e\nu$	0.0202	0.0216 ± 0.0016	1.0693 ± 0.0792
$D^0 \to K^{-}e\nu$	0.0364	0.0350 ± 0.0009	0.9315 ± 0.0247
$D^0 \to \pi^{-}e\nu$	0.0037	0.0028 ± 0.0002	0.7568 ± 0.0459
$D^0 \to \rho^{-}e\nu$	0.0029	0.0019 ± 0.0004	0.6552 ± 0.1379
$D^0 \to K^{*-}\mu\nu$	0.0198	0.0194 ± 0.0025	0.9798 ± 0.1263
$D^0 \to K^{-}\mu\nu$	0.0322	0.0327 ± 0.0013	1.0155 ± 0.0404
$D^0 \to \pi^{-}\mu\nu$	0.0037	0.0024 ± 0.0002	0.6378 ± 0.0649
$D^0 \to \rho^{-}\mu\nu$	0.0029	0.0019 ± 0.0004	0.6552 ± 0.1379
$D^+ \to K^{*0}e\nu$	0.0480	0.0556 ± 0.0032	1.1583 ± 0.0667
$D^+ \to K^{0}e\nu$	0.0670	0.0860 ± 0.0050	1.2836 ± 0.0667
$D^+ \to \pi^{0}e\nu$	0.0031	0.0044 ± 0.0007	1.4194 ± 0.2258
$D^+ \to \rho^{0}e\nu$	0.0022	0.0022 ± 0.0004	1.000 ± 0.1818
$D^+ \to \omega e\nu$	0.0029	0.0016 ± 0.0007	0.5517 ± 0.2414
$D^+ \to K^{*0}\mu\nu$	0.0430	0.0550 ± 0.0050	1.2791 ± 0.1163
$D^+ \to K^{0}\mu\nu$	0.0700	0.0960 ± 0.0080	1.3714 ± 0.1143
$D^+ \to \pi^{0}\mu\nu$	0.0064	0.0044 ± 0.0007	0.6875 ± 0.1094
$D^+ \to \rho^{0}\mu\nu$	0.0027	0.0025 ± 0.0005	0.9259 ± 0.1852
$D^+ \to \omega\mu\nu$	0.0029	0.0016 ± 0.0007	0.5517 ± 0.2222
$D_s^+ \to \phi\ell\nu$	0.0200	0.0249 ± 0.0028	1.2450 ± 0.2400
$D_s^+ \to \eta\ell\nu$	0.0260	0.0320 ± 0.0060	1.2308 ± 0.2308
$D_s^+ \to \eta'\ell\nu$	0.0089	0.0111 ± 0.0034	1.2472 ± 0.3820

Table 14: [**Semileptonic** D **decay BF weight**] D semileptonic decay branching fractions and weights.

Decay mode	SP5/SP6	isospin averaged BF	Weight
$B^+ \to \bar{X}^0\ell^+\nu$	0.106088	0.1112 ± 0.0019	1.0480 ± 0.0179
$B^0 \to X^-\ell^+\nu$	0.106100	0.1038 ± 0.0018	0.9785 ± 0.0167

Table 15: [$B \to \ell$ **decay BF weight**] Inclusive $B \to x\ell\nu$ branching fractions.

and

$$\mathcal{B}(B^+ \to X\ell\nu) = \frac{1}{1+t_{+0}}\mathcal{B}(B \to X\ell\nu) = 0.1038 \pm 0.0018 \tag{185}$$

Weights are given by the ratios of new and old branching fractions as summarized in Table 15. These weights are applied to lepton side of each candidate.

9.1.3 Inclusive $B \to D(D_s)$ decay branching fractions

In UncorrDL, UncorrCL, CascL, FakeL and CombRL backgrounds, we correct the branching fraction of inclusive $B \to D(D_s)$ decays using measured branching fractions from [25][26]. We apply weights to D of $D\ell$ candidates, and for cascade lepton backgrounds such as $B \to D(D_s)X$ and $D(D_s) \to X'\ell\nu$, we also apply weights to the $B \to D(D_s)X$ part. $B \to D_s$ weights come in only the cascade case.
The $D(D_s)$ momentum dependent weights are given in Tables 16 - 21. If we apply these weights to, for example, UncorrDL ($B^+(B^-) \to D^0$), we get a distribution shown in Figure 16. The corrected histogram is fluctuated because of the statistical fluctuation of the weights. To smooth up the fluctuation, we use a 4th order polynomial to approximate the $B \to D$ weights as a function of D momentum and 3rd order polynomial to approximate the $B \to D_s$ weights as a function of D_s momentum. The distributions of weights and fit results are given in Figures 17 - 19. Using the fitted coefficients, the weight w of $B \to D$ is given by

$$\begin{aligned} w = &(\text{Coefficient0}) + (\text{Coefficient1})p_D + (\text{Coefficient2})p_D^2 \\ &+(\text{Coefficient3})p_D^3 + (\text{Coefficient4})p_D^4 \end{aligned} \tag{186}$$

where p_D is the D momentum. In Figure 20, the same UncorrDL ($B^+(B^-) \to D^0$) using this fit resut is given. The corrected distribution is much smoother and retains similar character as the original distribution.

p_D (GeV)	$B^- \to D^0$	$\bar{B}^0 \to D^0$
0.00 - 0.15	0.0600 ± 0.1217	0.0275 ± 0.3010
0.15 - 0.30	0.4667 ± 0.1217	0.6083 ± 0.2374
0.30 - 0.45	0.9800 ± 0.1241	0.7300 ± 0.2080
0.45 - 0.60	0.8600 ± 0.1040	0.9036 ± 0.1863
0.60 - 0.75	1.1022 ± 0.1014	1.0000 ± 0.1779
0.75 - 0.90	1.2491 ± 0.1005	1.0125 ± 0.1677
0.90 - 1.05	1.1431 ± 0.0904	1.1791 ± 0.1516
1.05 - 1.20	1.1770 ± 0.0879	1.2500 ± 0.1519
1.20 - 1.35	1.1306 ± 0.0839	1.1738 ± 0.1451
1.35 - 1.50	1.1324 ± 0.0760	1.3902 ± 0.1516
1.50 - 1.65	1.0128 ± 0.0680	1.1245 ± 0.1214
1.65 - 1.80	1.1371 ± 0.0788	0.9500 ± 0.1538
1.80 - 1.95	0.9544 ± 0.0676	1.0423 ± 0.1600
1.95 - 2.10	0.9673 ± 0.0708	1.0850 ± 0.1789
2.10 - 2.25	0.8850 ± 0.0765	1.4500 ± 0.5274
2.25 - 2.40	0.8154 ± 0.1101	

Table 16: [p_D **dependent** $B \to D^0$ **weight (Correlated)**] D^0 momentum dependent, inclusive $B \to D^0$ branching fraction weights (Correlated). These are based on the branching fractions given in [25][26].

p_D (GeV)	$B^+ \to D^0$	$B^0 \to D^0$
0.00 - 0.15	0.1333 ± 0.1374	0.0938 ± 0.2519
0.15 - 0.30	0.4500 ± 0.1521	0.5114 ± 0.2636
0.30 - 0.45	0.7143 ± 0.1738	0.5172 ± 0.2695
0.45 - 0.60	0.9391 ± 0.1962	1.2000 ± 0.3265
0.60 - 0.75	1.2833 ± 0.2083	1.2214 ± 0.3465
0.75 - 0.90	1.4182 ± 0.2186	1.2424 ± 0.3601
0.90 - 1.05	2.3733 ± 0.3206	0.8182 ± 0.3961
1.05 - 1.20	1.1077 ± 0.2837	0.5647 ± 0.4759
1.20 - 1.35	0.8571 ± 0.4247	1.6000 ± 0.8412
1.35 - 1.50	2.9000 ± 1.1180	-3.5333 ± 1.9887
1.50 - 1.65	0.0833 ± 0.8580	3.4615 ± 2.6315
1.65 - 1.80	6.6667 ± 3.0732	6.3333 ± 8.3333

Table 17: [p_D **dependent** $B \to D^0$ **weight (Uncorrelated)**] D^0 momentum dependent, inclusive $B \to D^0$ branching fraction weights (Uncorrelated). These are based on the branching fractions given in [25][26].

p_D (GeV)	$B^- \to D^+$	$\bar{B}^0 \to D^+$
0.00 - 0.20	0.5588 ± 0.2712	0.1333 ± 0.2007
0.02 - 0.40	0.6484 ± 0.2190	0.6790 ± 0.2351
0.40 - 0.60	1.0438 ± 0.2285	0.3702 ± 0.1820
0.60 - 0.80	1.0402 ± 0.2032	0.6823 ± 0.1581
0.80 - 1.00	0.6615 ± 0.1655	0.6279 ± 0.1347
1.00 - 1.20	0.7929 ± 0.1585	0.7489 ± 0.1286
1.20 - 1.40	0.7056 ± 0.1557	1.0468 ± 0.1417
1.40 - 1.60	0.5414 ± 0.1762	1.0208 ± 0.1027
1.60 - 1.80	0.8313 ± 0.2642	1.0036 ± 0.1241
1.80 - 2.00	0.7500 ± 0.4413	1.0453 ± 0.1255
2.00 - 2.20	0.3333 ± 0.5429	0.8105 ± 0.1175
2.20 - 2.40		0.8490 ± 0.1514

Table 18: [p_D **dependent** $B \to D^+$ **weight (Correlated)**] D^+ momentum dependent, inclusive $B \to D^+$ branching fraction weights (Correlated). These are based on the branching fractions given in [25][26].

p_D (GeV)	$B^+ \to D^+$	$B^0 \to D^+$
0.00 - 0.20	0.1935 ± 0.1962	0.1613 ± 0.1962
0.20 - 0.40	0.2273 ± 0.2293	0.6462 ± 0.4440
0.40 - 0.60	1.0400 ± 0.3078	0.9315 ± 0.5156
0.60 - 0.80	0.0789 ± 0.2654	0.1143 ± 0.5157
0.80 - 1.00	0.8088 ± 0.3175	-0.0984 ± 0.5590
1.00 - 1.20	1.3400 ± 0.3795	1.7674 ± 0.9047
1.20 - 1.40	0.0800 ± 0.4948	-0.6667 ± 1.4414
1.40 - 1.60	0.5000 ± 1.3463	2.0000 ± 5.2100
1.60 - 1.80	2.1429 ± 1.4070	5.5000 ± 5.2633
1.80 - 2.00	2.0000 ± 2.2361	-8.0000 ± 6.4275

Table 19: [p_D **dependent** $B \to D^+$ **weight (Uncorrelated)**] D^+ momentum dependent, inclusive $B \to D^+$ branching fraction weights (Uncorrelated). These are based on the branching fractions given in [25][26].

p_D (GeV)	$B^- \to D_s^+$	$B^+ \to D_s^+$
0.00 - 0.34	-0.3810 ± 0.8624	1.3529 ± 0.5153
0.34 - 0.68	0.0938 ± 0.5703	0.0800 ± 0.2335
0.68 - 1.02	1.9167 ± 0.9904	0.7037 ± 0.2253
1.02 - 1.36	4.3333 ± 1.8295	0.4167 ± 0.1179
1.36 - 1.70	2.0000 ± 2.2804	0.9397 ± 0.1682
1.70 - 2.04	2.3333 ± 2.4267	1.2171 ± 0.2320

Table 20: [p_{D_s} **dependent** $B^\pm \to D_s^+$ **weight**] D_s^+ momentum dependent, inclusive $B^\pm \to D_s^+$ branching fraction weights. These are based on the branching fractions given in [25][26].

p_D (GeV)	$\bar{B}^0 \to D_s^+$	$B^0 \to D_s^+$
0.00 - 0.34	-1.0000 ± 0.6353	0.1579 ± 0.4243
0.34 - 0.68	1.8000 ± 1.2272	1.1800 ± 0.4847
0.68 - 1.02	0.1111 ± 1.4449	1.3714 ± 0.4006
1.02 - 1.36	5.8750 ± 2.8264	0.6917 ± 0.2072
1.36 - 1.70	-1.1250 ± 3.6443	1.1164 ± 0.2357
1.70 - 2.04	4.0000 ± 4.6690	1.2800 ± 0.3134

Table 21: [p_{D_s} **dependent** $B^0(\bar{B}^0) \to D_s^+$ **weight**] D_s^+ momentum dependent, inclusive $B^0(\bar{B}^0) \to D_s^+$ branching fraction weights. These are based on the branching fractions given in [25][26].

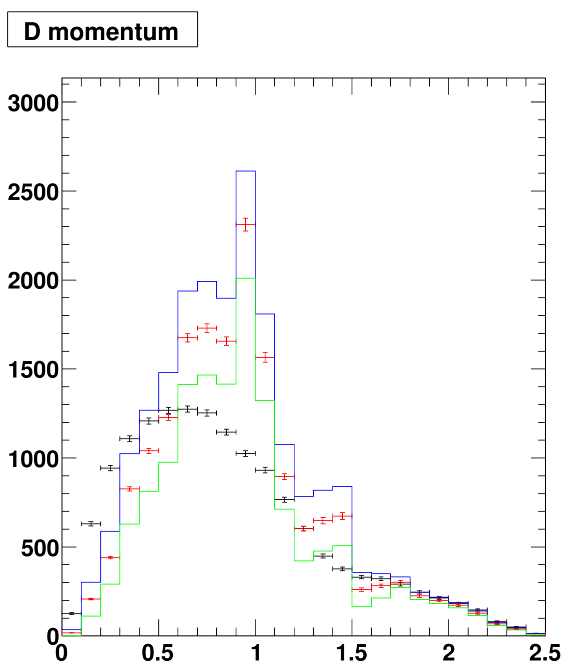


Figure 16: [**UncorrDL BF correction by Table**] Effect of BpBm UncorrDL background BF correction using weights given in tables. The black lines with error bars are before correction, and the red lines with error bars are after correction. Blue and green histograms show the range of uncertainty on the red histogram due to the uncertainties on inclusive $B \to D$ weights.

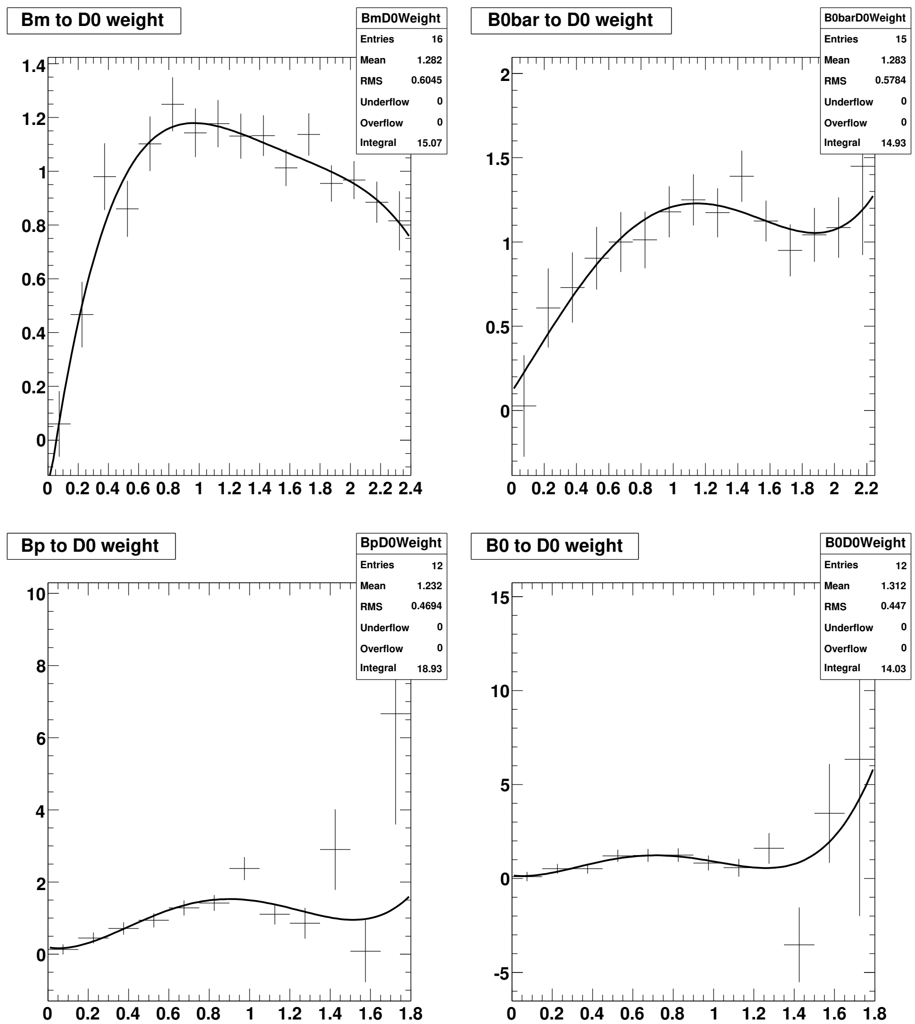


Figure 17: [**Fit for** $B \to D^0$ **weight**] 4th order polynomial fit for $B \to D^0$ weight. Top-left : $B^- \to D^+$, top-right : $\bar{B}^0 \to D^0$, bottom-left : $B^+ \to D^0$ and bottom-right : $B^0 \to D^0$

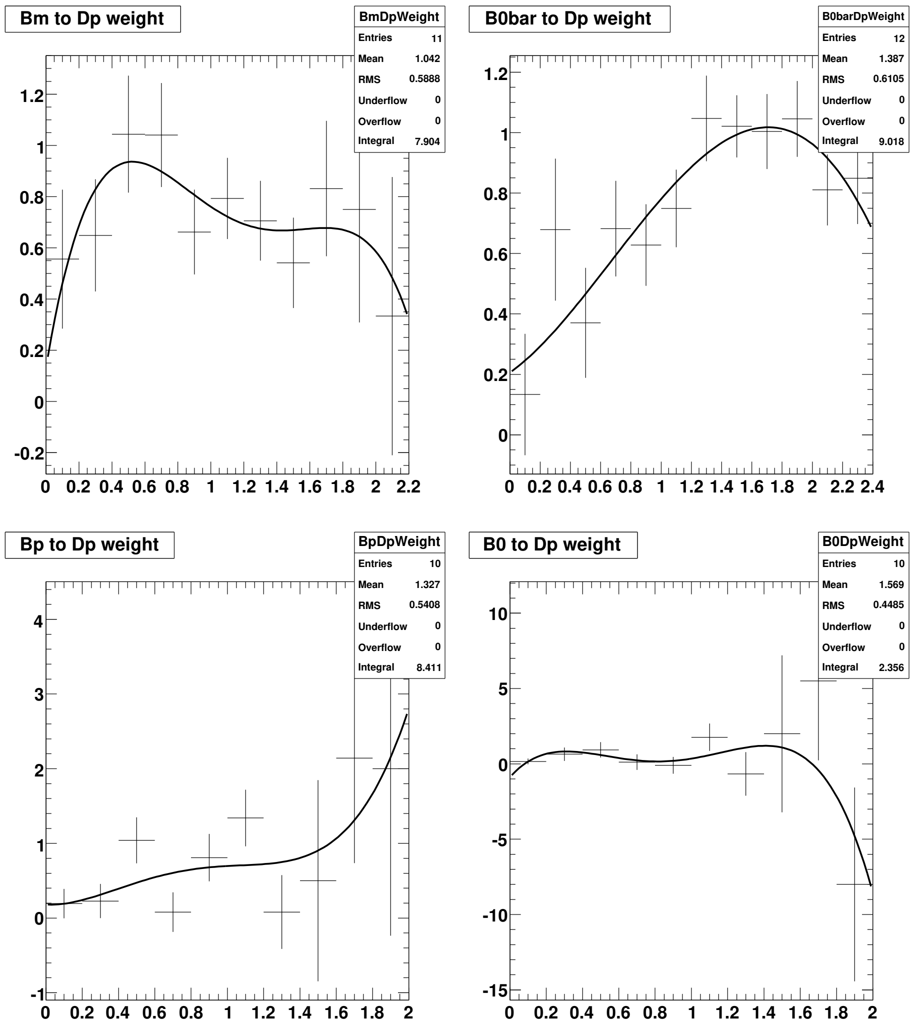


Figure 18: [**Fit for $B \to D^+$ weight**] 4th order polynomial fit for $B \to D^+$ weight. Top-left : $B^- \to D^+$, top-right : $\bar{B}^0 \to D^+$, bottom-left : $B^+ \to D^+$ and bottom-right : $B^0 \to D^+$

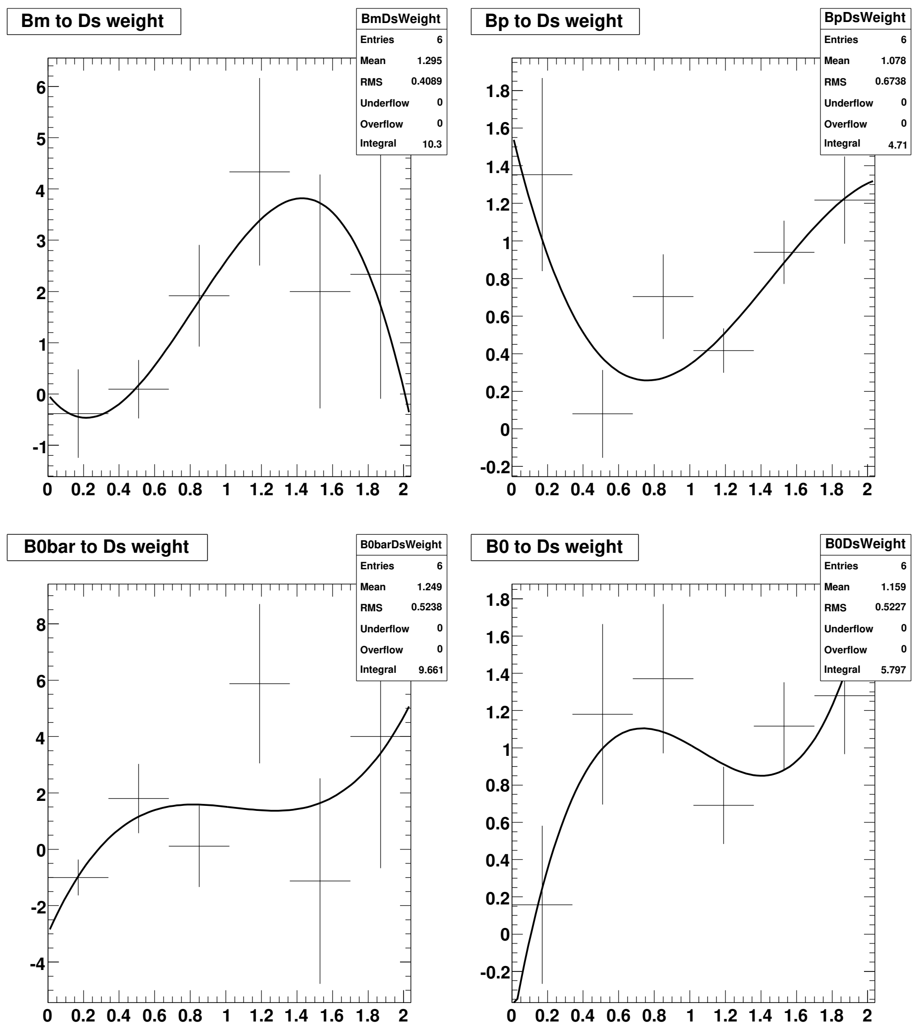


Figure 19: [**Fit for** $B \to D_s^+$ **weight**] 3rd order polynomial fit for $B \to D_s^+$ weight. Top-left : $B^- \to D_s^+$, top-right : $B^+ \to D_s^+$, bottom-left : $\bar{B}^0 \to D_s^+$ and bottom-right : $B^0 \to D_s^+$

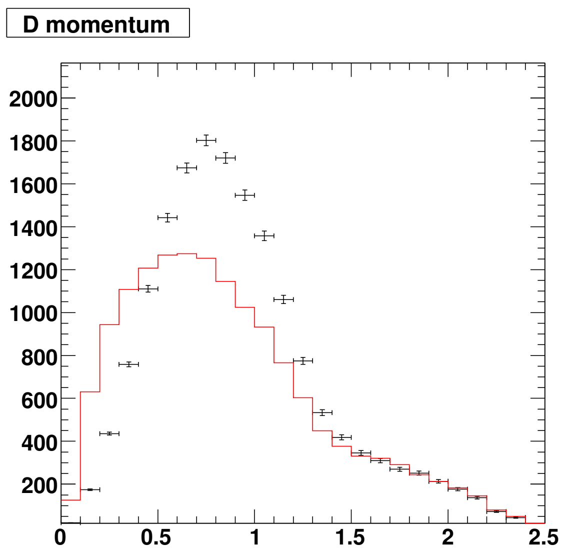


Figure 20: [**UncorrDL BF correction by Fitting**] Effect of BpBm UncorrDL background BF correction using a 3rd order polynomial. The red histogram is before correction, and the black lines with error bars are after correction.

9.2 D^{**} decay branching fractions

The values used in SP5/SP6 MC are summarized in Appendix D. We adopt the following new values in our fit. These are not applied in candidate level. Thus, these values goes into the coefficients in the fit. See details in section 4. For D_0^*, D_1 and D_1', based on isospin symmetry

$$\begin{aligned}
\mathcal{B}(D_0^{*+} \to D^0\pi^+) &= 2\mathcal{B}(D_0^{*+} \to D^+\pi^0) \\
\mathcal{B}(D_0^{*0} \to D^+\pi^-) &= 2\mathcal{B}(D_0^{*0} \to D^0\pi^0) \\
\mathcal{B}(D_1^{+} \to D^{*0}\pi^+) &= 2\mathcal{B}(D_1^{+} \to D^{*+}\pi^0) \\
\mathcal{B}(D_1^{0} \to D^{*+}\pi^-) &= 2\mathcal{B}(D_1^{0} \to D^{*0}\pi^0) \\
\mathcal{B}(D_1'^{+} \to D^{*0}\pi^+) &= 2\mathcal{B}(D_1'^{+} \to D^{*+}\pi^0) \\
\mathcal{B}(D_1'^{0} \to D^{*+}\pi^-) &= 2\mathcal{B}(D_1'^{0} \to D^{*0}\pi^0)
\end{aligned} \tag{187}$$

Thus, we use

$$\begin{aligned}
&\mathcal{B}(D_0^{*+} \to D^0\pi^+) = \mathcal{B}(D_0^{*0} \to D^+\pi^-) = \mathcal{B}(D_1^{+} \to D^{*0}\pi^+) = \mathcal{B}(D_1^{0} \to D^{*+}\pi^-) \\
&= \mathcal{B}(D_1'^{+} \to D^{*0}\pi^+) = \mathcal{B}(D_1'^{0} \to D^{*+}\pi^-) = 0.6667
\end{aligned} \tag{188}$$

and

$$\begin{aligned}
&\mathcal{B}(D_0^{*+} \to D^+\pi^0) = \mathcal{B}(D_0^{*0} \to D^0\pi^0) = \mathcal{B}(D_1^{+} \to D^{*+}\pi^0) = \mathcal{B}(D_1^{0} \to D^{*0}\pi^0) \\
&= \mathcal{B}(D_1'^{+} \to D^{*+}\pi^0) = \mathcal{B}(D_1'^{0} \to D^{*0}\pi^0) = 0.3333
\end{aligned} \tag{189}$$

Here we asuumed that D^{**} decays 100 % into $D^{(*)}\pi$. However there is an observation of $D_1 \to D\pi^+\pi^-$ decays [27]. We consider this in later sections.
For D_2^*, from isospin symmetry

$$\begin{aligned}
\mathcal{B}(D_2^{*+} \to D^0\pi^+) &= 2\mathcal{B}(D_2^{*+} \to D^+\pi^0) \\
\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) &= 2\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0) \\
\mathcal{B}(D_2^{*0} \to D^+\pi^-) &= 2\mathcal{B}(D_2^{*0} \to D^0\pi^0) \\
\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-) &= 2\mathcal{B}(D_2^{*0} \to D^{*0}\pi^0)
\end{aligned} \tag{190}$$

From PDG [12]

$$\frac{\mathcal{B}(D_2^{*+} \to D^0\pi^+)}{\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+)} = 1.9 \pm 1.1 \pm 0.3 \tag{191}$$

$$\frac{\mathcal{B}(D_2^{*0} \to D^+\pi^-)}{\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)} = 2.3 \pm 0.6 \tag{192}$$

By isospin symmetry, these two numbers are supposed to be same. Thus, we take an average and get

$$\begin{aligned}
f_{D_2^*} \equiv \frac{\mathcal{B}(D_2^{*+} \to D^0\pi^+)}{\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+)} &= \frac{\mathcal{B}(D_2^{*0} \to D^+\pi^-)}{\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)} \\
&= 2.2 \pm 0.64
\end{aligned} \tag{193}$$

Babar measure [15]

$$\frac{\mathcal{B}(D_2^{*+} \to D^0\pi^+) + \mathcal{B}(D_2^{*0} \to D^-\pi^+)}{\mathcal{B}(D_2^{*+} \to D^{(*)0}\pi^+) + \mathcal{B}(D_2^{*0} \to D^{(*)+}\pi^+)} = 0.62 \pm 0.03 \pm 0.02 \tag{194}$$

With isospin symmetry, this produce

$$f_{D_2^*} = 1.6 \pm 0.1 \tag{195}$$

We take an average over two, assuming maximum correlation, to get

$$f_{D_2^*} = 1.7 \pm 0.4 \tag{196}$$

Branching fractions are given by following formulae

$$\mathcal{B}(D_2^{*+} \to D^0\pi^+) = \mathcal{B}(D_2^{*0} \to D^+\pi^-) = \frac{2f_{D_2^*}}{3(1+f_{D_2^*})} \tag{197}$$

$$\mathcal{B}(D_2^{*+} \to D^+\pi^0) = \mathcal{B}(D_2^{*0} \to D^0\pi^0) = \frac{f_{D_2^*}}{3(1+f_{D_2^*})} \tag{198}$$

$$\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) = \mathcal{B}(D_2^{*0} \to D^{*+}\pi^-) = \frac{2}{3(1+f_{D_2^*})} \tag{199}$$

$$\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0) = \mathcal{B}(D_2^{*0} \to D^{*0}\pi^0) = \frac{1}{3(1+f_{D_2^*})} \tag{200}$$

9.3 $D^{(*)}$ decay branching fractions

The values used in SP5/SP6 MC are summarized in Appendix D. We adopt the following new values in our fit. These are not applied in candidate level. Thus, these values goes into the coefficients in the fit. See details in section 4. From PDG [12]

$$\begin{aligned}
&\mathcal{B}(D^0 \to K^-\pi^+) = 0.0389 \pm 0.0005 \\
&\mathcal{B}(D^+ \to K^-\pi^+\pi^+) = 0.0922 \pm 0.0021 \\
&\mathcal{B}(D^{*+} \to D^0\pi^+) = 0.677 \pm 0.005 \\
&\mathcal{B}(D^{*+} \to D^+\pi^0) = 0.307 \pm 0.005 \\
&\mathcal{B}(D^{*+} \to D^+\gamma) = 0.016 \pm 0.004
\end{aligned} \tag{201}$$

We also apply

$$\mathcal{B}(D^{*+} \to D^+\pi^0) + \mathcal{B}(D^{*+} \to D^+\gamma) = 1 - \mathcal{B}(D^{*+} \to D^0\pi^+) \tag{202}$$

9.4 An example

Here we give a specific example to illustrate BF re-weighting. In the case of

- $B^+ \to D^{*-}D^+K^+$
  with $D^{*-} \to \bar{D}^0\pi^-$ and $D^+ \to \bar{K}^0e^+\nu$
  and these $\bar{D}^0$ and e^+ form a Dl candidate as if $B^+ \to \bar{D}^0e^+\nu$

The number of this background is given by

$$\begin{aligned}&N = (\text{number of produced } B^+B^- \text{ pairs})\times \\ &\mathcal{B}(B^+ \to D^{*-}D^+K^+)\mathcal{B}(D^{*-} \to \bar{D}^0\pi^-)\mathcal{B}(D^+ \to \bar{K}^0e^+\nu)\mathcal{B}(D^0 \to K^-\pi^+)\end{aligned} \tag{203}$$

Thus, it is proportional to all branching fractions. This means that the weight is given by the ratio of new and old branching fractions. This also means that even if you do not know the details of the decay chain, if you know, for example, $D^+ \to \bar{K}^0e^+\nu$ is included in the chain, you can still apply a weight only for the known part.
In above example, for the lepton side, we apply

- Inclusive $B^+ \to D^+$ BF weight (section 9.1.3)
- Semileptonic $D^+ \to \bar{K}^0e^+\nu$ BF weight (section 9.1.1)

and for the D side, we apply

- $D^{*-} \to \bar{D}^0\pi^-$ decay BF weight (section 9.3)
- $D^0 \to K^-\pi^+$ decay BF weight (section 9.3)

Total weight is given by the product of these 4 weights. The formar two weights are applied in the histogram making process and the latter two weights are applied in a part of fitting program before going into the actual fitting process.

9.5 Effect of branching fraction re-weighting

Figure 21 - 23 show the lepton momentum, D momentum and $\cos\theta_{B-Dl}$ after branching fraction re-weighting. These are the projections of 3D histograms. If you compare these with figure 13 - 15, you can see improvements of data-MC agreement.

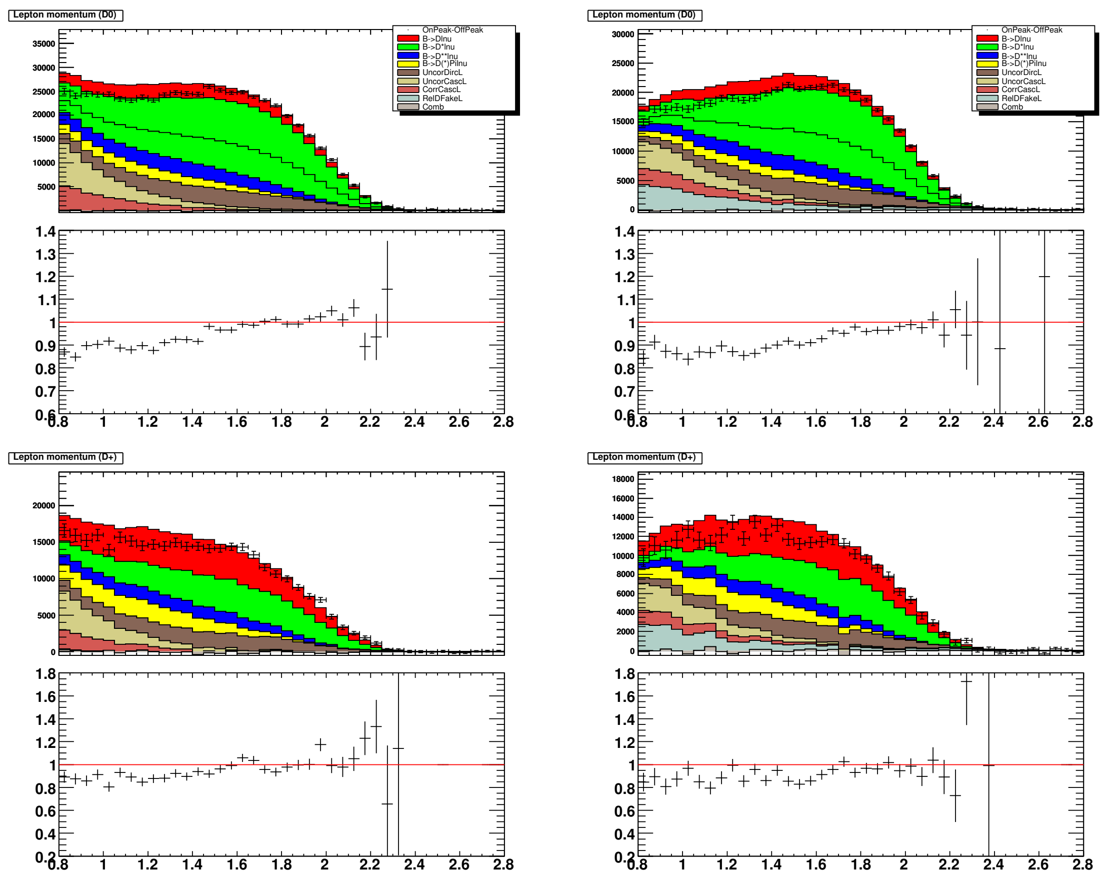


Figure 21: [p_ℓ **after MC re-weighting (Electron vs muon)**] Lepton momentum plot after branching fraction re-weighting and D mass sideband subtraction in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

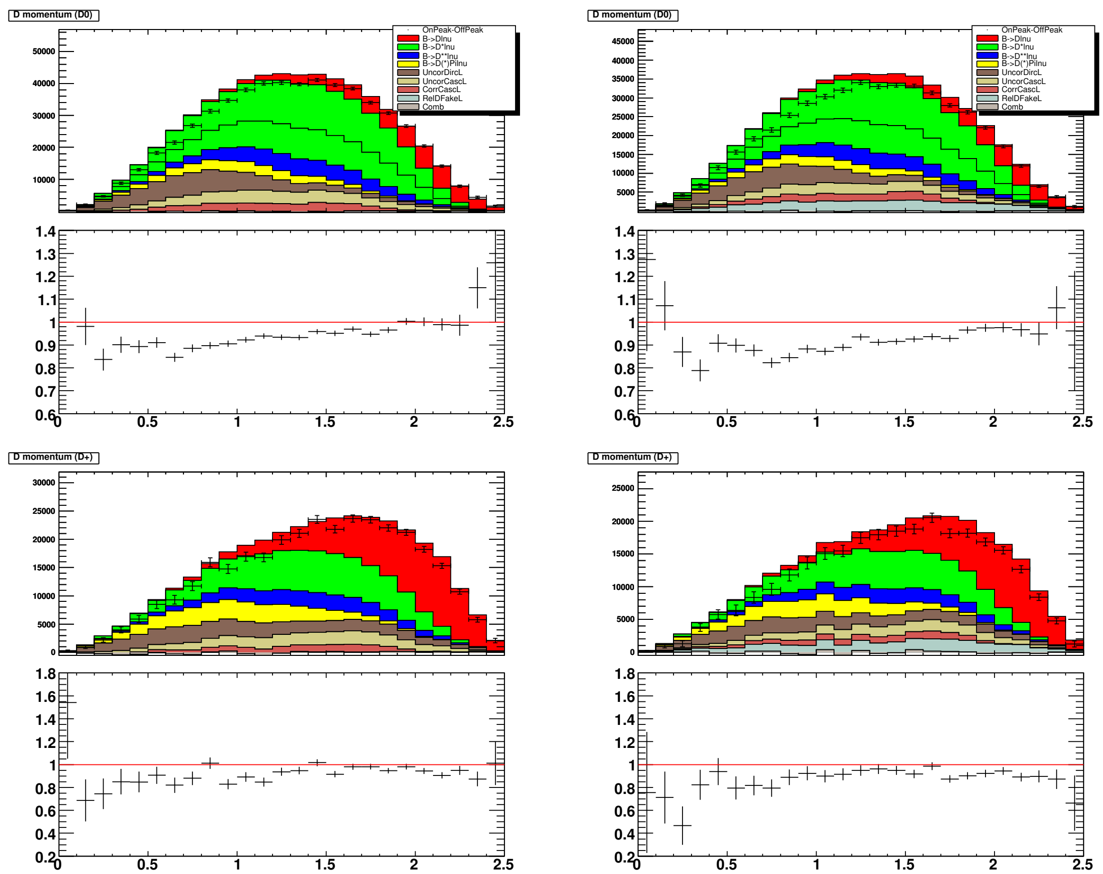


Figure 22: [p_D **after MC re-weighting (Electron vs muon)**] D momentum plot after branching fraction re-weighting and D mass sideband subtraction in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

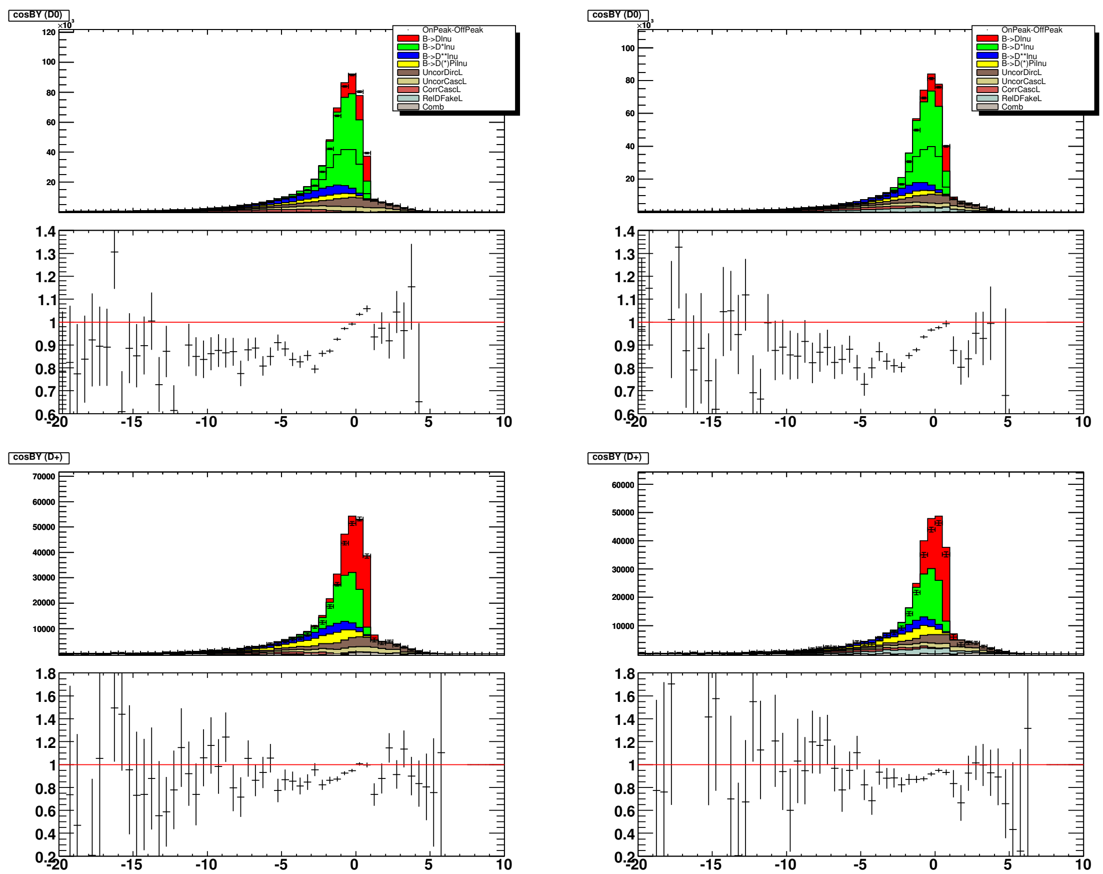


Figure 23: [$\cos\theta_{B-Dl}$ **after MC re-weighting (Electron vs muon)**] $\cos\theta_{B-Dl}$ plot after branching fraction re-weighting and D mass sideband subtraction in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

9.6 Beam energy re-weighting

The beam energy of data is not well simulated by MC. There is a dedicated study of data beam energy using fully reconstructed B tags [30]. We can use the result to make MC beam energy distribution similar to data beam energy distribution. Table 22 show the mean and width of beam energy distributions of data and MC averaged over each Runs.
Beam energies of data and MC, as a fuction of energy E, are given by Gaussians

$$\exp\left[-\frac{(E_{\text{data}}-E)^2}{2\sigma_{\text{data}}^2}\right], \quad \exp\left[-\frac{(E_{\text{MC}}-E)^2}{2\sigma_{\text{MC}}^2}\right] \tag{204}$$

We take ratios og these two Gaussinas to get weight

$$W^{\text{beam}}(E) = \xi \exp\left[-\frac{(E_{\text{data}}-E)^2}{2\sigma_{\text{data}}^2}\right] \Big/ \exp\left[-\frac{(E_{\text{MC}}-E)^2}{2\sigma_{\text{MC}}^2}\right] \tag{205}$$

where ξ is the normalization factor to make the integrals of original and weighted distributions equal.
The weight $W^{\text{beam}}(E)$ can be read as the ratio of numbers of data and MC events at beam energy E. Thus, for example, if the weight is larger than 1 which means we need to increase the number of MC events by the factor $W^{\text{beam}}(E)$. This is exactly what we did above. This $W^{\text{beam}}(E)$ needs to be multiplied by other weights to be integrated into total weight W_{jk}^{MC}. $W^{\text{beam}}(E)$ does not depend on MC mode (j) but depends on the event (k) because beam enrgy E differ event by event.
This re-weigting modify lepton memontum, D momentum and $\cos\theta_{B-Dl}$ distributions.

9.6.1 OffPeak data beam energy re-weighting

OffPeak data is taken at the energy 39MeV below the OnPeak data energy. Thus, we need to scale lepton enrgy and D energy by a factor r_{beam} to match OnPeak data. The factor is given by

$$r_{\text{beam}} = \frac{E_{\text{data}}^{OnPeak}}{E_{\text{data}}^{OffPeak}} = \frac{E_{\text{data}}}{E_{\text{data}} - 0.039 \text{ GeV}} \tag{206}$$

where E_{data} is given in Table 22.

Run	data [30]		SP5/SP6 MC		Normalization factor (ξ)
	mean (E_{data})	width (σ_{data})	mean (E_{MC})	width (σ_{MC})	
Run1	10.577618	0.00499	10.5778	0.00481	0.963934
Run2	10.578668	0.00501	10.5778	0.00481	0.960076
Run3	10.578393	0.00487	10.5778	0.00481	0.987636
Run4	10.577699	0.00492	10.5778	0.00481	0.977638

Table 22: [**Beam energy**] Average beam energy and width of each Run. Normalization factor ξ is also shown.

10 Form Factor (FF) Re-weighting (W_{jk}^{MC})

We use SP5/SP6 MC for our analysis. However, there have been some improvements of form factors (FF) since SP5/SP6 MC were generated. To take those improvements into account, form factor re-weighting is necessary. See also Appendix I for details of form factors.

10.1 Normalization

When we change parameters of form factors, we need to make sure that total rates stay same. Thus, we need a normalization factor, R_N, such that

$$\Gamma = \int \frac{d\Gamma(\text{old FF})}{dw} dw = R_N \int \frac{d\Gamma(\text{new FF})}{dw} dw \tag{207}$$

In actual integration the Gaussian quadrature method is used.

10.2 $B \to D\ell\nu$

In this mode, ISGW2 model [31][32] is used in MC. ISGW2 is a type of constituent quark models. We need to change it to HQET (Heavy Quark Effective Theory) model [33][34]. The ISGW2 model gives form factors $f_+(q^2)$ and $f_-(q^2)$, and the HQET model gives $h_+(w)$ and $h_-(w)$, where w is the velocity transfer (see Appendix H). These form factors are related by

$$h_+ = \frac{1}{R}\left(f_+ + \frac{1-r}{1+r} f_-\right) \tag{208}$$

$$h_- = \frac{1}{R}\left(\frac{1-r}{1+r} f_+ + f_-\right) \tag{209}$$

where

$$R = \frac{2\sqrt{m_B m_D}}{m_B + m_D} \tag{210}$$

and

$$r = \frac{m_D}{m_B} \tag{211}$$

q^2 is related to w by

$$w = \frac{m_B^2 + m_D^2 - q^2}{2m_B m_D} \quad \text{or} \quad q^2 = m_B^2 + m_D^2 - 2m_B m_D w \tag{212}$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D\ell\nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{3/2}\, \mathcal{J}_D(w) \tag{213}$$

where

$$\mathcal{J}_D(w) = [(1+r)h_+(w) - (1-r)h_-(w)]^2 \tag{214}$$

In the ISGW2 model,

$$\begin{aligned}
&(1+r)h_+ - (1-r)h_- \\
&\quad = (1+r)\frac{1}{R}\left(f_+ + \frac{1-r}{1+r}f_-\right) - (1-r)\frac{1}{R}\left(\frac{1-r}{1+r}f_+ + f_-\right) \\
&\quad = \frac{1+r}{R}\frac{(1+r)^2-(1-r)^2}{(1+r)^2}f_+ = \frac{1+r}{R}\frac{4r}{(1+r)^2}f_+ = \frac{1+r}{R}R^2 f_+ \\
&\quad = (1+r)Rf_+
\end{aligned} \tag{215}$$

Thus

$$\mathcal{J}_D^{ISGW2} = [(1+r)Rf_+]^2 \tag{216}$$

In HQET model, we can set

$$\begin{aligned}
h_+(w) &= h_+(1)[1-\rho_D^2(w-1)] \\
h_-(w) &= 0
\end{aligned} \tag{217}$$

where ρ_D^2 is the slope. $h_+(1)$ can be given by a Lattice QCD calculation [35]

$$h_+(1) = 1.074 \pm 0.018 \pm 0.016 = 1.074 \pm 0.024 \tag{218}$$

Thus

$$\mathcal{J}_D^{HQET} = (1+r)^2[h_+(1)]^2[1-\rho_D^2(w-1)]^2 \tag{219}$$

By using Equation (212), we can express f_+ as a function of w. Then, the weight is given by

$$W_{jk}^{MC}(w_k) = \frac{\mathcal{J}_D^{HQET}}{\mathcal{J}_D^{ISGW2}}R_N = \frac{[h_+(1)]^2[1-\rho_D^2(w-1)]^2}{[Rf_+]^2}R_N \tag{220}$$

where, the normalization factor, R_N, is given by

$$\begin{aligned}
&\int_0^{1.590} (w^2-1)^{3/2}\,[Rf_+]^2\,dw \\
&\qquad = R_N \int_1^{1.590} (w^2-1)^{3/2}[h_+(1)]^2[1-\rho_D^2(w-1)]^2 dw
\end{aligned} \tag{221}$$

Or, we can use the CLN parametrization proposed by Caprini, Lellouch and Neubert [36]

$$h_+(w) = h_+(1)[1 - 8\rho_D^2 z + (51\rho_D^2 - 10)z^2 - (252\rho_D^2 - 84)z^3] \tag{222}$$

where

$$z = \frac{\sqrt{w+1}-\sqrt{2}}{\sqrt{w+1}+\sqrt{2}} \tag{223}$$

In this case, the weight is given by

$$W_{jk}^{MC}(w_k) = \frac{[h_+(1)]^2[1 - 8\rho_D^2 z + (51\rho_D^2 - 10)z^2 - (252\rho_D^2 - 84)z^3]^2}{[Rf_+]^2}R_N \tag{224}$$

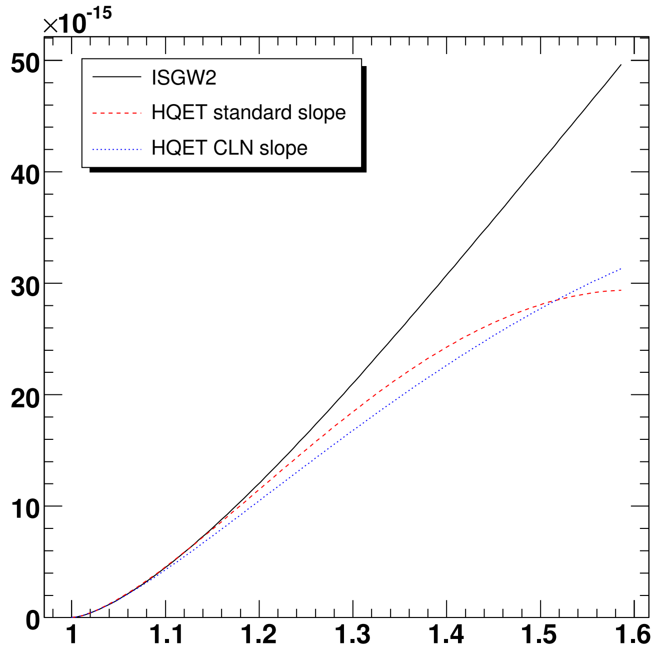


Figure 24: [$B \to D\ell\nu$ **FF**] The differential decay rate $d\Gamma(B \to D\ell\nu)/dw$ as a function of w. This is in case that the standard slope is $\rho_D^2 = 0.8$ and the CLN slope is $\rho_D^2 = 1.17$

where, the normalization factor, R_N, is given by

$$\int_0^{1.590} dw \ (w^2-1)^{3/2} \left[Rf_+\right]^2$$
$$= R_N \int_1^{1.590} dw \ (w^2-1)^{3/2} [h_+(1)]^2 \times \left[1 - 8\rho_D^2 z + (51\rho_D^2 - 10)z^2 - (252\rho_D^2 - 84)z^3\right]^2 \quad (225)$$

Figure 24 shows the differential decay rate $d\Gamma(B \to D\ell\nu)/dw$ as a function of w. The black line shows the ISGW2 model, the red dashed line shows the HQET model with the standard slope in case of $\rho_D^2 = 0.8$, and the blue dotted line shows the HQET model with the CLN slope in case of $\rho_D^2 = 1.17$ [24]. Figure 25 shows the distribution of weight W^{MC}_{jk} in the case of the CLN slope of $\rho_D^2 = 1.17$, corresponding to the normalization factor $R_N = 1.363$. BpBm MC is used to create the plot. The change in the spectrum of lepton momentum, D momentum and $\cos\theta_{B-Dl}$ is given in Figure 26 - 28.

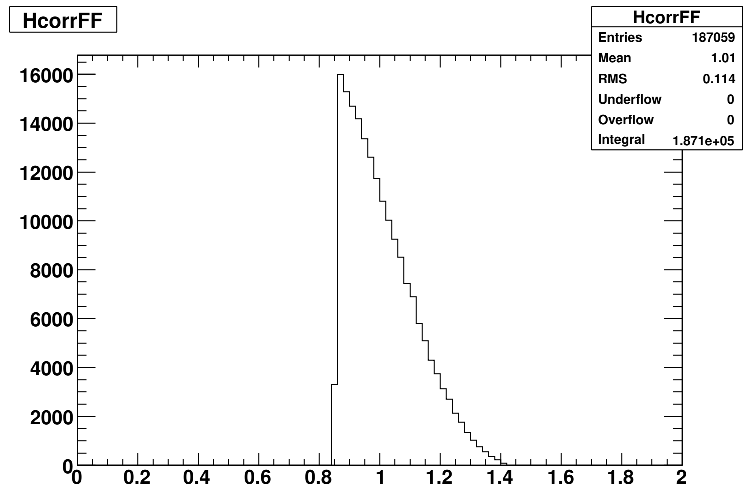


Figure 25: [$B \to D\ell\nu$ **FF weight**] The distribution of weight W^{MC}_{jk} for $B \to D\ell\nu$ decays This is made from BpBm MC.

10.3 $B \to D^*\ell\nu$

We need three kinematic variables to describe this decay. We usually chose $\cos\theta_\ell$, $\cos\theta_V$ and χ as shown in the figure 29. The definition of these variables are

- $\cos\theta_\ell$: the cosine of the angle between the directions of "the ℓ in the W rest frame" and "the W in the B rest frame".
- $\cos\theta_V$: the cosine of the angle between the directions of "the D in the D^* rest frame" and "the D^* in the B rest frame".
- χ : the azimuthal angle between the planes formed by "W-ℓ system' ' and "$D^* - D$ system".

HQET model is used and the differential decay rate is given by [33][34]

$$
\begin{aligned}
\frac{d\Gamma(B\to D^*\ell\nu)}{dw d\cos\theta_V d\cos\theta_\ell d\chi} = & \frac{3G_F^2}{4(4\pi)^4}|V_{cb}|^2 m_B m_{D^*}^2\sqrt{w^2-1}(1-2wr+r^2)\times \\
& [(1-\cos\theta_\ell)^2\sin^2\theta_V|H_+(w)|^2 \\
& +(1+\cos\theta_\ell)^2\sin^2\theta_V|H_-(w)|^2 \\
& +4\sin^2\theta_\ell\cos^2\theta_V|H_0(w)|^2 \\
& -4\sin\theta_\ell(1-\cos\theta_\ell)\sin\theta_V\cos\theta_V\cos\chi H_+(w)H_0(w) \\
& +4\sin\theta_\ell(1+\cos\theta_\ell)\sin\theta_V\cos\theta_V\cos\chi H_-(w)H_0(w) \\
& -2\sin^2\theta_\ell\sin^2\theta_V\cos2\chi H_+(w)H_-(w)]
\end{aligned}
\tag{226}
$$

where r is the mass ratio

$$
r = \frac{m_{D^*}}{m_B} \tag{227}
$$

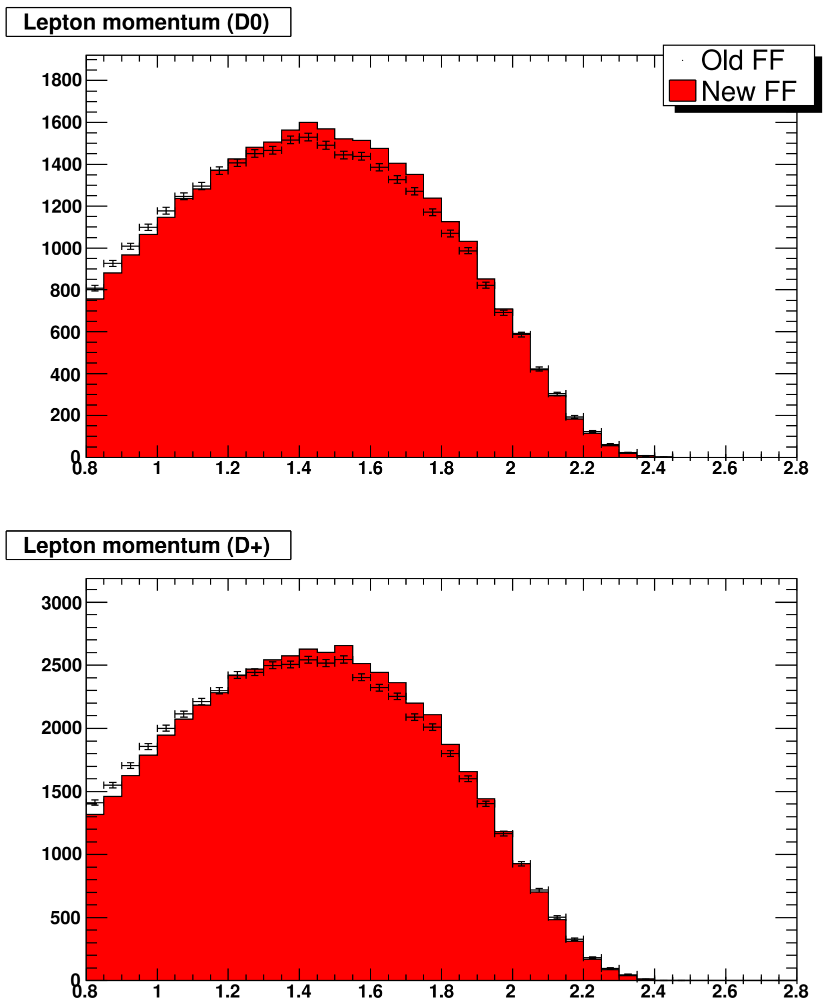


Figure 26: [p_ℓ **after** $B \to D\ell\nu$ **re-weighting**] Lepton momentum spectrum of $B^+ \to \bar{D}^0\ell\nu$ decays (top) and $B^0 \to D^-\ell\nu$ decays (bottom). Black line is ISGW2 model and red histogram is HQET model with CLN slope $\rho_D^2 = 1.17$

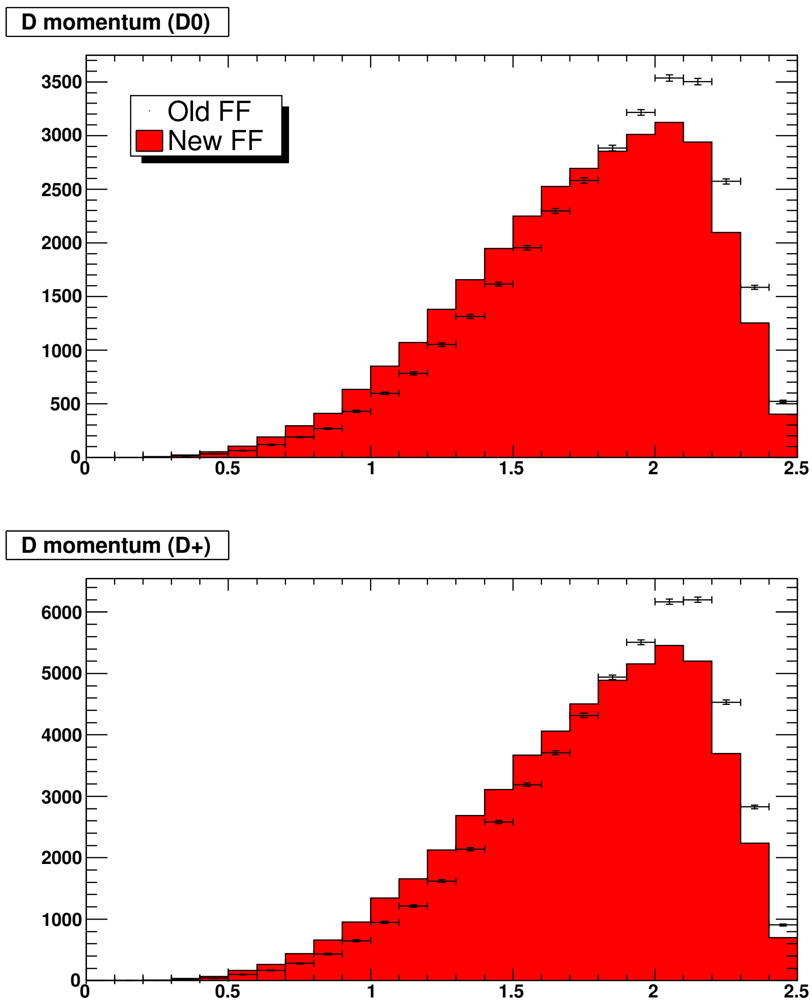


Figure 27: [p_D **after** $B \to D\ell\nu$ **re-weighting**] D momentum spectrum of $B^+ \to \bar{D}^0\ell\nu$ decays (top) and $B^0 \to D^-\ell\nu$ decays (bottom). Black line is ISGW2 model and red histogram is HQET model with CLN slope $\rho^2_D = 1.17$

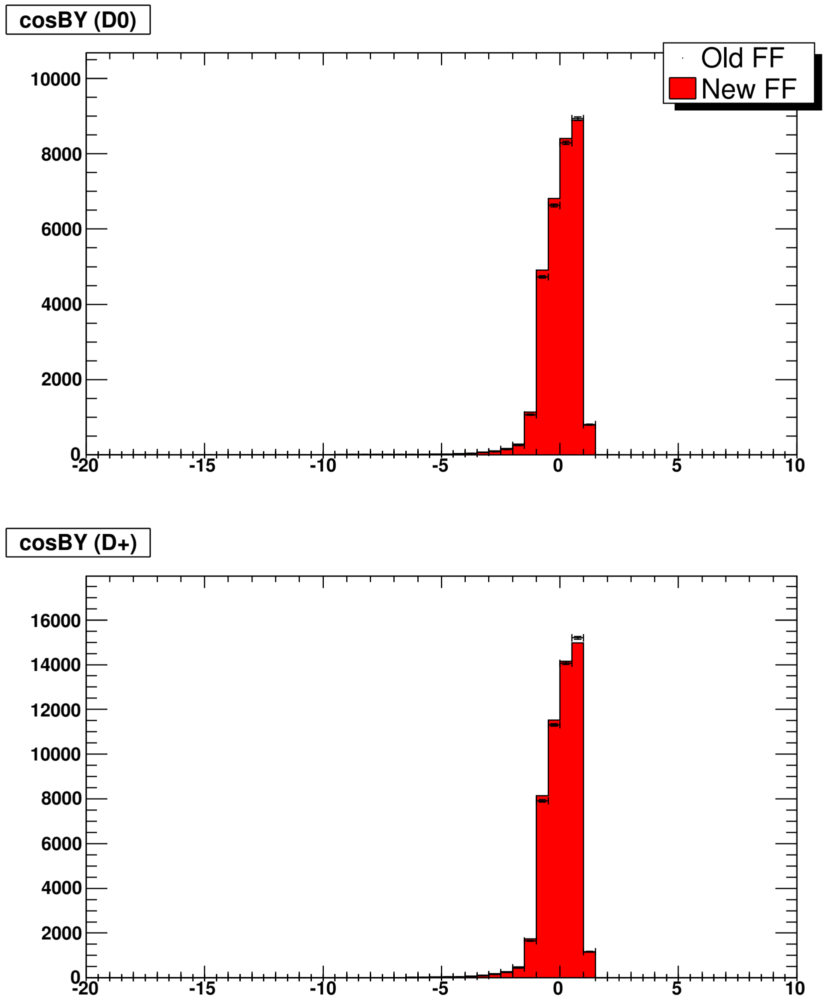


Figure 28: [$\cos\theta_{B-Dl}$ **after** $B \to D\ell\nu$ **re-weighting**] $\cos\theta_{B-Dl}$ of $B^+ \to \bar{D}^0\ell\nu$ decays (top) and $B^0 \to D^-\ell\nu$ decays (bottom). Black line is ISGW2 model and red histogram is HQET model with CLN slope $\rho_D^2 = 1.17$

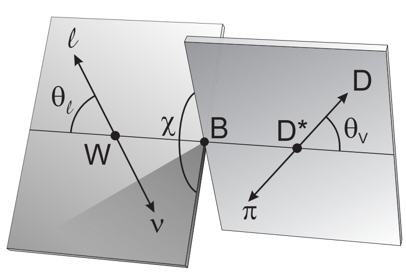


Figure 29: [$B \to D^*\ell\nu$ **decay geometry**] Geometry of $B \to D^*\ell\nu$ decays.

$H_i(w)$ are

$$H_i = -m_B \frac{R^*(1-r^2)(w+1)}{2\sqrt{1-2wr+r^2}} h_{A_1}(w) \widetilde{H}_i(w) \tag{228}$$

where R^* is

$$R^* = \frac{2\sqrt{m_B m_{D^*}}}{m_B + m_{D^*}} \tag{229}$$

and $\widetilde{H}_i(w)$ are

$$\begin{aligned} \widetilde{H}_\pm(w) &= \tfrac{\sqrt{1-2wr+r^2}}{1-r}\left(1 \mp \sqrt{\tfrac{w-1}{w+1}} R_1(w)\right) \\ \widetilde{H}_0(w) &= 1 + \tfrac{w-1}{1-r}\left(1 - R_2(w)\right) \end{aligned} \tag{230}$$

were R_1 and R_2 are the form factor ratios. To make the notation simple, we define

$$\begin{aligned} & c_L \equiv \cos\theta_\ell \\ & c_V \equiv \cos\theta_V \\ & c_\chi \equiv \cos\chi \\ & \tilde{h}_\pm(w) \equiv (1-r)\widetilde{H}_\pm(w) = \sqrt{1-2wr+r^2}\left(1 \mp \sqrt{\tfrac{w-1}{w+1}} R_1(w)\right) \\ & \tilde{h}_0(w) \equiv (1-r)\widetilde{H}_0(w) = (1-r) + (w-1)\left(1 - R_2(w)\right) = (w-r) - (w-1)R_2(w) \end{aligned} \tag{231}$$

then (see Appendix I.4)

$$\frac{d\Gamma(B \to D^*\ell\nu)}{dw dc_V dc_L d\chi} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2} \times \frac{9}{64\pi}(w+1)^2 \ \mathcal{J}_{FF} \tag{232}$$

where

$$\begin{aligned} \mathcal{J}_{FF}(R_1, R_2, \rho^2) \;=\; & \left[h_{A_1}(w)\right]^2 \times \\ & [(1-c_L)^2(1-c_V^2)\tilde{h}_+^2 + (1+c_L)^2(1-c_V^2)\tilde{h}_-^2 \\ & +4(1-c_L^2)c_V^2\tilde{h}_0^2 - 2(1-c_L^2)(1-c_V^2)(2c_\chi^2-1)\tilde{h}_+\tilde{h}_- \\ & -4\sqrt{1-c_L^2}(1-c_L)\sqrt{1-c_V^2}c_V c_\chi \tilde{h}_+\tilde{h}_0 \\ & +4\sqrt{1-c_L^2}(1+c_L)\sqrt{1-c_V^2}c_V c_\chi \tilde{h}_-\tilde{h}_0] \end{aligned} \tag{233}$$

In SP5/SP6 MC, the values of parameters are

$$\begin{aligned} R_1^{MC} &= 1.18 \\ R_2^{MC} &= 0.72 \\ \left(\rho^2\right)^{MC} &= 0.92 \end{aligned} \tag{234}$$

with

$$\begin{aligned} R_1(w) &= R_1^{MC} \\ R_2(w) &= R_2^{MC} \\ h_{A_1}(w) &= h_{A_1}(1)\left[1 - \left(\rho^2\right)^{MC}(w-1)\right] \end{aligned} \tag{235}$$

where $h_{A_1}(1)$ can be givn by a Lattice calculation [37][38]. The values from a recent Babar measurement [39][40] are

$$\begin{aligned} R_1 &= 1.417 \pm 0.061 \pm 0.044 \\ R_2 &= 0.836 \pm 0.037 \pm 0.022 \\ \rho^2 &= 1.179 \pm 0.048 \pm 0.028 \end{aligned} \tag{236}$$

where the CLN parametrization [36] is used

$$\begin{aligned} R_1(w) &= R_1 - 0.12(w-1) + 0.05(w-1)^2 \\ R_2(w) &= R_2 + 0.11(w-1) - 0.06(w-1)^2 \\ h_{A_1}(w) &= h_{A_1}(1)\left[1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3\right] \end{aligned} \tag{237}$$

where

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}} \tag{238}$$

We also use the CLN parametrization in our fit. The weight is given by

$$W_{jk}^{MC}(w_k) = \frac{\mathcal{J}_{FF}(R_1, R_2, \rho^2)}{\mathcal{J}_{FF}(R_1^0, R_2^0, \rho_0^2)} \tag{239}$$

To calculate the normalization factor R_N, we need to integrate over c_V, c_L and χ. Thus, it is easier to use $d\Gamma/dw$ (see Appendix I.4)

$$\frac{d\Gamma(B \to D^* \ell\nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2} \mathcal{J}_{WD}(R_1, R_2, \rho^2) \tag{240}$$

where

$$\mathcal{J}_{WD}(R_1, R_2, \rho^2) = (w+1)^2 [h_{A_1}]^2 (\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2) \tag{241}$$

Thus, the normalization factor is given by

$$\begin{aligned} &\int_1^{1.504} dw\ \ (w^2-1)^{1/2} \mathcal{J}_{WD}(R_1^0, R_2^0, \rho_0^2) \\ &= R_N \int_1^{1.504} dw\ \ (w^2-1)^{1/2} \mathcal{J}_{WD}(R_1, R_2, \rho^2) \end{aligned} \tag{242}$$

Figure 30 shows the differential decay rate $d\Gamma(B \to D^* \ell\nu)/dw$ as a function of w. The red dashed line shows the HQET model with the old parameters of R_1, R_2 and ρ^2, and the blue dotted line shows the HQET model with new parameters. Figure 31 shows the distribution of weight W_{jk}^{MC} for BpBm MC corresponding to the normalization factor $R_N = 1.044$. The change in the spectrum of lepton momentum, D momentum and $\cos\theta_{B-Dl}$ is given in Figure 32 - 34.

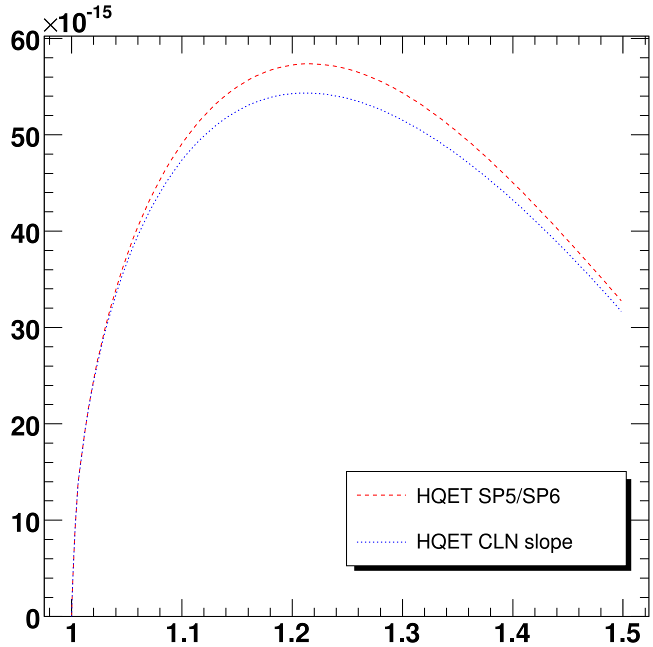


Figure 30: [$B \to D^*\ell\nu$ **FF**] The differential decay rate $d\Gamma(B \to D^*\ell\nu)/dw$ as a function of w.

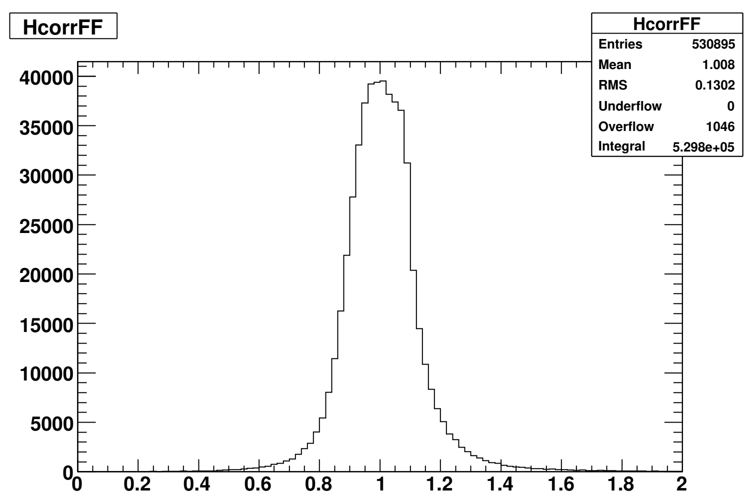


Figure 31: [$B \to D^*\ell\nu$ **FF weight**] The distribution of weight W^{MC}_{jk} for $B \to D^*\ell\nu$ decays This is made from BpBm MC.

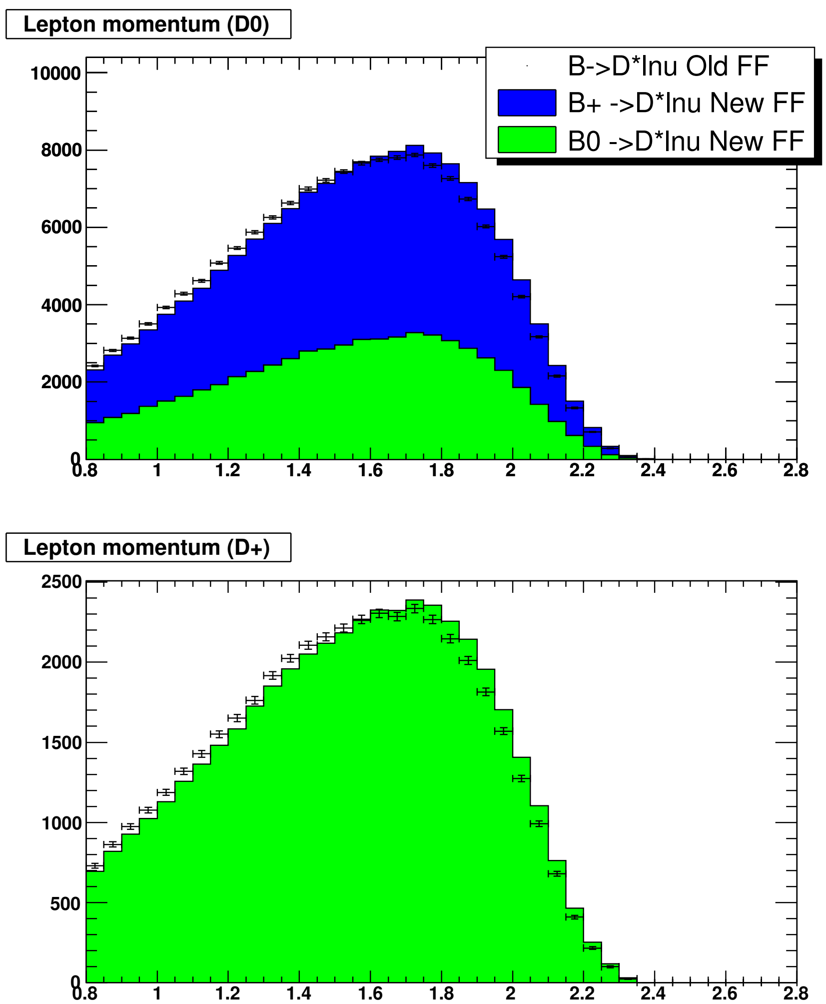


Figure 32: [p_ℓ **after** $B \to D^*\ell\nu$ **re-weighting**] Lepton momentum spectrum of $B \to D^*\ell\nu$ decays. The top is D^0 and the bottom is D^+. Black line is with old FF parameters and histogram is with new FF parameters. Green is the contribution from B^+ and blue is from B^0.

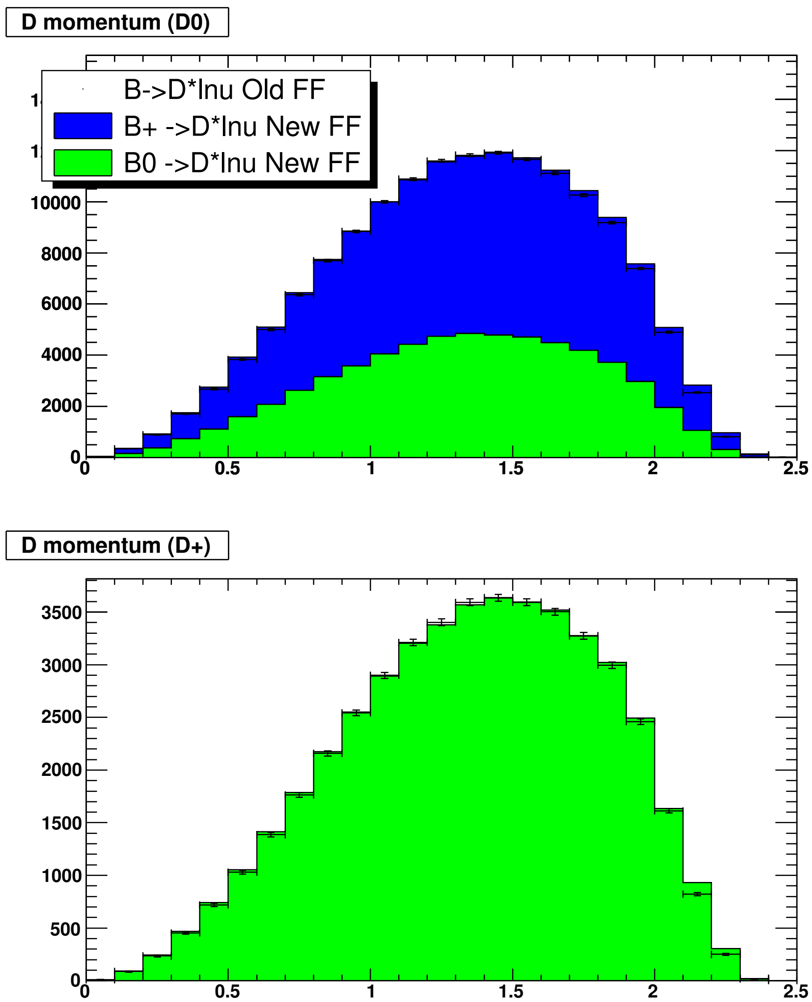


Figure 33: [p_D **after** $B \to D^* \ell \nu$ **re-weighting**] D momentum spectrum of $B \to D^* \ell \nu$ decays. The top is D^0 and the bottom is D^+. Black line is with old FF parameters and histogram is with new FF parameters. Green is the contribution from B^+ and blue is from B^0.

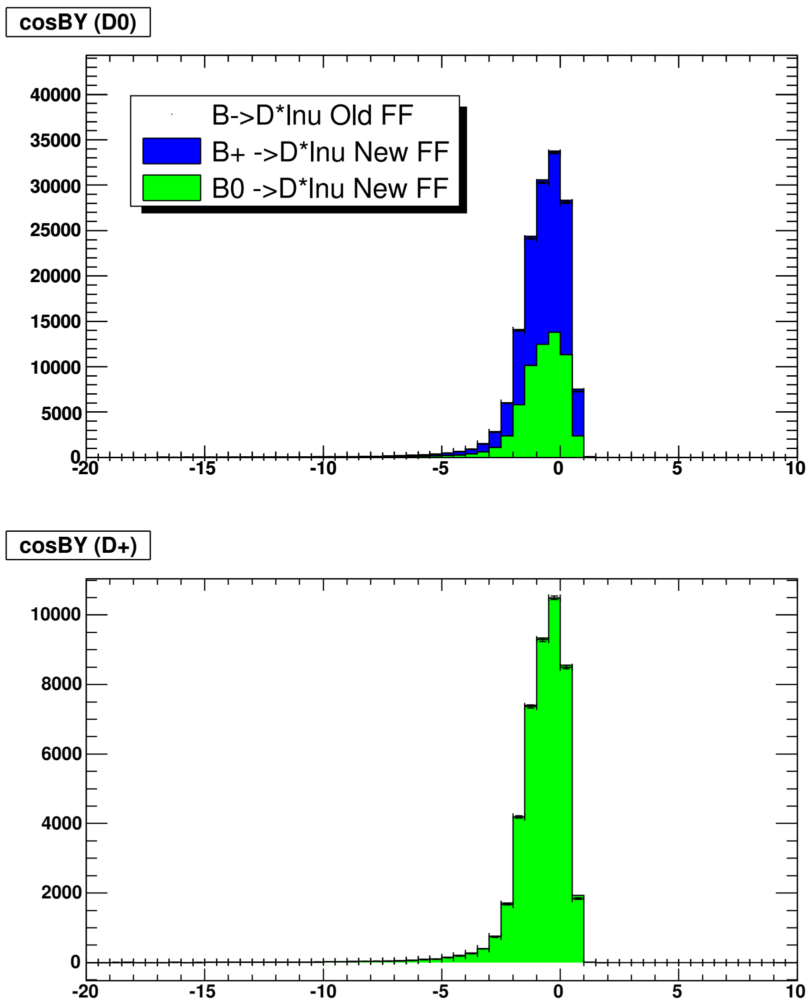


Figure 34: [$\cos\theta_{B-Dl}$ **after** $B \to D^*\ell\nu$ **re-weighting**] $\cos\theta_{B-Dl}$ of $B \to D^*\ell\nu$ decays. The top is D^0 and the bottom is D^+. Black line is with old FF parameters and histogram is with new FF parameters. Green is the contribution from B^+ and blue is from B^0.

10.4 $B \to D^{**}\ell\nu$

In SP5/SP6 MC, ISGW2 model is used. We would like to use HQET based LLSW model [41][42]. Details of LLSW model is given in Appendix I. We use the Approximation B1 to calculate the form factors and the Approximation B2 to estimate systematic uncertainties. The differential decay rates are given as functions of w and θ. This θ is the angle between the charged lepton and the charmed meson in the rest frame of the virtual W boson. Thus $\theta = \pi - \theta_\ell$, e.g.,

$$\cos\theta = \cos(\pi - \theta_\ell) = -\cos\theta_\ell \tag{243}$$

10.4.1 $B \to D_0^*\ell\nu$

The ISGW2 form factors $u_+(q^2)$ and $u_-(q^2)$ are related to the LLSW form factors $g_+(w)$ and $g_-(w)$

$$g_+ = \frac{1}{R}\left(u_+ + \frac{1-r}{1+r}u_-\right), \quad g_- = \frac{1}{R}\left(\frac{1-r}{1+r}u_+ + u_-\right) \tag{244}$$

where

$$R = \frac{2\sqrt{m_B m_{D_0^*}}}{m_B + m_{D_0^*}}, \quad r = \frac{m_{D_0^*}}{m_B} \tag{245}$$

The differential decay rate is given by

$$\frac{d^2\Gamma_{D_0^*}}{dw d\cos\theta} = \frac{G_F^2|V_{cb}|^2 m_B^5}{64\pi^3} r^3(w^2-1)^{3/2}\, \mathcal{I}_{D_0^*}(w,\theta) \tag{246}$$

where

$$\mathcal{I}_{D_0^*}(w,\theta) = (1-\cos^2\theta)[(1+r)g_+ - (1-r)g_-]^2 \tag{247}$$

The weight is given by the ratio of $\mathcal{I}_{D_0^*}$ part.

$$W_{jk}^{MC}(w_k) = \frac{\mathcal{I}_{D_0^*}^{LLSW}}{\mathcal{I}_{D_0^*}^{ISGW2}} R_N \tag{248}$$

Integrated by θ, the differential decay rate is

$$\frac{d\Gamma(B \to D_0^*\ell\nu)}{dw} = \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3(w^2-1)^{3/2}\, \mathcal{J}_{D_0^*}(w) \tag{249}$$

where

$$\mathcal{J}_{D_0^*}(w) = [(1+r)g_+ - (1-r)g_-]^2 \tag{250}$$

Thus, the normalization factor, R_N, is given by

$$\int_0^{1.327} (w^2-1)^{3/2}\mathcal{J}_{D_0^*}^{ISGW2} dw = R_N \int_1^{1.327} (w^2-1)^{3/2}\mathcal{J}_{D_0^*}^{LLSW} dw \tag{251}$$

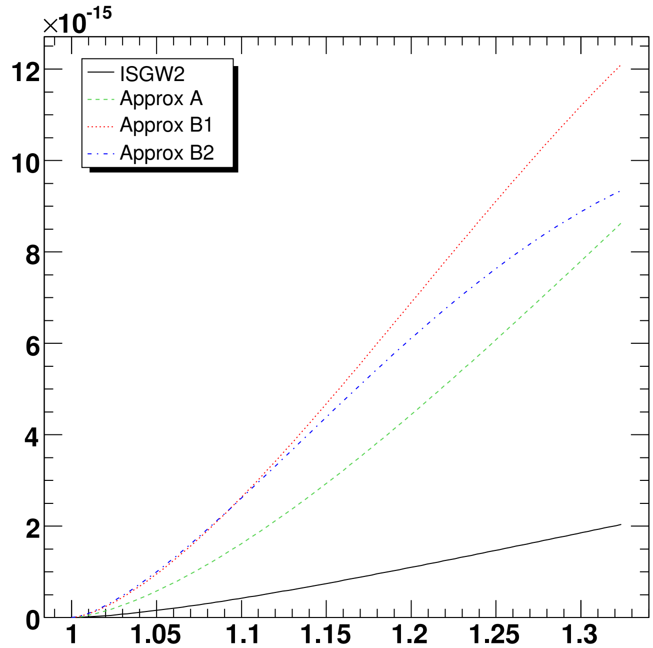


Figure 35: [$B \to D_0^* \ell \nu$ **FF**] The differential decay rate $d\Gamma(B \to D_0^* \ell \nu)/dw$ as a function of w.

Figure 35 shows the differential decay rate $d\Gamma(B \to D_0^* \ell \nu)/dw$ as a function of w. The black line shows the ISGW2 model, the other lines show the LLSW model; the green dashed line is the Approximation A, the red dotted line is the Approximation B1 and the blue dot-dashed line is the Approximation B2. We use the Approximation B1 in our fit. Figure 36 shows the distribution of weight W^{MC}_{jk} in the case of the Approximation B1, corresponding to the normalization factor $R_N = 0.1624$. BpBm MC is used to create the plot.

10.4.2 $B \to D_1' \ell \nu$

The ISGW2 form factors q, l and c_+ are related to the LLSW form factors g_A, g_{V_1}, g_{V_2} and g_{V_3}.

$$g_A = R m_B (1+r) q \tag{252}$$

$$g_{V_1} = \frac{2}{R m_B (1+r)} l \tag{253}$$

$$g_{V_3} + r g_{V_2} = R m_B (1+r) c_+ \tag{254}$$

where

$$R = \frac{2\sqrt{m_B m_{D_1'}}}{m_B + m_{D_1'}}, \quad r = \frac{m_{D_1'}}{m_B} \tag{255}$$

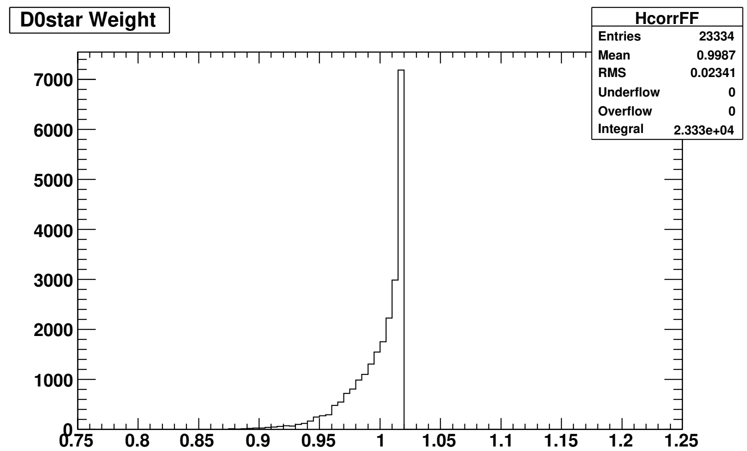


Figure 36: [$B \to D_0^* \ell \nu$ **FF weight**] The distribution of weight W_{jk}^{MC} for $B \to D_0^* \ell \nu$ decays This is made from BpBm MC.

The differential decay rate is given by

$$\frac{d^2\Gamma_{D_1'}}{dw d\cos\theta} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{64\pi^3} r^3 (w^2-1)^{1/2}\, \mathcal{I}_{D_1'}(w,\theta) \tag{256}$$

where

$$\begin{aligned}\mathcal{I}_{D_1'}(w,\theta) = {} & (1-\cos^2\theta)[(w-r)g_{V_1} + (w^2-1)(g_{V_3}+rg_{V_2})]^2 \\ & +(1-2rw+r^2)[(1+\cos^2\theta)(g_{V_1}^2 + (w^2-1)g_A^2) - 4\cos\theta\sqrt{w^2-1}g_{V_1}g_A]\end{aligned} \tag{257}$$

The weight is given by the ratio of $\mathcal{I}_{D_1'}$ part.

$$W_{jk}^{MC}(w_k) = \frac{\mathcal{I}_{D_1'}^{LLSW}}{\mathcal{I}_{D_1'}^{ISGW2}} R_N \tag{258}$$

Integrated by θ, the differential decay rate is

$$\frac{d\Gamma(B \to D_1 \ell \nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2}\, \mathcal{J}_{D_1'}(w) \tag{259}$$

where

$$\begin{aligned}\mathcal{J}_{D_1'}(w) = {} & \left[(w-r)g_{V_1} + (w^2-1)(g_{V_3}+rg_{V_2})\right]^2 \\ & +2(1-2rw+r^2)[g_{V_1}^2 + (w^2-1)g_A^2]\end{aligned} \tag{260}$$

Thus, the normalization factor, R_N, is given by

$$\int_0^{1.316} dw (w^2-1)^{1/2} \mathcal{J}_{D_1'}^{ISGW2} = R_N \int_1^{1.316} dw (w^2-1)^{1/2} \mathcal{J}_{D_1'}^{LLSW} \tag{261}$$

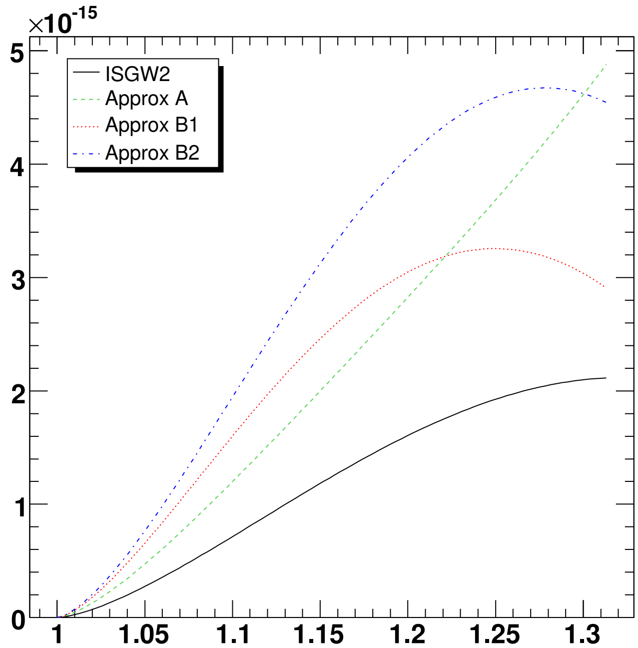


Figure 37: [$B \to D_1' \ell\nu$ **FF**] The differential decay rate $d\Gamma(B \to D_1' \ell\nu)/dw$ as a function of w.

Figure 37 shows the differential decay rate $d\Gamma(B \to D_1' \ell\nu)/dw$ as a function of w. The black line shows the ISGW2 model, the other lines show the LLSW model; the green dashed line is the Approximation A, the red dotted line is the Approximation B1 and the blue dot-dashed line is the Approximation B2. We use the Approximation B1 in our fit. Figure 38 shows the distribution of weight W_{jk}^{MC} in the case of the Approximation B1, corresponding to the normalization factor $R_N = 0.5488$. BpBm MC is used to create the plot.

10.4.3 $B \to D_1 \ell\nu$

The ISGW2 form factors v, r and s_+ are related to the LLSW form factor f_A, f_{V_1}, f_{V_2} and f_{V_3}.

$$f_A = R m_B (1 + r) v \tag{262}$$

$$f_{V_1} = \frac{2}{R m_B (1 + r)} r \tag{263}$$

$$f_{V_3} + r f_{V_2} = R m_B (1 + r) s_+ \tag{264}$$

where

$$R = \frac{2\sqrt{m_B m_{D_1}}}{m_B + m_{D_1}}, \quad r = \frac{m_{D_1}}{m_B} \tag{265}$$

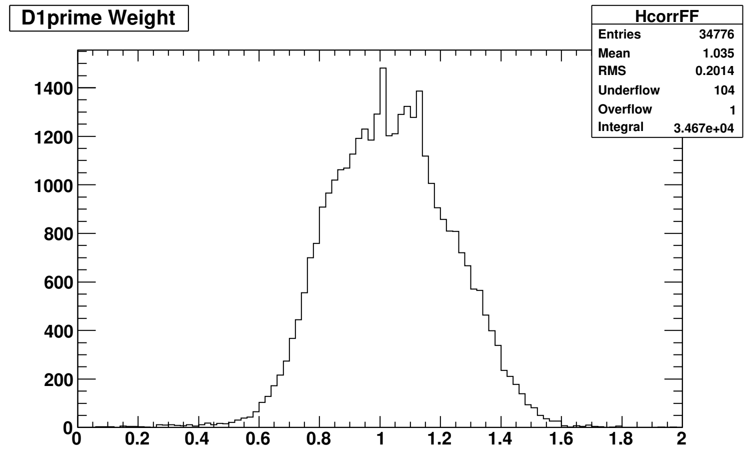


Figure 38: [$B \to D_1' \ell \nu$ **FF weight**] The distribution of weight W_{jk}^{MC} for $B \to D_1' \ell \nu$ decays This is made from BpBm MC.

The differential decay rate is given by

$$\frac{d^2\Gamma_{D_1}}{dw d\cos\theta} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{64\pi^3} r^3 (w^2-1)^{1/2}\, \mathcal{I}_{D_1}(w,\theta) \tag{266}$$

where

$$\begin{aligned}\mathcal{I}_{D_1}(w,\theta) &= (1-\cos^2\theta)[(w-r)f_{V_1} + (w^2-1)(f_{V_3} + r f_{V_2})]^2 \\ &+ (1-2rw+r^2)[(1+\cos^2\theta)(f_{V_1}^2 + (w^2-1)f_A^2) - 4\cos\theta\sqrt{w^2-1} f_{V_1} f_A]\end{aligned} \tag{267}$$

The weight is given by the ratio of $\mathcal{I}_{D_1}$ part.

$$W_{jk}^{MC}(w_k) = \frac{\mathcal{I}_{D_1}^{LLSW}}{\mathcal{I}_{D_1}^{ISGW2}} R_N \tag{268}$$

Integrated by θ, the differential decay rate is

$$\frac{d\Gamma(B \to D_1 \ell\nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2}\, \mathcal{J}_{D_1}(w) \tag{269}$$

where

$$\begin{aligned}\mathcal{J}_{D_1}(w) &= \left[(w-r)f_{V_1} + (w^2-1)(f_{V_3} + r f_{V_2})\right]^2 \\ &+ 2(1-2rw+r^2)[f_{V_1}^2 + (w^2-1)f_A^2]\end{aligned} \tag{270}$$

Thus, the normalization factor, R_N, is given by

$$\int_0^{1.320} dw (w^2-1)^{1/2} \mathcal{J}_{D_1}^{ISGW2} = R_N \int_1^{1.320} dw (w^2-1)^{1/2} \mathcal{J}_{D_1}^{LLSW} \tag{271}$$

Figure 39 shows the differential decay rate $d\Gamma(B \to D_1 \ell\nu)/dw$ as a function of w. The black line shows the ISGW2 model, the other lines show the LLSW model; the green

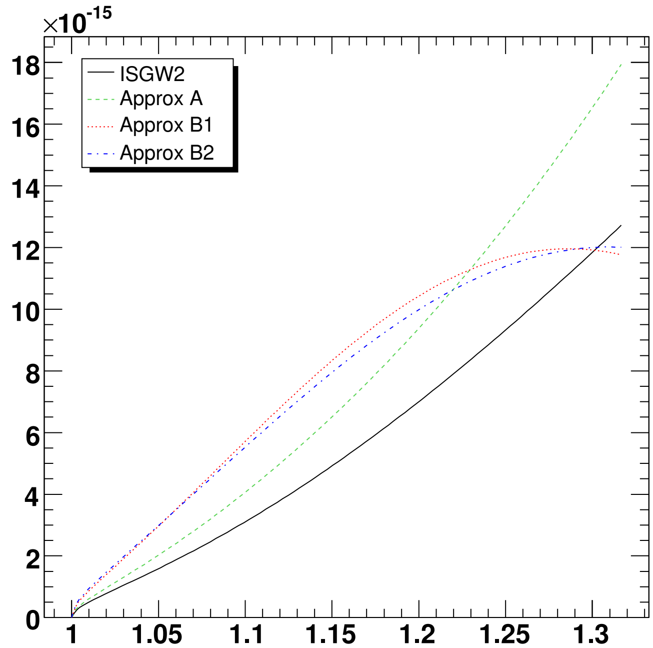


Figure 39: [$B \to D_1\ell\nu$ **FF**] The differential decay rate $d\Gamma(B \to D_1\ell\nu)/dw$ as a function of w.

dashed line is the Approximation A, the red dotted line is the Approximation B1 and the blue dot-dashed line is the Approximation B2. We use the Approximation B1 in our fit. Figure 40 shows the distribution of weight W^{MC}_{jk} in the case of the Approximation B1, corresponding to the normalization factor $R_N = 0.7330$. BpBm MC is used to create the plot.

10.4.4 $B \to D_2^*\ell\nu$

The ISGW2 form factors h, k and b_+ are related to the LLSW form factor k_V, k_{A_1}, k_{A_2} and k_{A_3}.

$$k_V = Rm_B^2(1-r)h \tag{272}$$

$$k_{A_1} = \frac{2}{R(1-r)}k \tag{273}$$

$$k_{A_3} + rk_{A_2} = Rm_B^2(1-r)b_+ \tag{274}$$

where

$$R = \frac{2\sqrt{m_B m_{D_2^*}}}{m_B + m_{D_2^*}}, \quad r = \frac{m_{D_2^*}}{m_B} \tag{275}$$

The differential decay rate is given by

$$\frac{d^2\Gamma_{D_2^*}}{dw\,d\cos\theta} = \frac{G_F^2|V_{cb}|^2 m_B^5}{64\pi^3} r^3(w^2-1)^{3/2}\,\frac{1}{2}\,\mathcal{I}_{D_2^*}(w,\theta) \tag{276}$$

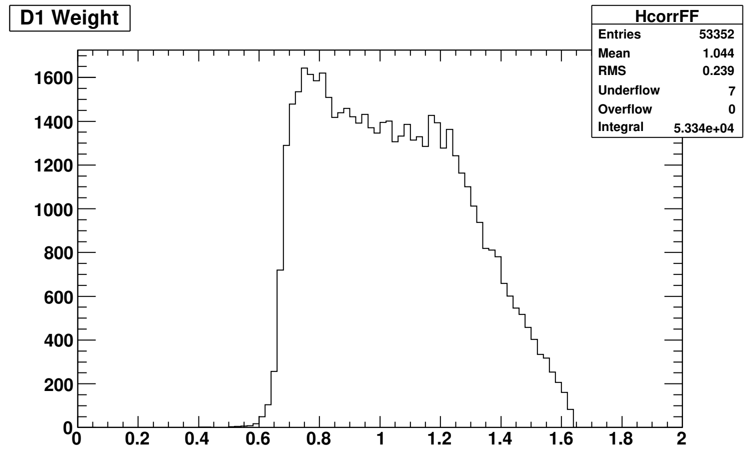


Figure 40: [$B \to D_1 \ell \nu$ **FF weight**] The distribution of weight W_{jk}^{MC} for $B \to D_1 \ell \nu$ decays This is made from BpBm MC.

where

$$\mathcal{I}_{D_2^*}(w,\theta) = \frac{4}{3}(1-\cos^2\theta)[(w-r)k_{A_1} + (w^2-1)(k_{A_3}+rk_{A_2})]^2$$
$$+(1-2rw+r^2)[(1+\cos^2\theta)(k_{A_1}^2 + (w^2-1)k_V^2) - 4\cos\theta\sqrt{w^2-1}k_{A_1}k_V] \quad (277)$$

The weight is given by the ratio of $\mathcal{I}_{D_2^*}$ part

$$W_{jk}^{MC}(w_k) = \frac{\mathcal{I}_{D_2^*}^{LLSW}}{\mathcal{I}_{D_2^*}^{ISGW2}} R_N \quad (278)$$

Integrated by θ, decay rate is

$$\frac{d\Gamma(B \to D_2^* \ell \nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{3/2} \, \mathcal{J}_{D_2^*}(w) \quad (279)$$

where

$$\mathcal{J}_{D_2^*}(w) = \frac{2}{3}\left[(w-r)k_{A_1} + (w^2-1)(k_{A_3}+rk_{A_2})\right]^2$$
$$+2(1-2rw+r^2)[k_{A_1}^2 + (w^2-1)k_V^2] \quad (280)$$

Thus, the normalization factor, R_N, is given by

$$\int_0^{1.306} dw (w^2-1)^{3/2} \mathcal{J}_{D_2^*}^{ISGW2} = R_N \int_1^{1.306} dw (w^2-1)^{3/2} \mathcal{J}_{D_2^*}^{LLSW} \quad (281)$$

Figure 41 shows the differential decay rate $d\Gamma(B \to D_2^* \ell \nu)/dw$ as a function of w. The black line shows the ISGW2 model, the other lines show the LLSW model; the green dashed line is the Approximation A, the red dotted line is the Approximation B1 and the blue dot-dashed line is the Approximation B2. We use the Approximation B1 in our fit. Figure 42 shows the distribution of weight W_{jk}^{MC} in the case of the Approximation B1, corresponding to the normalization factor $R_N = 0.7490$. BpBm MC is used to create the plot.

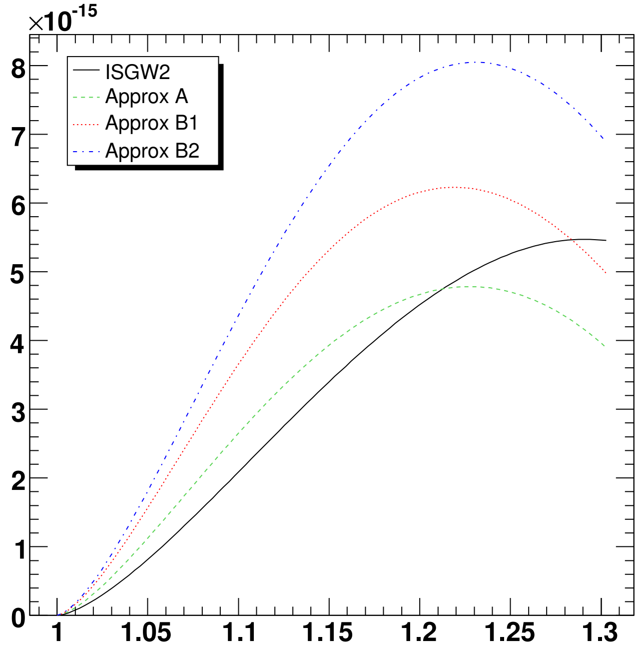


Figure 41: [$B \to D_2^*\ell\nu$ **FF**] The differential decay rate $d\Gamma(B \to D_2^*\ell\nu)/dw$ as a function of w.

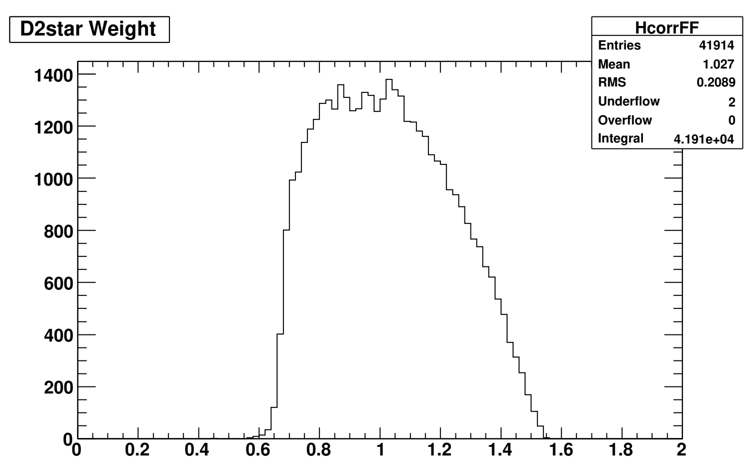


Figure 42: [$B \to D_2^*\ell\nu$ **FF weight**] The distribution of weight W_{jk}^{MC} for $B \to D_2^*\ell\nu$ decays This is made from BpBm MC.

10.4.5 Effect on kinematic variables

The change in the spectrum of lepton momentum, D momentum and $\cos\theta_{B-Dl}$ is given in Figure 43 - 45.

10.5 $B \to D^{(*)}\pi\ell\nu$ (non-resonant)

In SP5/SP6 Goiry-Roberts model [43][44] is used. We keep this model because there is no better alternative.

10.5.1 $B \to D\pi\ell\nu$

$$W_{jk}^{Sig}(w_k) = 1 \tag{282}$$

10.5.2 $B \to D^{*}\pi\ell\nu$

$$W_{jk}^{Sig}(w_k) = 1 \tag{283}$$

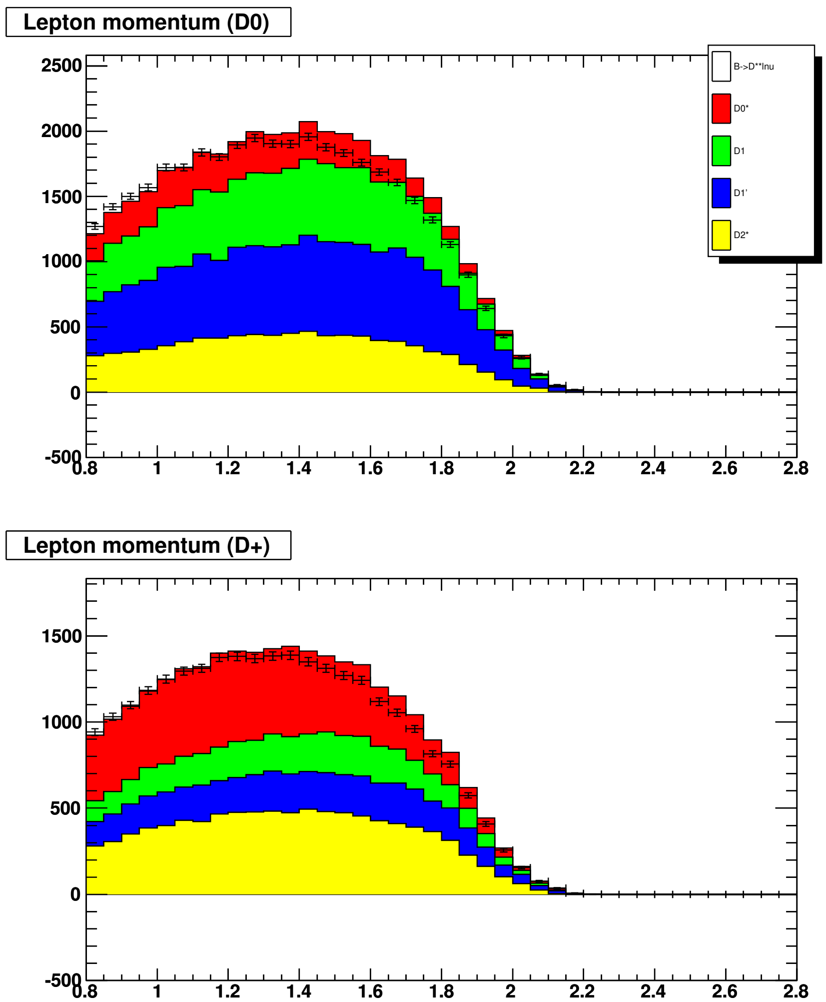


Figure 43: [p_ℓ **after** $B \to D^{**}\ell\nu$ **re-weighting**] Lepton momentum spectrum of $B \to D^{**}\ell\nu$ decays. Black line is ISGW2 model and colored histogram is HQET model. Red is D_0^*, green is D_1, blue is D_1' and yellow is D_2^*.

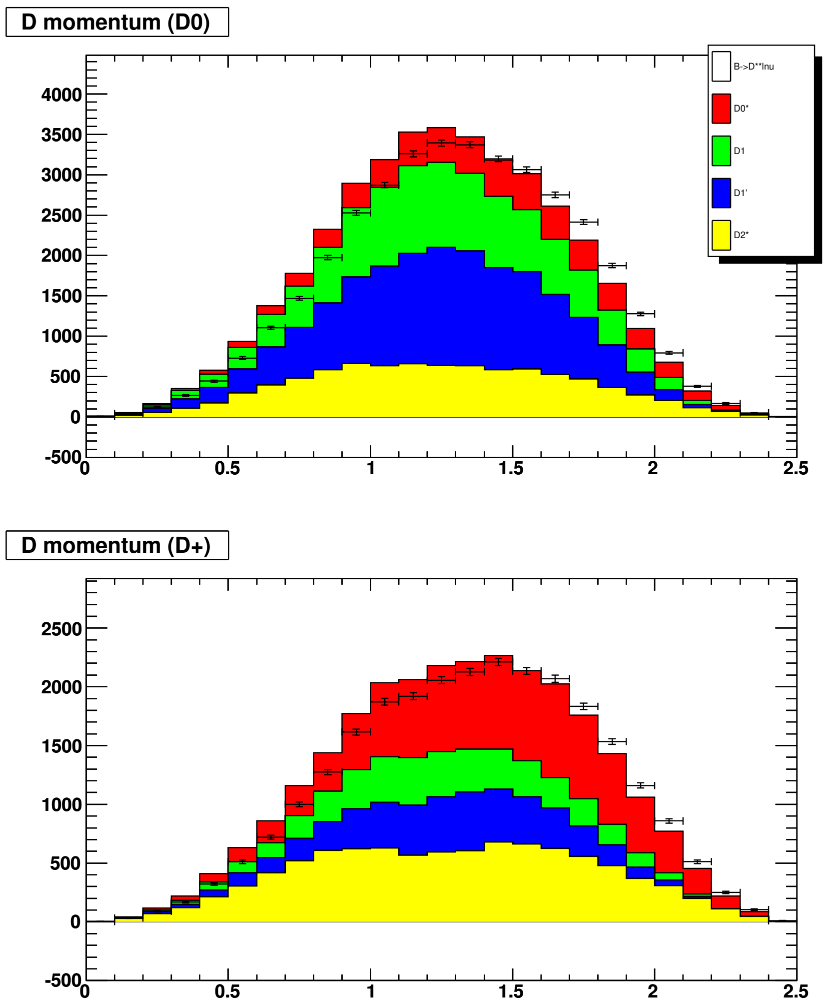


Figure 44: [p_D **after** $B \to D^{**}\ell\nu$ **re-weighting**] D momentum spectrum of $B \to D^{**}\ell\nu$ decays. Black line is ISGW2 model and colored histogram is HQET model. Red is D_0^*, green is D_1, blue is D_1' and yellow is D_2^*.

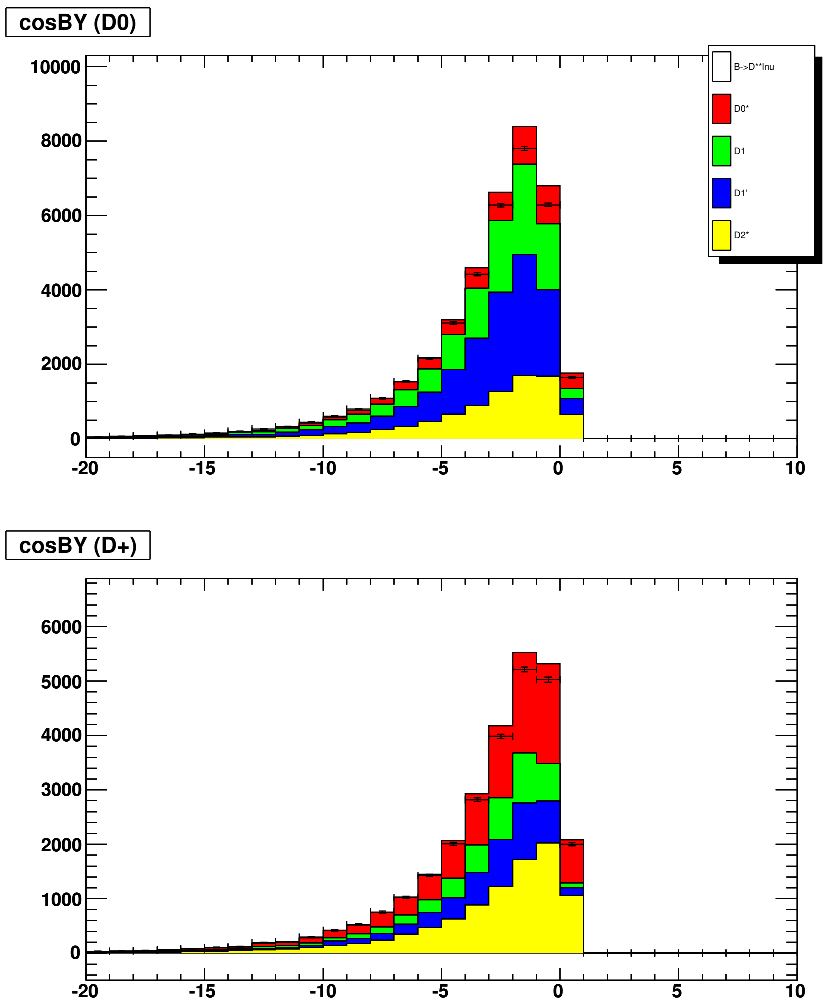


Figure 45: [$\cos\theta_{B-Dl}$ **after** $B \to D^{**}\ell\nu$ **re-weighting**] $\cos\theta_{B-Dl}$ of $B \to D^{**}\ell\nu$ decays. Black line is ISGW2 model and colored histogram is HQET model. Red is D_0^*, green is D_1, blue is D_1' and yellow is D_2^*.

10.6 Effect of FF re-weighting

Figure 46 - 48 show the lepton momentum, D momentum and $\cos\theta_{B-Dl}$ after FF re-weighting. To make these plots, we used

$$\begin{aligned} \rho_D^2 &= 1.17 \\ R_1 &= 1.417 \\ R_2 &= 0.836 \\ \rho^2 &= 1.179 \end{aligned} \tag{284}$$

with CLN parametrization and Approximation B1 as described above. We also used the B decay branching fraction given in section 5.2. These are the projections of 3D histograms. If you compare these with figure 21 - 23, you can see improvements of data-MC agreement.

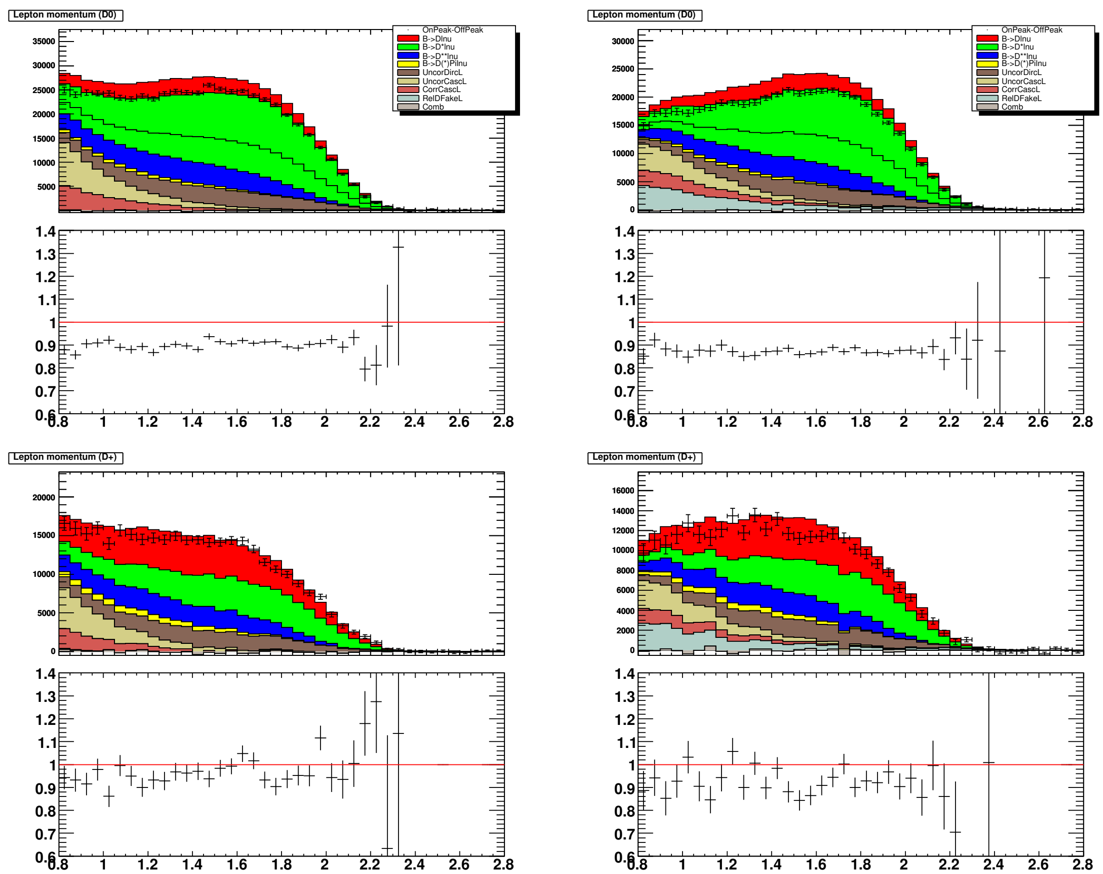


Figure 46: [p_ℓ **after FF re-weighting (Electron vs muon)**] Lepton momentum plot after form factor re-weighting and D mass sideband subtraction in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

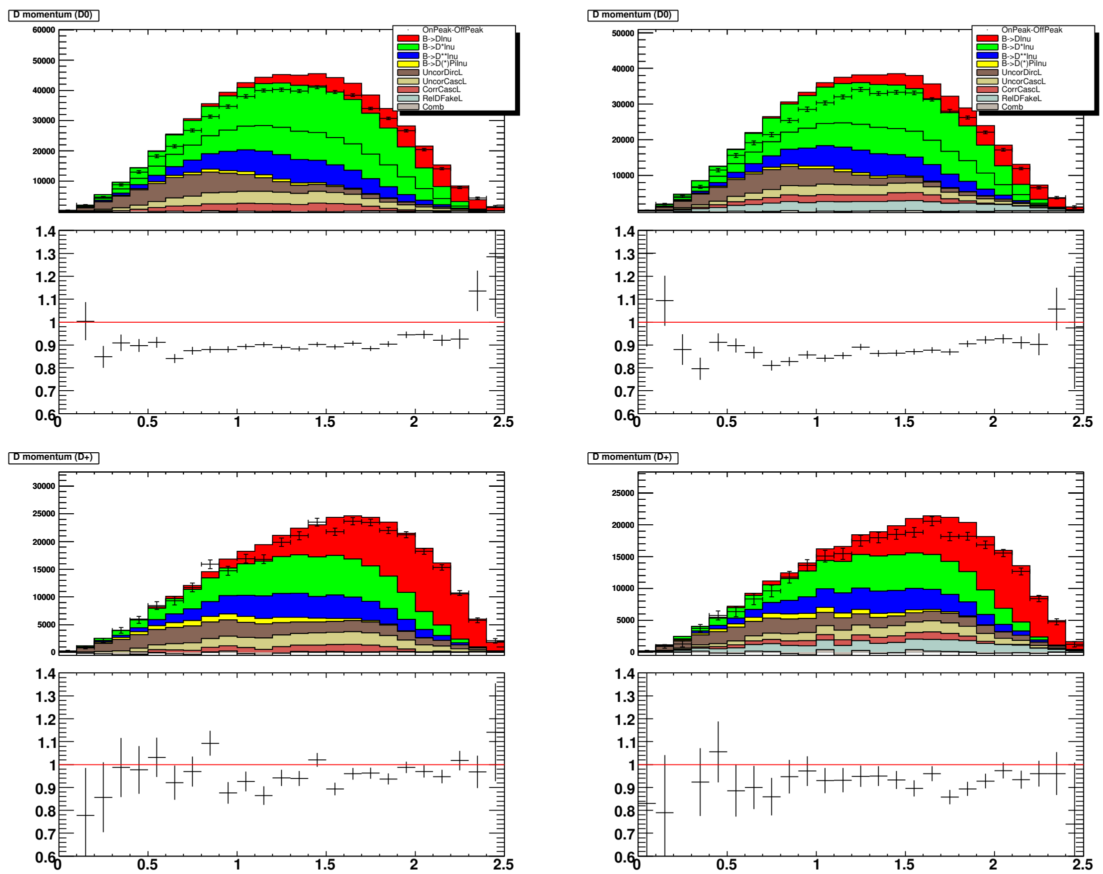


Figure 47: [p_D **after FF re-weighting (Electron vs muon)**] D momentum plot after form factor re-weighting and D mass sideband subtraction in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

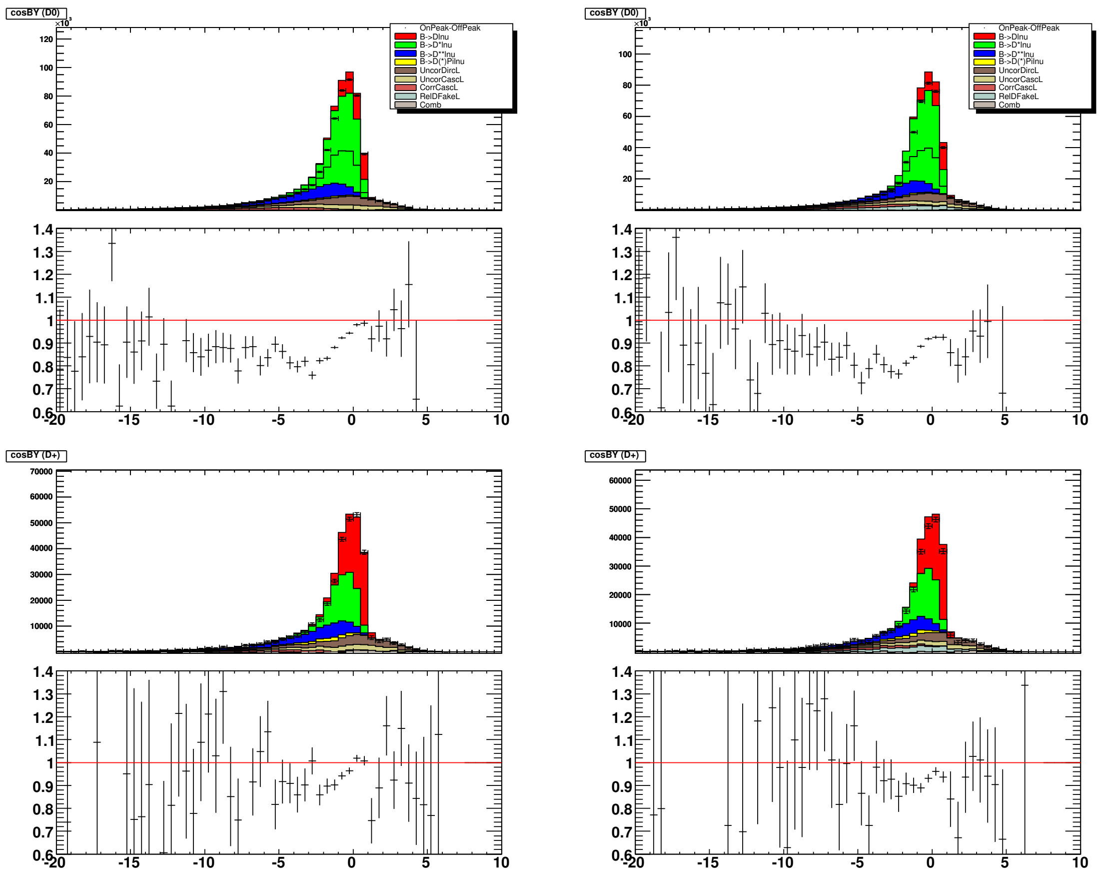


Figure 48: [$\cos\theta_{B-Dl}$ **after FF re-weighting (Electron vs muon)**] $\cos\theta_{B-Dl}$ plot after form factor re-weighting and D mass sideband subtraction in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

11 FF parameter fitting technique

11.1 $B \to D\ell\nu$ slope fitting

To fit for $B \to D\ell\nu$ slope, it is not computationally efficient to go through the calculation of the weight event by event. The following is a better way to do fitting.

11.1.1 Number of events

We may rewrite the weight (Equation (220))

$$W_{jk}^{MC}(w_k) = \left[\frac{[h_+(1)]^2}{[Rf_+]^2} - \frac{2(w-1)[h_+(1)]^2}{[Rf_+]^2}\rho_D^2 + \frac{(w-1)^2[h_+(1)]^2}{[Rf_+]^2}\left(\rho_D^2\right)^2\right] R_N \quad (285)$$

Hence, we define

$$W_{jk}^{MC}(w_k) \equiv \left[W_{jk}^0(w_k) - W_{jk}^1(w_k)\rho_D^2 + W_{jk}^2(w_k)\left(\rho_D^2\right)^2\right] R_N \quad (286)$$

Then, the number of events is

$$\begin{aligned} N_{ij}^{MC} = \Bigg[& \sum_l \left(r_l^{Lumi} \sum_{k\in i,j,l} r_k^{Trk} r_k^{KPID} r_k^{LPID} W_{jk}^0 \right) \\ & - \sum_l \left(r_l^{Lumi} \sum_{k\in i,j,l} r_k^{Trk} r_k^{KPID} r_k^{LPID} W_{jk}^1 \right) \rho_D^2 \\ & + \sum_l \left(r_l^{Lumi} \sum_{k\in i,j,l} r_k^{Trk} r_k^{KPID} r_k^{LPID} W_{jk}^2 \right) \left(\rho_D^2\right)^2 \Bigg] R_N \end{aligned} \quad (287)$$

Hence

$$N_{ij}^{MC} \equiv \left[N_{ij}^{MC0} - N_{ij}^{MC1}\rho_D^2 + N_{ij}^{MC2}\left(\rho_D^2\right)^2\right] R_N \quad (288)$$

Instead of going through event by event in fitting, first we create these three histograms (N_{ij}^{MC0}, N_{ij}^{MC1} and N_{ij}^{MC2}), and then use those histograms in fitting.
In another parametrization (Equation (222)).

$$\begin{aligned} & 1 - 8\rho_D^2 z + (51\rho_D^2 - 10)z^2 - (252\rho_D^2 - 84)z^3 \\ & = (1 - 10z^2 + 84z^3) - (8z - 51z^2 + 252z^3)\rho_D^2 \end{aligned} \quad (289)$$

Thus, by replacing

$$\begin{aligned} & 1 \to (1 - 10z^2 + 84z^3) \\ & (w-1) \to (8z - 51z^2 + 252z^3) \end{aligned} \quad (290)$$

we can use the same method

11.1.2 Weighted events and statistical uncertainty

Sine our events are weighted, calculation of statistical uncertainty is a little bit tricky. Suppose the weight of the k-th event or candidate is W_k and it is given by

$$W_k = F_{1k} - F_{2k}\rho_D^2 + F_{3k}\left(\rho_D^2\right)^2 \tag{291}$$

The number of events after D mass sideband subtraction is given by

$$\begin{aligned}\sum_k^{peak} W_k - \sum_k^{sideband} W_k = &\left(\sum_k^{peak} F_{1k} - \sum_k^{sideband} F_{1k}\right) \\ &- \left(\sum_k^{peak} F_{2k} - \sum_k^{sideband} F_{2k}\right)\rho_D^2 + \left(\sum_k^{peak} F_{3k} - \sum_k^{sideband} F_{3k}\right)\left(\rho_D^2\right)^2\end{aligned} \tag{292}$$

Thus, the number of events is still given by

$$N = \left[N_1 - N_2\rho_D^2 + N_3\left(\rho_D^2\right)^2\right] R_N \tag{293}$$

The square of statistical uncertainty is calculated by

$$\begin{aligned}\sum_k^{peak} W_k^2 + \sum_k^{sideband} W_k^2 = &\sum_k^{peak}\left(F_{1k} - F_{2k}\rho_D^2 + F_{3k}\left(\rho_D^2\right)^2\right)^2 \\ &+ \sum_k^{sideband}\left(F_{1k} - F_{2k}\rho_D^2 + F_{3k}\left(\rho_D^2\right)^2\right)^2\end{aligned} \tag{294}$$

Thus, no need to sum peak and sideband separately

$$\begin{aligned}\sum_k W_k^2 &= \sum_k \left(F_{1k} - F_{2k}\rho_D^2 + F_{3k}\left(\rho_D^2\right)^2\right)^2 \\ &= \sum_k F_{1k}^2 - \sum_k 2F_{1k}F_{2k}\rho_D^2 + \sum_k \left(2F_{1k}F_{3k} + F_{2k}^2\right)\left(\rho_D^2\right)^2 \\ &\quad - \sum_k 2F_{2k}F_{3k}\left(\rho_D^2\right)^3 + \sum_k F_{3k}^2\left(\rho_D^2\right)^4\end{aligned} \tag{295}$$

Hence, we need 5 histograms to calculate the square of statistical uncertainty

$$\sigma^2 = \left[N_1 - N_2\rho_D^2 + N_3\left(\rho_D^2\right)^2 - N_4\left(\rho_D^2\right)^3 + N_5\left(\rho_D^2\right)^4\right] R_N^2 \tag{296}$$

Note that in the above argumant, the sideband scale factor introduced in section 6 Equation (171), is included in the weight W_k (F_{nk}).

11.2 $B \to D^*\ell\nu$ parameter fitting

We may rewrite the $\mathcal{J}_{FF}(R_1, R_2, \rho^2)$. From equation (233),

$$
\begin{aligned}
&\mathcal{J}_{FF}(R_1, R_2, \rho^2) \\
&= [F_1 - F_2\rho^2 + F_3(\rho^2)^2]\,[G_1 + G_2R_1 + G_3R_2 + G_4R_1^2 + G_5R_2^2 + G_6R_1R_2] \\
&= F_1G_1 - F_2G_1\rho^2 + F_3G_1(\rho^2)^2 \\
&\quad +F_1G_2R_1 - F_2G_2R_1\rho^2 + F_3G_2R_1(\rho^2)^2 \\
&\quad +F_1G_3R_2 - F_2G_3R_2\rho^2 + F_3G_3R_2(\rho^2)^2 \\
&\quad +F_1G_4R_1^2 - F_2G_4R_1^2\rho^2 + F_3G_4R_1^2(\rho^2)^2 \\
&\quad +F_1G_5R_2^2 - F_2G_5R_2^2\rho^2 + F_3G_5R_2^2(\rho^2)^2 \\
&\quad +F_1G_6R_1R_2 - F_2G_6R_1R_2\rho^2 + F_3G_6R_1R_2(\rho^2)^2
\end{aligned}
\tag{297}
$$

where, using the CLN parametrization of the slope,

$$
\begin{aligned}
F_1 &= (1-15z^2+91z^3)^2 \\
F_2 &= 2(1-15z^2+91z^3)(8z-53z^2+231z^3) \\
F_3 &= (8z-53z^2+231z^3)^2 \\
G_6 &= (-4)\sqrt{1-c_L^2}(1-c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(w-1)\sqrt{\tfrac{w-1}{w+1}} \\
&\quad +4\sqrt{1-c_L^2}(1+c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(-1)(w-1)\sqrt{\tfrac{w-1}{w+1}} \\
G_4 &= (1-c_L)^2(1-c_V^2)(1-2wr+r^2)\tfrac{w-1}{w+1} \\
&\quad +(1+c_L)^2(1-c_V^2)(1-2wr+r^2)\tfrac{w-1}{w+1} \\
&\quad +(-2)(1-c_L^2)(1-c_V^2)(2c_\chi^2-1)(1-2wr+r^2)(-1)\tfrac{w-1}{w+1} \\
G_2 &= (1-c_L)^2(1-c_V^2)(1-2wr+r^2)(-2)\sqrt{\tfrac{w-1}{w+1}} \\
&\quad +(1+c_L)^2(1-c_V^2)(1-2wr+r^2)(+2)\sqrt{\tfrac{w-1}{w+1}} \\
&\quad +(-4)\sqrt{1-c_L^2}(1-c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(-1)(w-r)\sqrt{\tfrac{w-1}{w+1}} \\
&\quad +4\sqrt{1-c_L^2}(1+c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(w-r)\sqrt{\tfrac{w-1}{w+1}} \\
&\quad +2G_4A_1+G_6A_2 \\
G_5 &= 4(1-c_L^2)c_V^2(w-1)^2 \\
G_3 &= 4(1-c_L^2)c_V^2(-2)(w-r)(w-1) \\
&\quad +(-4)\sqrt{1-c_L^2}(1-c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(-1)(w-1) \\
&\quad +4\sqrt{1-c_L^2}(1+c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(-1)(w-1) \\
&\quad +2G_5A_2+G_6A_1 \\
G_1 &= (1-c_L)^2(1-c_V^2)(1-2wr+r^2) \\
&\quad +(1+c_L)^2(1-c_V^2)(1-2wr+r^2) \\
&\quad +4(1-c_L^2)c_V^2(w-r)^2 \\
&\quad +(-2)(1-c_L^2)(1-c_V^2)(2c_\chi^2-1)(1-2wr+r^2) \\
&\quad +(-4)\sqrt{1-c_L^2}(1-c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(w-r) \\
&\quad +4\sqrt{1-c_L^2}(1+c_L)\sqrt{1-c_V^2}c_Vc_\chi\sqrt{1-2wr+r^2}(w-r) \\
&\quad +(G_2-2G_4A_1-G_6A_2)A_1-(G_3-2G_5A_2-G_6A_1)A_2+G_4A_1^2+G_5A_2^2+G_6A_1A_2
\end{aligned}
\tag{298}
$$

where

$$
\begin{aligned}
A_1 &= -0.12(w-1)+0.05(w-1)^2 \\
A_2 &= 0.11(w-1)-0.06(w-1)^2
\end{aligned}
\tag{299}
$$

Thus, same as previous section, we can, first, make 18 histograms corresponding to the 18 terms in the above $\mathcal{J}_{FF}(R_1, R_2, \rho^2)$. Then we can calculate the number of events N

$$
\begin{aligned}
N = [&N_1 - N_2\rho^2 + N_3(\rho^2)^2 \\
&+N_4R_1 - N_5R_1\rho^2 + N_6R_1(\rho^2)^2 \\
&+N_7R_2 - N_8R_2\rho^2 + N_9R_2(\rho^2)^2 \\
&+N_{10}R_1^2 - N_{11}R_1^2\rho^2 + N_{12}R_1^2(\rho^2)^2 \\
&+N_{13}R_2^2 - N_{14}R_2^2\rho^2 + N_{15}R_2^2(\rho^2)^2 \\
&+N_{16}R_1R_2 - N_{17}R_1R_2\rho^2 + N_{18}R_1R_2(\rho^2)^2]\, R_N
\end{aligned}
\tag{300}
$$

The square of statistical uncertainty can also be calculated in the same way.

$$
\begin{aligned}
&[F_1 - F_2\rho^2 + F_3(\rho^2)^2]^2\,[G_1 + G_2R_1 + G_3R_2 + G_4R_1^2 + G_5R_2^2 + G_6R_1R_2]^2 \\
&= [F_1^2 - 2F_1F_2\rho^2 + (2F_1F_3 + F_2^2)(\rho^2)^2 - 2F_2F_3(\rho^2)^3 + F_3^2(\rho^2)^4] \\
&\quad [G_1^2 + 2G_1G_2R_1 + 2G_1G_3R_2 \\
&\quad +(2G_1G_4 + G_2^2)R_1^2 + (2G_1G_5 + G_3^2)R_2^2 + 2(G_1G_6 + G_2G_3)R_1R_2 \\
&\quad +2G_2G_4R_1^3 + 2G_3G_5R_2^3 + 2(G_2G_6 + G_3G_4)R_1^2R_2 + 2(G_2G_5 + G_3G_6)R_1R_2^2 \\
&\quad +G_4^2R_1^4 + G_5^2R_2^4 + 2G_4G_6R_1^3R_2 + 2G_5G_6R_1R_2^3 + (2G_4G_5 + G_6^2)R_1^2R_2^2]
\end{aligned}
\tag{301}
$$

Thus, we need $5 \times 15 = 75$ histograms to calculate the statistical uncertainty.

12 Test Fit

12.1 Binning

We use the following binning :

- lepton momentum in the CM frame p^*_ℓ
  (6 bins (GeV)) : 0.8, 1.05, 1.3, 1.55, 1.8, 2.05, 2.3
- D momentum in the CM frame p^*_D
  (5 bins (GeV)) : 0, 0.5, 1, 1.5, 2.0, 2.5
- $\cos\theta_{B-Dl}$ (see below)
  (5 bins) : -20, -5.5, -2, 0, 2, 10

The binning is decided in the valance of two opposing requirements :

- It needs to be as fine as possible to preserve the shape of original distribution.
- It needs to be as coarse as necessary to have a good statistics per bin.

The binning of p^*_ℓ is finer to get finer lepton spectrum because lepton moentum spectrum is well modeled in MC.

12.2 Summary of the fitting configuration

In this fit, we are trying to fit for three branching fractions and $\Upsilon(4S)$ decay BF ratio

- $\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$
- $\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)$
- f_{+0}

with isospin constraints

- $\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^-\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}_0\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^{*-}_0\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^0_1\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^-_1\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}_1\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D'^-_1\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}_2\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^{*-}_2\ell^+\nu)$
- $\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)$
  $= t_{+0}\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)$

- $\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)$
  $= t_{+0}\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)$

and constraints on relative branching fractions

- $f_{D_2^*/D_1} \equiv \frac{\mathcal{B}(B^+\to\bar{D}_2^{*0}\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^0\ell^+\nu)} = 0.78 \pm 0.17$
- $f_{D_0^* D\pi T/D_1 D_2^* T} \equiv \frac{\mathcal{BT}(B^+\to\bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 0.77 \pm 0.19$
- $f_{D_1' D^*\pi T/D_1 D_2^* T} \equiv \frac{\mathcal{BT}(B^+\to\bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 0.98 \pm 0.24$
- $f_{D\pi/D_0^*} \equiv \frac{\mathcal{B}(B^+\to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_0^{*0}\ell^+\nu)} = 0.42 \pm 0.28$
- $f_{D^*\pi/D_1'} \equiv \frac{\mathcal{B}(B^+\to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^{*0}\ell^+\nu)} = 0.071 \pm 0.050$

and by floating background components

- $C^{D^0}_{UncorDL}$
- $C^{D^0}_{UncorCL}$
- $C^{D^0}_{CascL}$
- $C^{D^+}_{UncorDL}$
- $C^{D^+}_{UncorCL}$
- $C^{D^+}_{CascL}$

but fixing background components

- $C^{D^0}_{FakeL} = 1$
- $C^{D^+}_{FakeL} = 1$

In addition, we are fitting for form factor parameters

- $B \to D\ell\nu$ decay FF slope : ρ_D^2
- $B \to D^*\ell\nu$ decay FF slope : ρ^2
- $B \to D^*\ell\nu$ decay FF ratio : R_1
- $B \to D^*\ell\nu$ decay FF ratio : R_2

by fixing

- $B \to D^{**}\ell\nu$ decay FF slope : $\hat{\tau}' = -1.5 \pm 0.5$
- Approximation B1 : $\tau_1 = 0, \tau_2 = 0$
  for systematics, Approximation B2 : $\tau_1 = \bar{\Lambda}\tau, \tau_2 = -\bar{\Lambda}'\tau$
  (see details of $B \to D^{**}\ell\nu$ decay FF in Appendix I)

Parameters	f_{+0} floated	f_{+0} fixed
ρ_D^2	1.239 ± 0.042 (3.3 %)	1.225 ± 0.042 (3.4 %)
ρ^2	1.316 ± 0.038 (2.9 %)	1.311 ± 0.039 (2.9 %)
R_1	1.408 ± 0.055 (3.9 %)	1.402 ± 0.055 (3.9 %)
R_2	0.764 ± 0.048 (6.3 %)	0.763 ± 0.049 (6.4 %)
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	0.02383 ± 0.00033 (1.4 %)	0.02360 ± 0.00029 (1.2 %)
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	0.05246 ± 0.00045 (0.87 %)	0.05284 ± 0.00037 (0.70 %)
$\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)$	0.02039 ± 0.00043 (2.1 %)	0.02025 ± 0.00043 (2.1 %)
f_{+0}	1.060 ± 0.016 (1.5 %)	1.037 (Fixed)
$C^{D^0}_{UncorDL}$	0.763 ± 0.034 (4.4 %)	0.762 ± 0.034 (4.5 %)
$C^{D^0}_{UncorCL}$	0.830 ± 0.035 (4.2 %)	0.838 ± 0.035 (4.2 %)
$C^{D^0}_{CascL}$	0.950 ± 0.081 (8.5 %)	0.952 ± 0.051 (8.5 %)
$C^{D^+}_{UncorDL}$	0.802 ± 0.105 (13.1 %)	0.786 ± 0.111 (14.1 %)
$C^{D^+}_{UncorCL}$	0.756 ± 0.072 (9.6 %)	0.740 ± 0.071 (9.7 %)
$C^{D^+}_{CascL}$	0.939 ± 0.130 (13.8 %)	0.928 ± 0.148 (16.0 %)
χ^2/ndof (P-value)	407/223 (5.0×10^{-14})	409/224 (4.2×10^{-14})

Table 23: [**Test fit results**] Test fit results.

12.3 Fit results

Fit results are shown in Table 23. Though we did a lot of corrections and re-weightings, χ^2 (P-value) is terrybly bad. We think this is still due to poor MC description of $B \to D^{**}\ell\nu$ and $B \to D^{(*)}\pi\ell\nu$ decays and backgrounds. Projected plots after fitting is shown in Figures 49-51. Significant amount of those components can be seen in these plots. In order to reduce data-MC disagreement, we try to reduce these components. Details are explained in the next section.

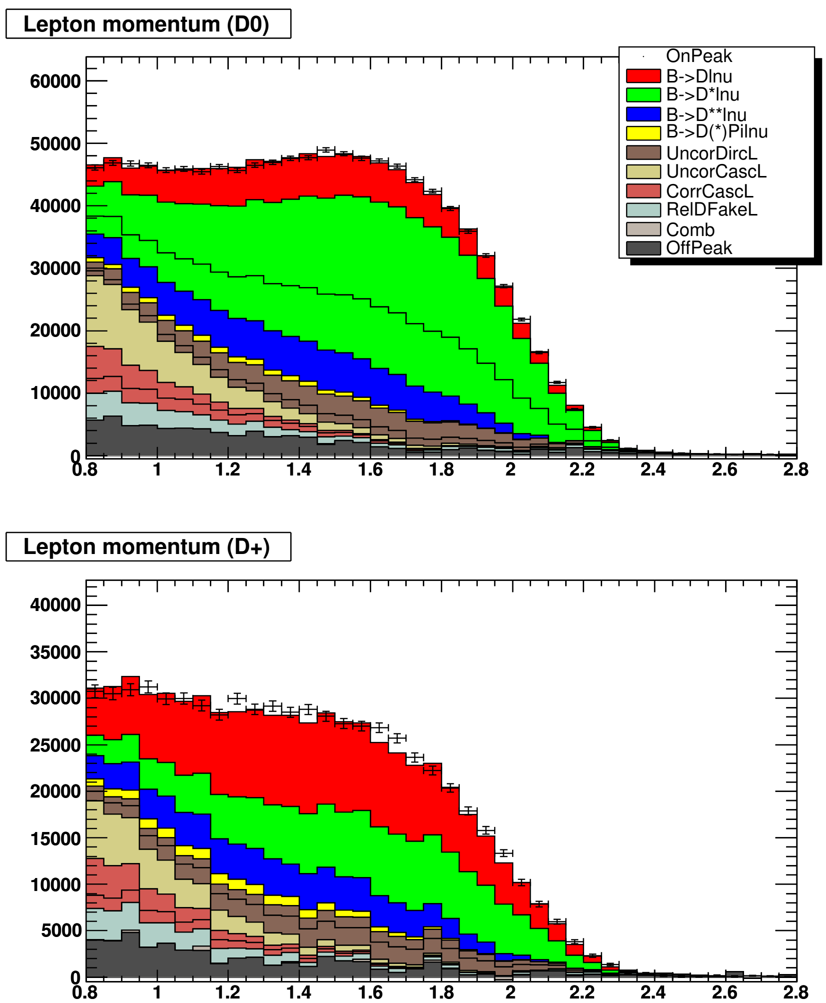


Figure 49: [p_ℓ **after fitting**] Lepton momentum plot after fitting. The top is D^0 and the bottom is D^+. Black is OnPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, light gray is Comb and dark gray is OffPeak. Comb is almost disappeared after D mass sideband subtraction.

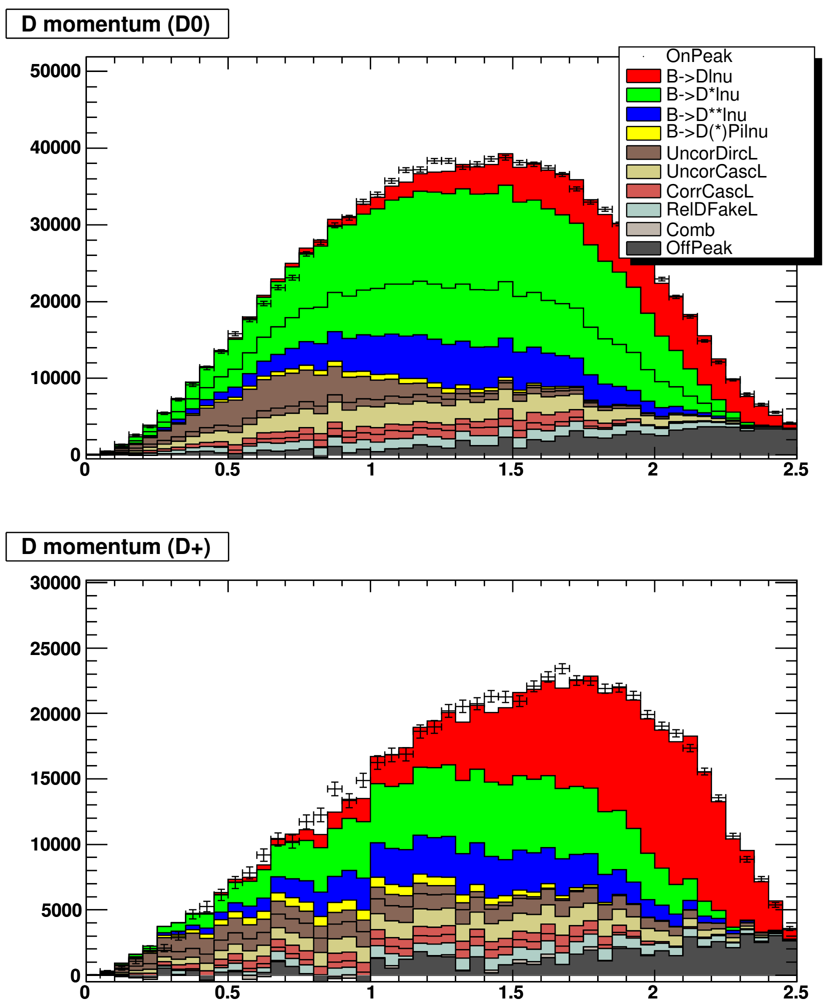


Figure 50: [p_D **after fitting**] D momentum plot after fitting. The top is D^0 and the bottom is D^+. Black is OnPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, light gray is Comb and dark gray is OffPeak. Comb is almost disappeared after D mass sideband subtraction.

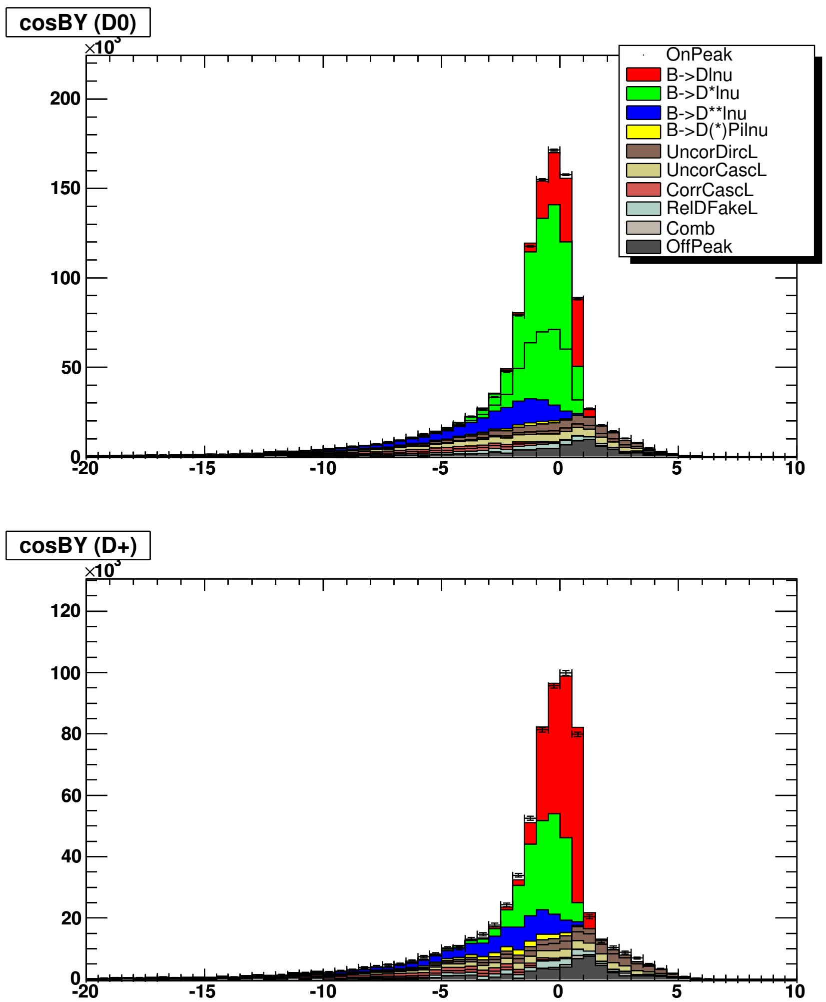


Figure 51: [$\cos\theta_{B-Dl}$ **after fitting**] $\cos\theta_{B-Dl}$ plot after fitting. The top is D^0 and the bottom is D^+. Black is OnPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, light gray is Com and dark gray is OffPeakb. Comb is almost disappeared after D mass sideband subtraction.

13 Kinematic cuts to improve χ^2

From this section we rename $\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)$ to $\mathcal{B}(B^+ \to \bar{D}^{(*)}\pi\ell^+\nu)$

13.1 lepton momentum, D momentum and $\cos\theta_{B-Dl}$ cuts

To improve χ^2, we apply lepton momentum, D momentum and $\cos\theta_{B-Dl}$ cuts. After many times of test fits we decided to apply cuts

- $p^*_\ell > 1.2$ GeV
- $p^*_D > 0.8$ GeV
- $-2 < \cos\theta_{B-Dl} < 1.1$

to get an acceptable χ^2 (or P-value). Upper end of $\cos\theta_{B-Dl}$ is set to 1.1 to avoid the sharpe edge at $\cos\theta_{B-Dl} = 1$. We also decided to use the following binning :

- lepton momentum in the CM frame p^*_ℓ
  (10 bins (GeV)) : 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.35
- D momentum in the CM frame p^*_D
  (9 bins (GeV)) : 0.8, 1.1, 1.35, 1.5, 1.65, 1.8, 1.95, 2.1, 2.25, 2.45
- $\cos\theta_{B-Dl}$
  (3 bins) : -2, -1, 0, 1.1

The partial rates pass this cuts were estimated in generator level. The ratio (partial rates)/(full rates) are 60.66 ± 0.07 % for $B \to D\ell\nu$ and 61.50 ± 0.07 % for $B \to D^*\ell\nu$. Since most of the background components are removed by the above cuts, it is meaningless to float background coefficienct s. Thus, we

- Fix all background coefficients to 1.

We also decided to

- Fix $f_{+0} = 1.037 \pm 0.029$

after many test fits. There has been missing signal BF compare to inclusive BF. Thus we

- Include $B \to D\pi\pi\ell\nu$ component

in our fit. Details of this component is explained in section 14.
Fit result with above configuration is given in Table 24. In this fit we fixed $\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell\nu) = 0.0070$ as a trial.

ρ_D^2	1.160 ± 0.038 (3.2 %)
ρ^2	1.350 ± 0.042 (3.0 %)
R_1	1.546 ± 0.065 (4.2 %)
R_2	0.635 ± 0.055 (8.6 %)
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	0.02332 ± 0.00025 (1.1 %)
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	0.05481 ± 0.00052 (0.9 %)
$\mathcal{B}(B^+ \to D^{(*)}\pi\ell\nu)$	0.01429 ± 0.00097 (6.8 %)
$\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell\nu)$	0.0070 (Fixed)
χ^2/ndof (P-value)	522/475 (0.067)

Table 24: [**Test fit result with kinematic cuts**] Fit results.

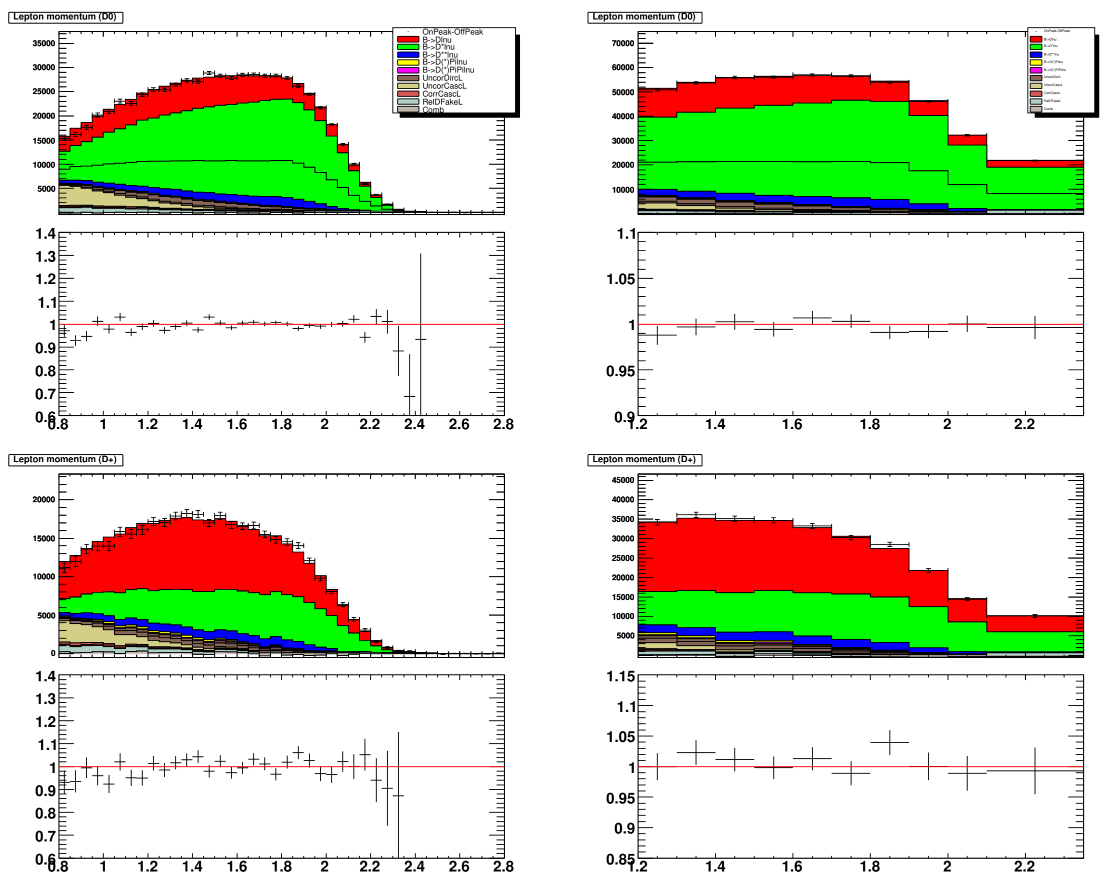


Figure 52: [p_ℓ **after fitting with kinematic cuts**] Lepton momentum plot after fitting. The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

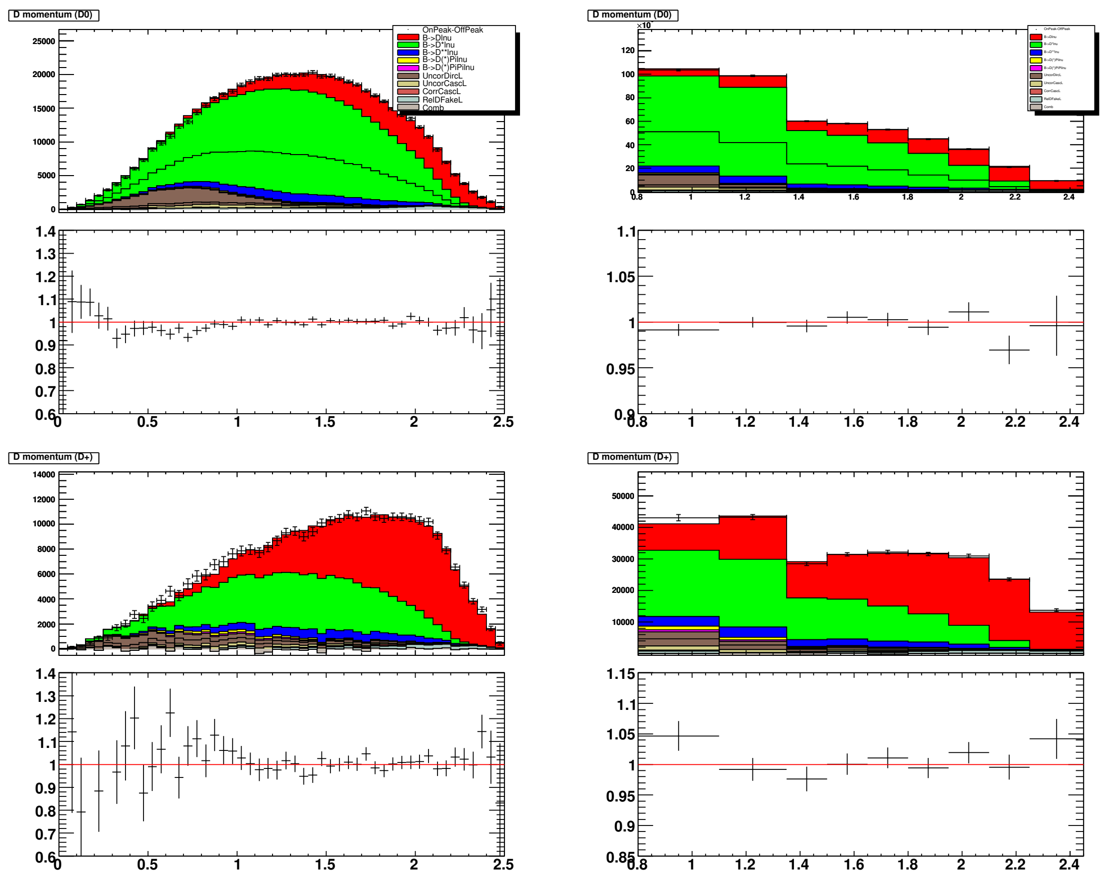


Figure 53: [p_D **after fitting with kinematic cuts**] D momentum plot after fitting. The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

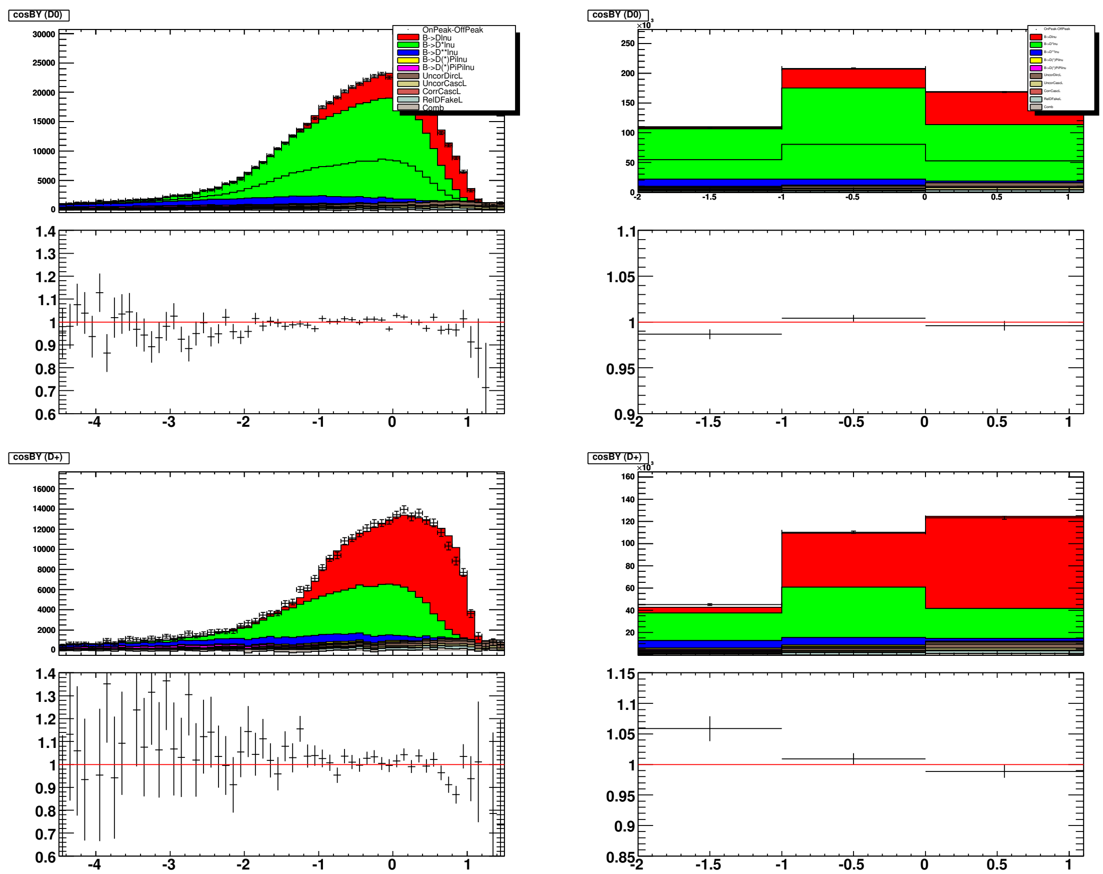


Figure 54: [$\cos\theta_{B-Dl}$ **after fitting with kinematic cuts**] $\cos\theta_{B-Dl}$ plot after fitting. The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

Parameters	Electron result	Muon result	NR
ρ_D^2	1.234 ± 0.045	1.089 ± 0.063	-1.75
ρ^2	1.351 ± 0.053	1.363 ± 0.065	+0.14
R_1	1.573 ± 0.084	1.560 ± 0.103	-0.10
R_2	0.632 ± 0.070	0.607 ± 0.087	-0.22
$\mathcal{B}(B \to D\ell\nu)$	0.02396 ± 0.00032	0.02283 ± 0.00037	-2.30
$\mathcal{B}(B \to D^*\ell\nu)$	0.05609 ± 0.00070	0.05376 ± 0.00076	-2.24
$\mathcal{B}(B \to D^{(*)}\pi\ell\nu)$	0.01302 ± 0.00134	0.01432 ± 0.00138	+0.68
χ^2/ndof (P-value)	467/469 (0.52)	484/466 (0.27)	13.9/7 (0.053)

Table 25: [**Electron vs Muon with kinematic cuts**] Fit results and normalized residuals (NR).

13.2 Elecron vs Muon

We fit electron and muon separately to see the their consistency. Result is shown in Table 25. The normalized residuals (NR) is defined by

$$\mathrm{NR} = \frac{(\text{Muon fitted value}) - (\text{Elctron fitted value})}{\sqrt{\sigma_e^2 + \sigma_\mu^2}} \tag{302}$$

The sum of the square of NR is given at the bottom of the Table. This correspont to a overall χ^2 of electron - muon difference. Overall eelectron - muon agreement is not bad but not rally good as well. Thus we decided to look at electron and muon separately.

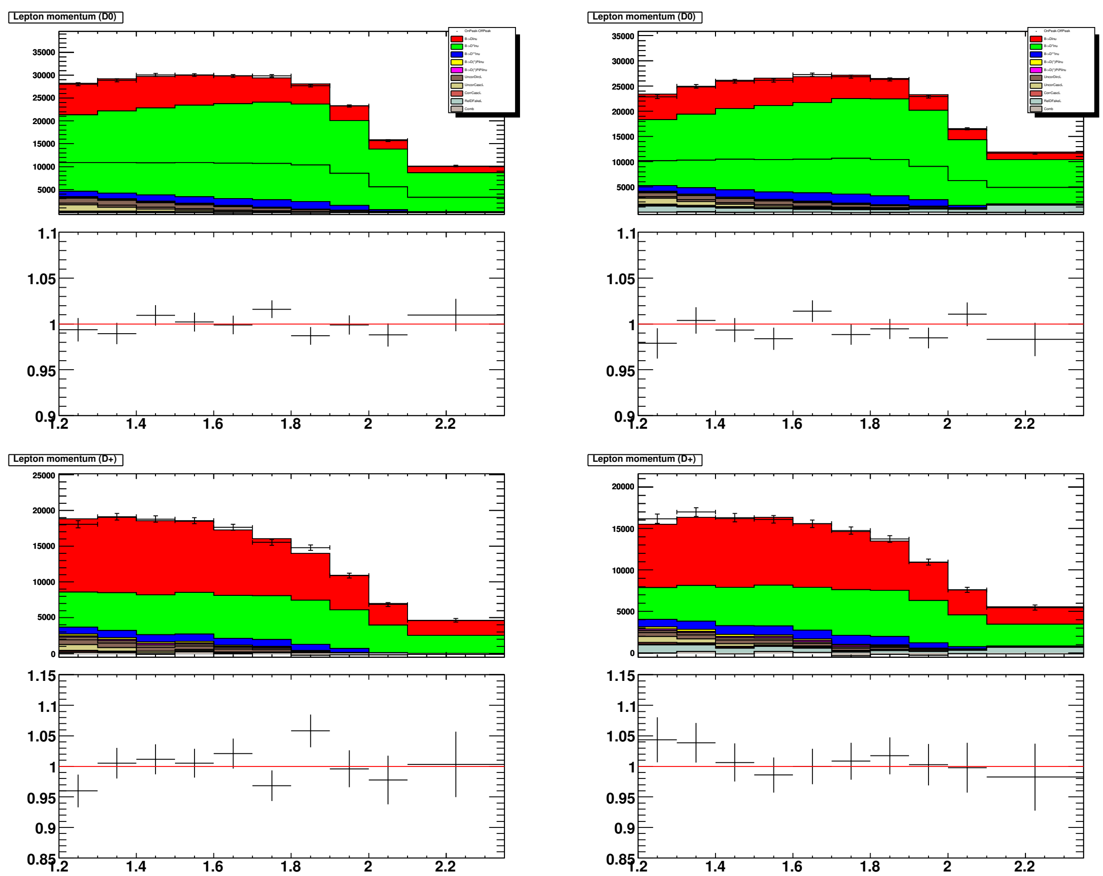


Figure 55: [p_ℓ **with kinematic cuts (Electron vs muon)**] Lepton momentum plot with fitted binning after fitting in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

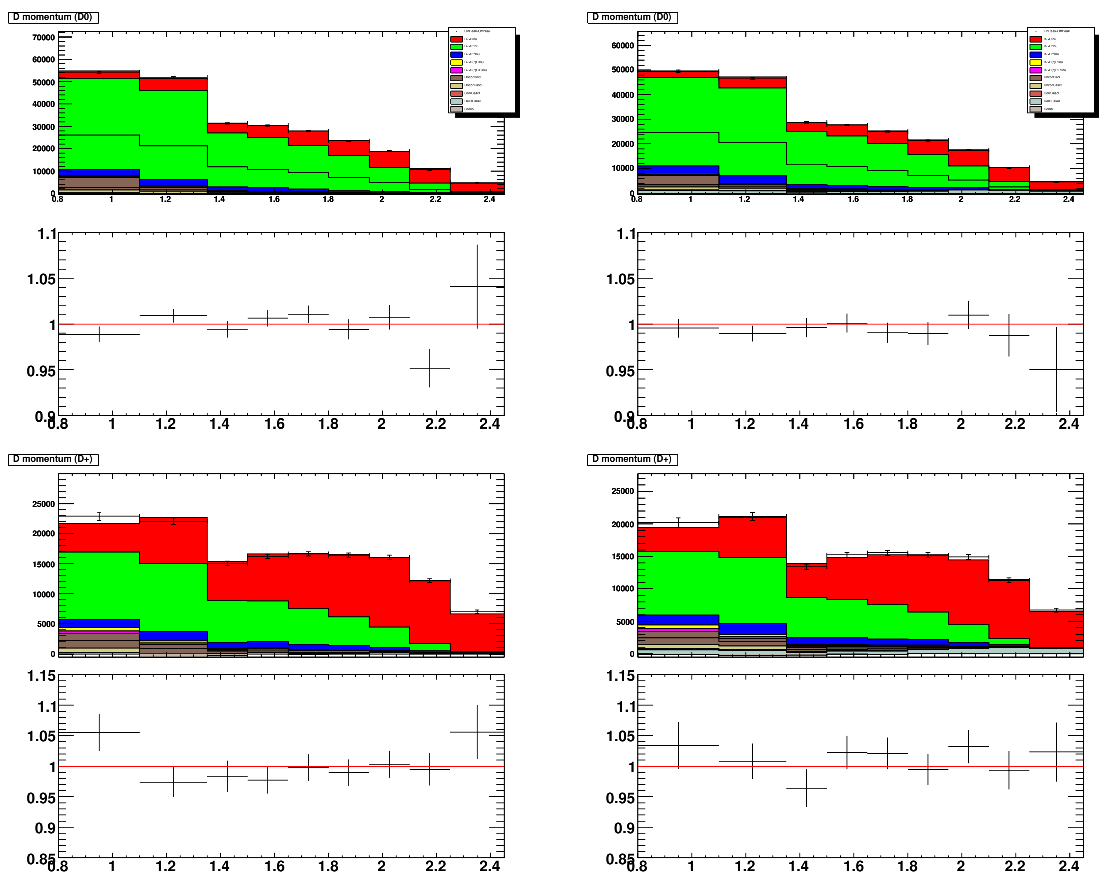


Figure 56: [p_D **with kinematic cuts (Electron vs muon)**] D momentum plot with fitted binning after fitting in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

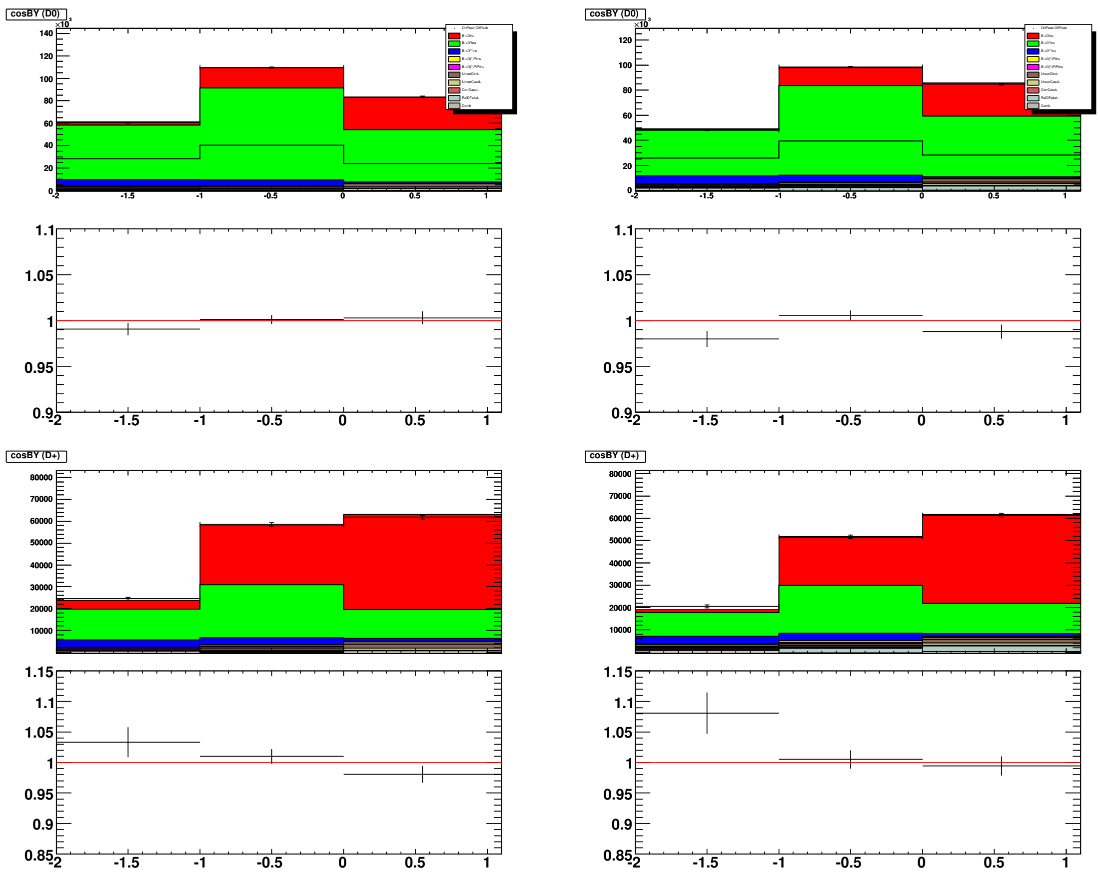


Figure 57: [$\cos\theta_{B-Dl}$ **with kinematic cuts (Electron vs muon)**] $\cos\theta_{B-Dl}$ plot with fitted binning after fitting in the case $\ell = e$ (Electron only) (left) and $\ell = \mu$ (Muon only) (right). The top is D^0 and the bottom is D^+. Black is OnPeak - OffPeak data. Red is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{**}\ell\nu$, yellow is non-resonant $B \to D^{(*)}\pi\ell\nu$, and the rest is background. Dark brown is UncorrDL, light brown is UncorrCL, dark red is CascL, blue gray is FakeL, and light gray is Comb. Comb is almost disappeared after D mass sideband subtraction.

14 $B \to D^{(*)}\pi\pi\ell\nu$ decays

Inclusive branching frcation (BF) $\mathcal{B}(B \to X_c\ell\nu$ is well measured. For exculsive BF, $\mathcal{B}(B \to D\ell\nu)$, $\mathcal{B}(B \to D^*\ell\nu)$ and $\mathcal{B}(B \to D^(*)\pi\ell\nu)$ are mesured. However, these three exclusive BF does not add up to inclusive BF. The most probable candidate of the missing component is $B \to D^{(*)}\pi\pi\ell\nu$ decays.

14.1 Event generation

The generic $B\bar{B}$ samples do not contain decays of the type $B \to X_c\ell\nu$ where X_c is a charm system that produces a $D^{(*)}$ and two pions. These decays must be modeled separately using EVTGEN. The X_c systems studied cover a range of spin states and decay mechanisms. They are listed in Table 26.

- The decay $B \to X_c\ell\nu$ is performed using the HQET matrix element, i.e. the X_c system is treated as a heavier D or D^* according to the spin. The mass and spin of the X_c state therefore determine the lepton momentum spectrum. The D momentum and the $\cos\theta_{D\ell}$ depend on the decay dynamics as well.
- The Xc states are decayed into D or D* (with equal branching fractions) and two pions according to phase space; the Xc* states are decayed the same way.
- The Yc and Yc* states, on the other hand, are decayed as two-body $D^{(*)}\rho$ states. The Yc(*) mass was chosen to be just above threshold for this decay.
- These various choices of spin, mass and decay mode serve as a reasonable sample of the potential $B \to D^{(*)}\pi\pi$ decay modes.

14.2 3D efficiency histogram

We can make 3D histograms with generated events, but we cannot use these histograms in our fit. We need to convert the histograms to reconstructed histograms. We make this realiize by the following steps

- Generate $B \to D\ell\nu$, $B \to D^*\ell\nu$ and $B \to D^{(*)}\pi\ell\nu$ events as was done in SP5/SP6.
- Create generated 3D signal histograms from the above events. We make 4 signal histograms :
  - $B^+ \to XD^0(\mathrm{or}\bar{D}^0)\ell\nu$
  - $B^+ \to XD^+(\mathrm{or}D^-)\ell\nu$
  - $B^0 \to XD^0(\mathrm{or}\bar{D}^0)\ell\nu$
  - $B^0 \to XD^0(\mathrm{or}D^-)\ell\nu$
- Make combined reconstructed signal histograms from existing histograms.

name	mass	width	$\max_D m$	charge	spin
Xc0	2.61	0.3	0.3	0	0
anti-Xc0	2.61	0.3	0.3	0	0
Xc+	2.61	0.3	0.3	+1	0
Xc-	2.61	0.3	0.3	-1	0
Xc*0	2.61	0.3	0.3	0	1
anti-Xc*0	2.61	0.3	0.3	0	1
Xc*+	2.61	0.3	0.3	+1	1
Xc*-	2.61	0.3	0.3	-1	1
Yc0	2.87	0.1	0.1	0	0
anti-Yc0	2.87	0.1	0.1	0	0
Yc+	2.87	0.1	0.1	+1	0
Yc-	2.87	0.1	0.1	-1	0
Yc*0	2.87	0.1	0.1	0	1
anti-Yc*0	2.87	0.1	0.1	0	1
Yc*+	2.87	0.1	0.1	+1	1
Yc*-	2.87	0.1	0.1	-1	1

Table 26: [X_c **systems**] X_c systems studied for systematic study.

- Make **3D efficiency histograms** by deviding reconstructed histograms by generated histograms.

- Apply the 3D efficiency histograms to generated $B \to D^{(*)}\pi\pi\ell\nu$ histograms to get reconstructed histograms.

The projection plots onto lepton momentum axis are shown in Figure 58 for electron and Figure 59 for muon in the 4 different cases listed above. In case of $B^+ \to XD^0(\text{or}\bar{D}^0)\ell\nu$, projection onto D momentum axis is also given in Figure 60 only for electron. Black points are reconstructed histograms and red lines are generated histograms. The bottom plots show efficiency. In principle, these are just ratio between the number of reconstructed events devided by the number of generated events in each bin. Thus, efficiency plots show

- (reconstruction efficiency)$\times$(geometrical acceptance) $\times$(event selection cuts)

Note that, $p^*_\ell > 0.8$ GeV and $-4.5 < \cos\theta_{B-Dl} < 1.5$ cuts are applied to make these plots.
We don't see any specific lepton momentum depedence or D momentum dependence. Relative efficiency between plots are reasonable. For example, $B^+ \to XD^+(\text{or}D^-)\ell\nu$ efficiency are lower because it requires one more chaged track to reconstruct D^+ than D^0.
Note that even if there is a overll normalization problem such as uncertainty on luminisity, it is canceled out when we apply the efficiency to generated $B \to D^{(*)}\pi\pi\ell\nu$, and will not go into final $B \to D^{(*)}\pi\pi\ell\nu$ histograms.

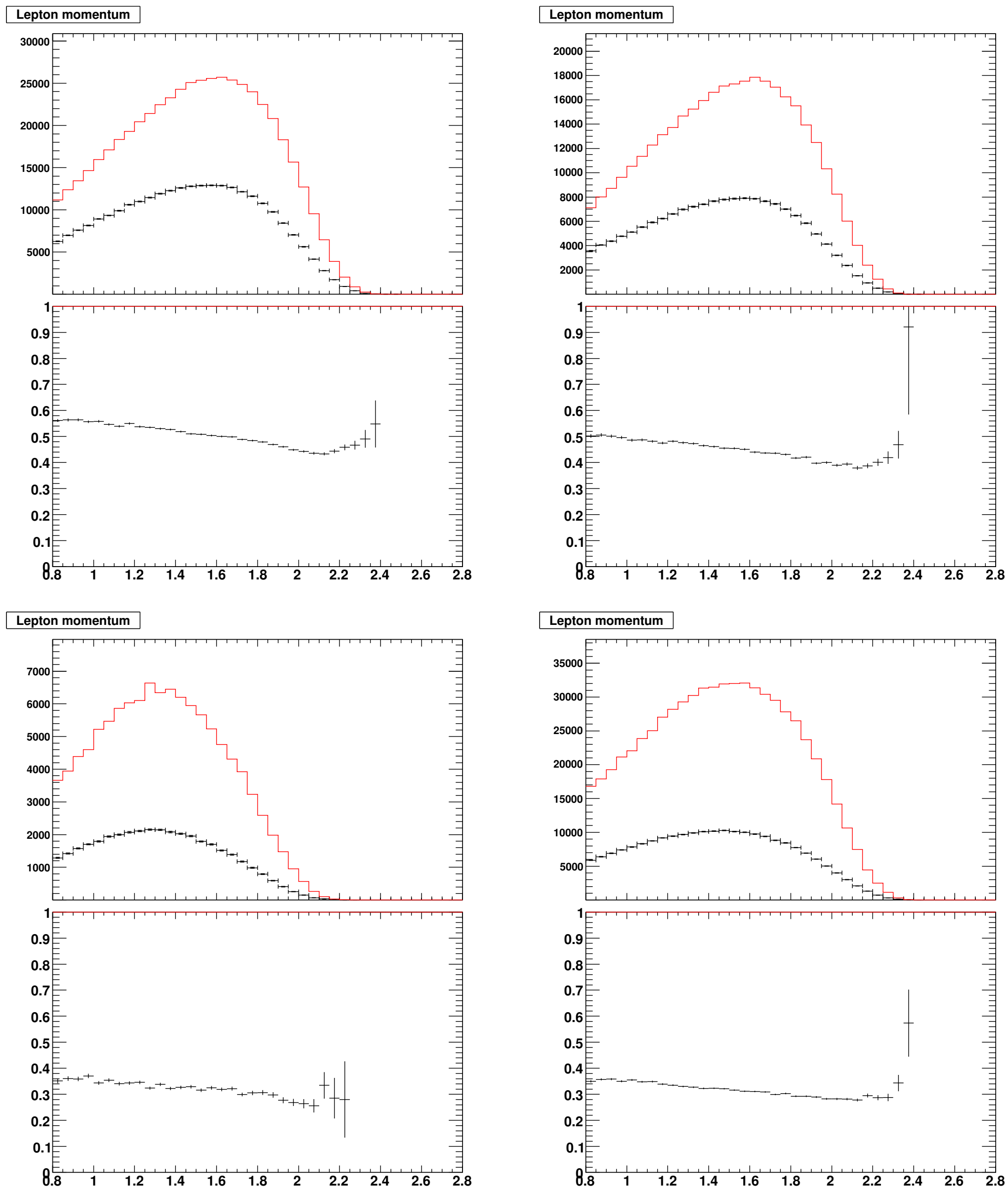


Figure 58: [**Efficiency histogram of electron : p_ℓ projection**] Efficiency histogram projection : $B^+ \to XD^0(\text{or}\bar{D}^0)\ell\nu$ (top-left), $B^0 \to XD^0(\text{or}\bar{D}^0)\ell\nu$ (top-right), $B^+ \to XD^+(\text{or}D^-)\ell\nu$ (bottom-left) and $B^0 \to XD^+(\text{or}D^-)\ell\nu$ (bottom-right). Black points are reconstructed histograms and red lines are generated histograms. The bottom plots show efficiency.

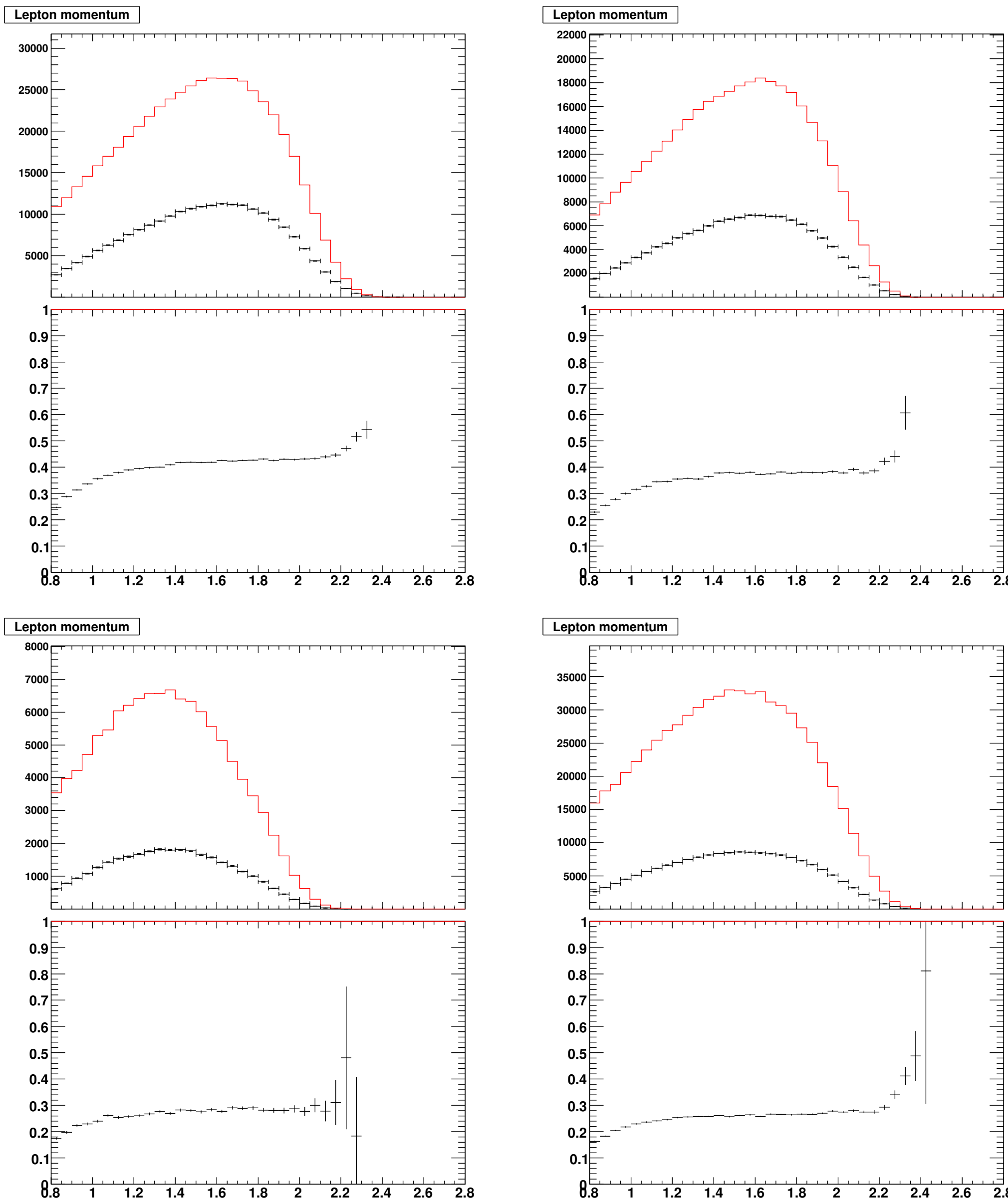


Figure 59: [**Efficiency histogram of muon : p_ℓ projection**] Efficiency histogram projection : $B^+ \to XD^0(\text{or}\bar{D}^0)\ell\nu$ (top-left), $B^0 \to XD^0(\text{or}\bar{D}^0)\ell\nu$ (top-right), $B^+ \to XD^+(\text{or}D^-)\ell\nu$ (bottom-left) and $B^0 \to XD^+(\text{or}D^-)\ell\nu$ (bottom-right). Black points are reconstructed histograms and red lines are generated histograms. The bottom plots show efficiency.

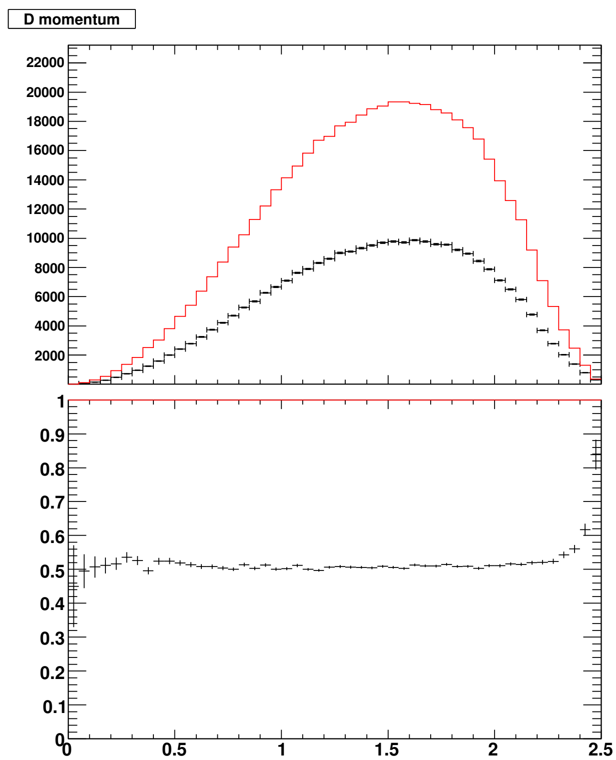


Figure 60: [**Efficiency histogram of electron : p_D projection ($B^+ \to XD^0(\mathrm{or}\bar{D}^0)\ell\nu$)**] Efficiency histogram projection. Black points are reconstructed histograms and red lines are generated histograms. The bottom plots show efficiency.

14.3 Isospin symmetry and $B \to D^{(*)}\pi\pi\ell\nu$ component

If we remove isospin constraints on $B \to D\ell\nu$ and $B \to D^*\ell\nu$ decays, and fix $B \to D^{(*)}\pi\ell\nu$ BF, fit results for electron are

$$\begin{aligned}\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) &= 0.0249 \pm 0.0006\\ \mathcal{B}(B^0 \to D^-\ell^+\nu) &= 0.0218 \pm 0.0004\\ \mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) &= 0.0533 \pm 0.0011\\ \mathcal{B}(B^0 \to D^{*-}\ell^+\nu) &= 0.0527 \pm 0.0015\end{aligned} \tag{303}$$

If we asuum systematic uncertainties are 1.5 times of statistical uncertainties, and using lifetime ratio $t_{+0} = 1.071 \pm 0.009$, we can calsulate (B^+ BF)/(B^0 BF)t_{+0}

$$\begin{aligned}B \to D\ell\nu &: 1.065 \pm 0.056\\ B \to D^*\ell\nu &: 0.944 \pm 0.060\end{aligned} \tag{304}$$

Thus, isospin symmetry holds. But, if we don't include $B \to D^{(*)}\pi\pi\ell\nu$ component in our fit, the same ratios are

$$\begin{aligned}B \to D\ell\nu &: 1.059 \pm 0.058\\ B \to D^*\ell\nu &: 0.895 \pm 0.069\end{aligned} \tag{305}$$

This is further away from isospin symmetry compare to the case where $B \to D^{(*)}\pi\pi\ell\nu$ component is included. This is another motivation to include $B \to D^{(*)}\pi\pi\ell\nu$ component in our fit.

15 D mass sideband subtraction revisited

We assume D mass sideband is flat when we do sideand subtraction. Here we check this assumption bin by bin.

15.1 Re-definition of D mass sideband

The both ends of D mass sideband defined in section 8.1 are too close to edges of acceppted D mass window of BToDlnu skim. Thus, we re-define D mass sideband. We also made D^+ peak region the same width as D^0.

- D^0 mass
  Peak : 1.840 - 1.888 GeV
  Side bands : 1.816 - 1.840 and 1.888 - 1.912 GeV

- D^+ mass
  Peak : 1.845 - 1.893 GeV
  Side bands : 1.821 - 1.845 and 1.893 - 1.917 GeV

15.2 Bin by bin D mass plots

The linearlity of the sideband region may depend on kinematic variables : lepton momentum, D momentum and $\cos\Theta_{B-D\ell}$. We plot OnPeak data D mass distribution with bin by bin of these kinematic variables. Binning are descibed in section 13.1. Plots are shown in Figures 61-63 for D^0 mass and in Figures 64-66 for D^+ mass. Thse Figures are in order of $\cos\Theta_{B-D\ell}$ bins. In one figure, plots are arranged in order of lepton momentum bins from top to bottom, and in oder of D momentum bins from left to right. No significant non-linearlity is obsedved from these plots.

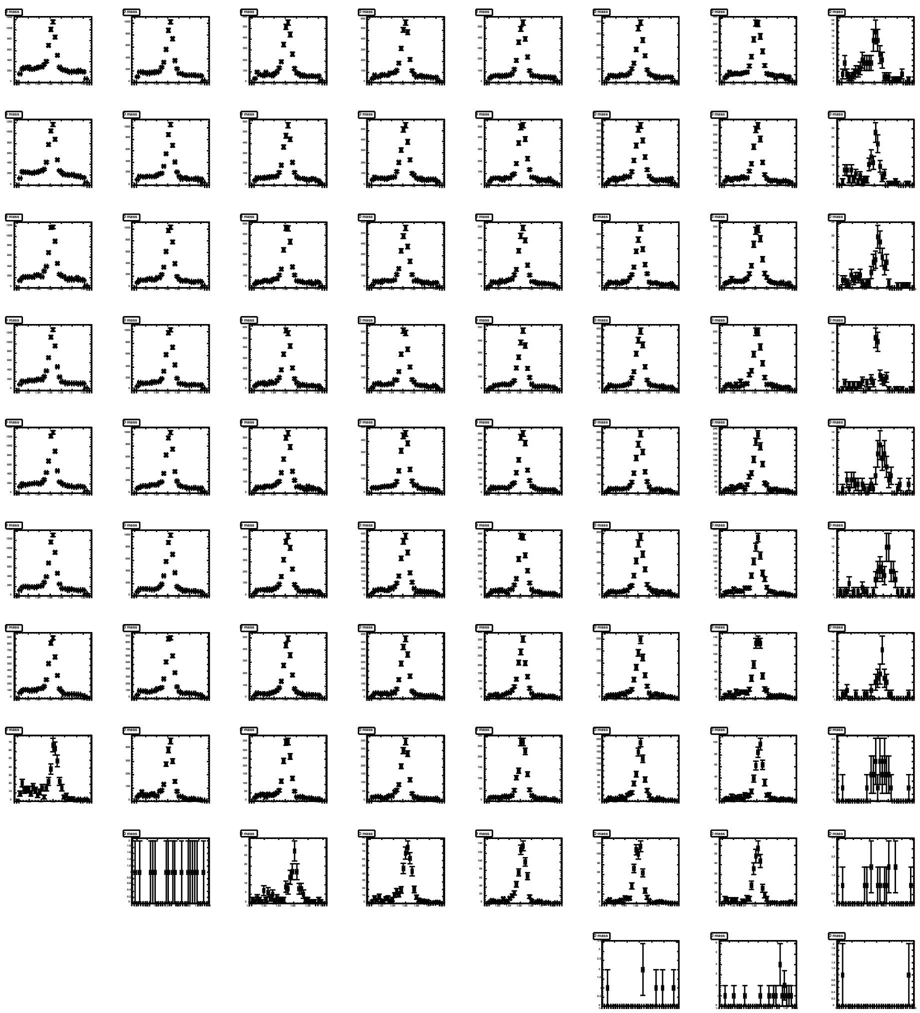


Figure 61: [**Bin by bin D^0 mass plots : OnPeak 1**] OnPeak D^0 mass plots of 1st $\cos\Theta_{B-D\ell}$ bin. From top to botton : in order of lepton momentum bins. From left to right : in order of D momentum bins. Plots are not shown for the bins whre there is no entry.

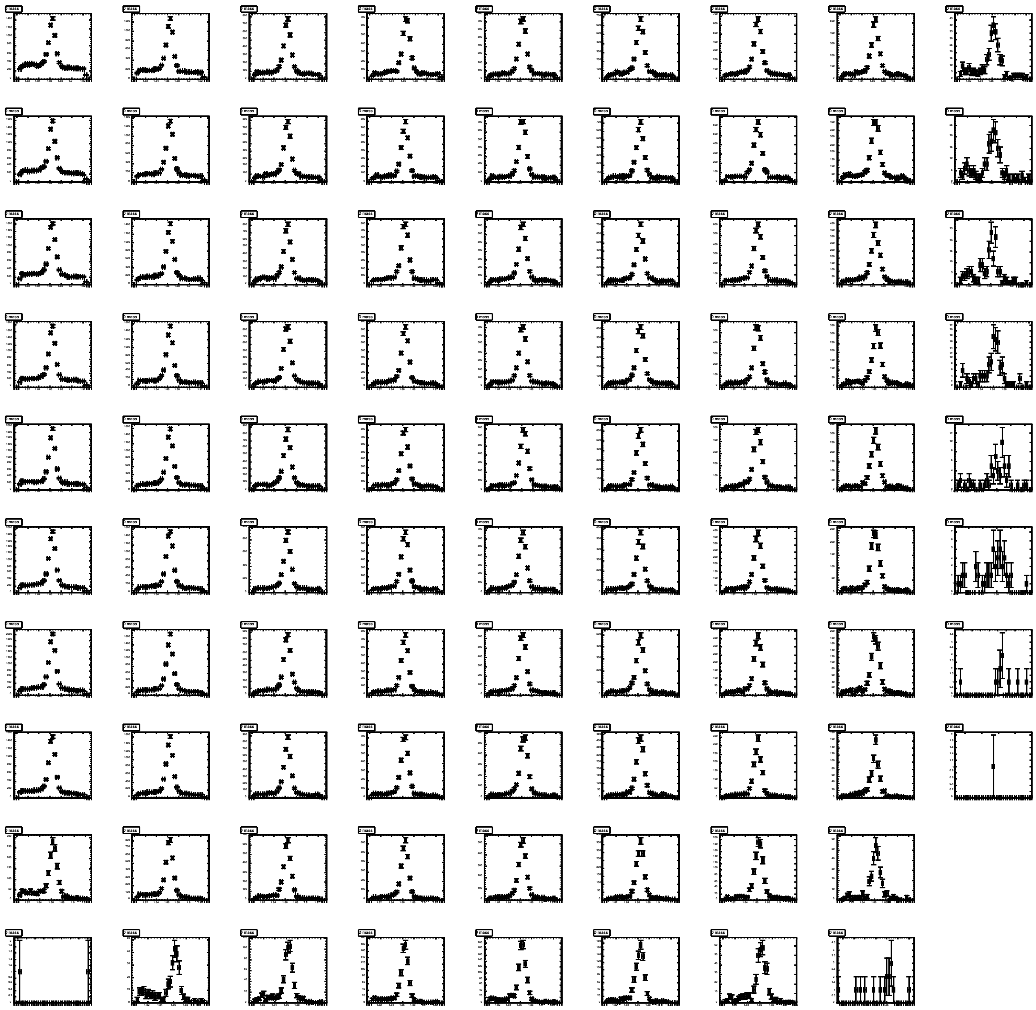


Figure 62: [**Bin by bin D^0 mass plots : OnPeak 2**] OnPeak D^0 mass plots of 2nd $\cos\Theta_{B-D\ell}$ bin. From top to botton : in order of lepton momentum bins. From left to right : in order of D momentum bins. Plots are not shown for the bins whre there is no entry.

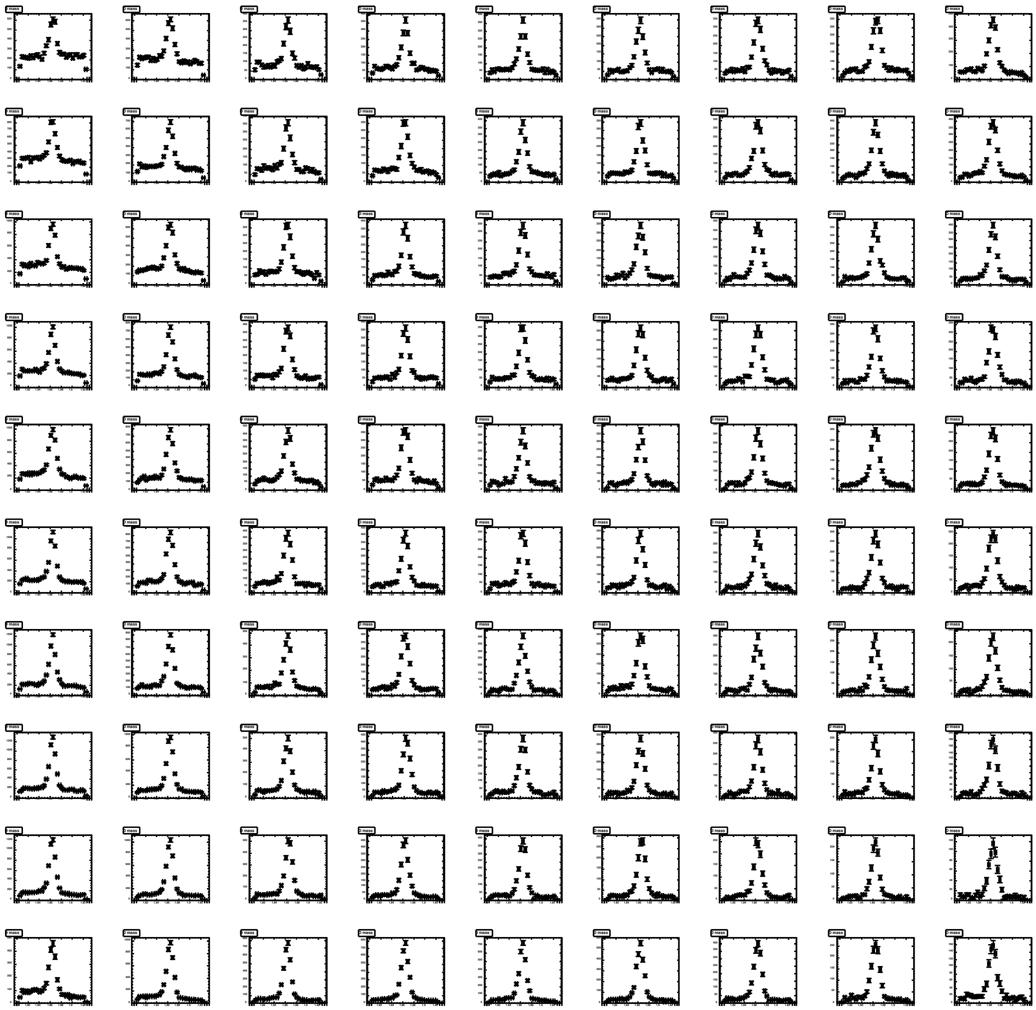


Figure 63: [**Bin by bin D^0 mass plots : OnPeak 3**] OnPeak D^0 mass plots of 3rd $\cos\Theta_{B-D\ell}$ bin. From top to botton : in order of lepton momentum bins. From left to right : in order of D momentum bins. Plots are not shown for the bins whre there is no entry.

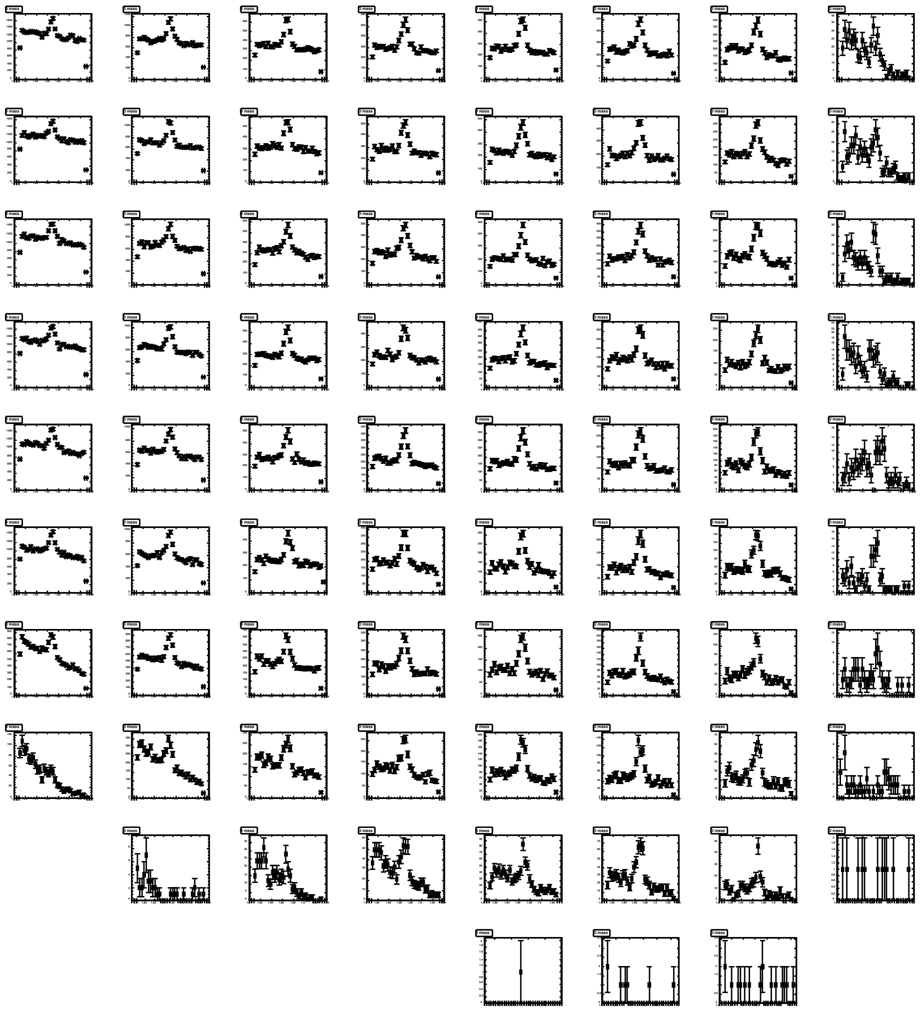


Figure 64: [**Bin by bin D^+ mass plots : OnPeak 1**] OnPeak D^+ mass plots of 1st $\cos\Theta_{B-D\ell}$ bin. From top to botton : in order of lepton momentum bins. From left to right : in order of D momentum bins. Plots are not shown for the bins whre there is no entry.

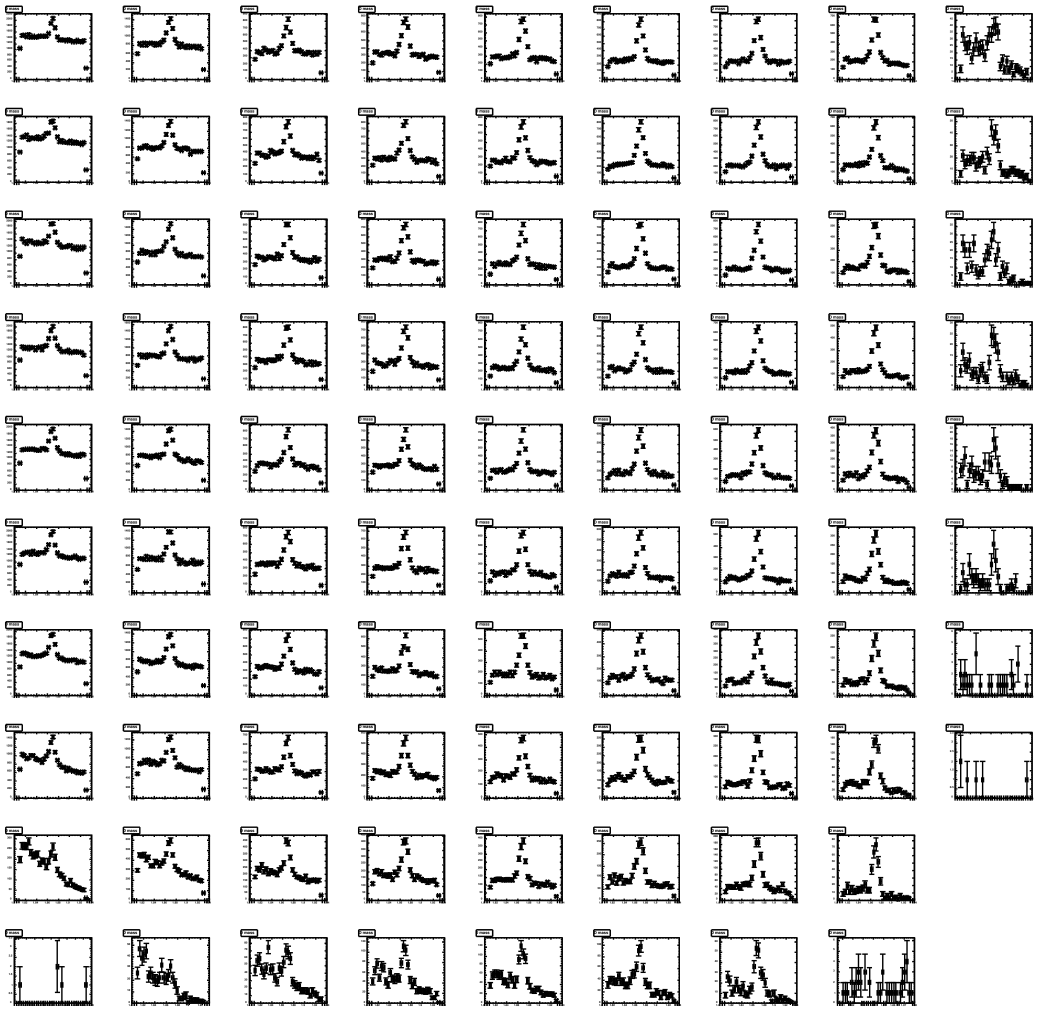


Figure 65: [**Bin by bin D^+ mass plots : OnPeak 2**] OnPeak D^+ mass plots of 2nd $\cos\Theta_{B-D\ell}$ bin. From top to botton : in order of lepton momentum bins. From left to right : in order of D momentum bins. Plots are not shown for the bins whre there is no entry.

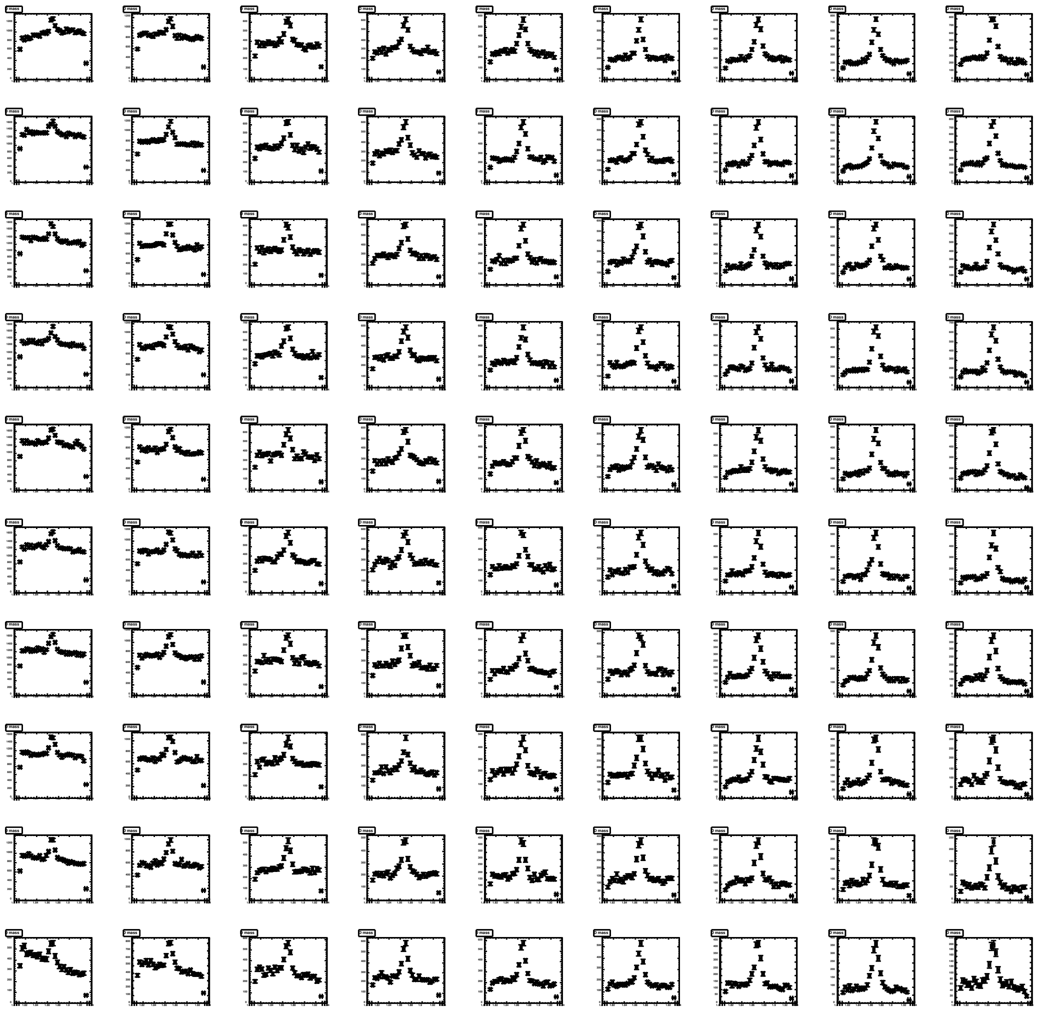


Figure 66: [**Bin by bin D^+ mass plots : OnPeak 3**] OnPeak D^+ mass plots of 3rd $\cos\Theta_{B-D\ell}$ bin. From top to botton : in order of lepton momentum bins. From left to right : in order of D momentum bins. Plots are not shown for the bins whre there is no entry.

16 Nominal fit configuration (Electron only)

16.1 $B \to D^{**}\ell\nu$ and non-resonant $B \to D^{(*)}\pi\ell\nu$ decay branching fractions

From PDG [12]

$$\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) = 0.0056 \pm 0.0013 \pm 0.0009 \tag{306}$$

Babar measure [15]

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) &= 0.00448 \pm 0.00026 \pm 0.00035 = 0.00448 \pm 0.000436 \\
\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu) &= 0.00354 \pm 0.00032 \pm 0.00054 = 0.00354 \pm 0.000627 \\
\mathcal{B}(B^0 \to D_1^- \ell^+\nu) &= 0.00364 \pm 0.00032 \pm 0.00049 = 0.00364 \pm 0.000585 \\
\mathcal{B}(B^0 \to D_2^{*-} \ell^+\nu) &= 0.00270 \pm 0.00035 \pm 0.00043 = 0.00270 \pm 0.000554
\end{aligned} \tag{307}$$

We take isospin average of these results assuming maximum correlation

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) &= 0.00419 \pm 0.000532 \\
\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu) &= 0.003215 \pm 0.0006104
\end{aligned} \tag{308}$$

Babar also measure by assuming isospin symmetry [45]

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) &= 0.0042 \pm 0.0004 \pm 0.0004 \\
\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+\nu) &= 0.0047 \pm 0.0006 \pm 0.0006 \\
\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu) &= 0.0029 \pm 0.0005 \pm 0.0003 \\
\mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+\nu) &= 0.0052 \pm 0.0007 \pm 0.0006
\end{aligned} \tag{309}$$

Combining these three results we obtain

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+\nu) &= 0.00466 \pm 0.00059 \\
\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+\nu) &= 0.0047 \pm 0.00085 \\
\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+\nu) &= 0.00306 \pm 0.00042 \\
\mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+\nu) &= 0.0052 \pm 0.00092
\end{aligned} \tag{310}$$

By summing these 4 BF

$$\mathcal{B}(B^+ \to \bar{D}^{**0} \ell^+\nu) = 0.0176 \pm 0.0014 \tag{311}$$

From PDG [12]

$$\begin{aligned}
\mathcal{B}(B^+ \to D^- \pi^+ \ell^+\nu) &= 0.0053 \pm 0.0009 \pm 0.0005 = 0.0053 \pm 0.0010 \\
\mathcal{B}(B^+ \to D^{*-} \pi^+ \ell^+\nu) &= 0.0064 \pm 0.0015 \pm 0.0002 = 0.0064 \pm 0.0015 \\
\mathcal{B}(B^0 \to \bar{D}^0 \pi^- \ell^+\nu) &= 0.0032 \pm 0.0009 \pm 0.0003 = 0.0032 \pm 0.00095 \\
\mathcal{B}(B^0 \to \bar{D}^{*0} \pi^- \ell^+\nu) &= 0.0065 \pm 0.0015 \pm 0.0005 = 0.0065 \pm 0.0016
\end{aligned} \tag{312}$$

We take isospin average by asuuming maximum correlation to get

$$\mathcal{B}(B^+ \to D^{(*)-}\pi^+\ell^+\nu) = 0.0111 \pm 0.0019 \tag{313}$$

Based on isospin symmetry we further get

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0167 \pm 0.0029 \tag{314}$$

Babar measure [4]

$$\begin{aligned}\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) &= 0.0152 \pm 0.0012 \pm 0.0010 \\ \mathcal{B}(B^0 \to \bar{D}^{(*)}\pi\ell^+\nu) &= 0.0137 \pm 0.0017 \pm 0.0010\end{aligned} \tag{315}$$

We take isospin average to get

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0149 \pm 0.0018 \tag{316}$$

We take an averge over two measurements to obtain

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0158 \pm 0.0017 \tag{317}$$

Comparing this to sum of $B^+ \to D^{**0}\ell^+\nu$ BF, there is only tiny space for non-resonant $\mathcal{B}(B^+ \to \bar{D}\pi\ell^+\nu)$. SP10 MC uses

$$\begin{aligned}\mathcal{B}(\text{non-resonant } B^+ \to D^-\pi^+\ell^+\nu) &= 0.0040 \pm 0.0024 \\ \mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^0\pi^0\ell^+\nu) &= 0.0020 \pm 0.0012 \\ \mathcal{B}(\text{non-resonant } B^+ \to D^{*-}\pi^+\ell^+\nu) &= 0.0012 \pm 0.0008 \\ \mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) &= 0.0006 \pm 0.0004\end{aligned} \tag{318}$$

However these are too big to fit in the tiny space. Thus we shrink them keeping same D^*/D ratio

$$\begin{aligned}\mathcal{B}(\text{non-resonant } B^+ \to D^-\pi^+\ell^+\nu) &= 0.0010 \pm 0.0010 \\ \mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^0\pi^0\ell^+\nu) &= 0.0005 \pm 0.0005 \\ \mathcal{B}(\text{non-resonant } B^+ \to D^{*-}\pi^+\ell^+\nu) &= 0.0003 \pm 0.0003 \\ \mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) &= 0.00015 \pm 0.00015\end{aligned} \tag{319}$$

Combining Equation (310) and (319), we use

- $\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) = 0.0047 \pm 0.0006$
- $\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu) = 0.0031 \pm 0.0004$
- $\mathcal{B}(B^+ \to \bar{D}_1^{\prime 0}\ell^+\nu) = 0.0047 \pm 0.0009$
- $\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) = 0.0052 \pm 0.0009$

- $\mathcal{B}$(non-resonant $B^+ \to D^-\pi^+\ell^+\nu) = 0.0010 \pm 0.0010$
- $\mathcal{B}$(non-resonant $B^+ \to \bar{D}^0\pi^0\ell^+\nu) = 0.0005 \pm 0.0005$
- $\mathcal{B}$(non-resonant $B^+ \to D^{*-}\pi^+\ell^+\nu) = 0.0003 \pm 0.0003$
- $\mathcal{B}$(non-resonant $B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) = 0.00015 \pm 0.00015$

Summing over all these BF, we get

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0196 \pm 0.0010 \tag{320}$$

This is 1.3 σ off from Equation(317).
We calculate the branching fraction ratios by assuming the above branching fractions are independent.

$$\begin{aligned}
f_{D_2^*/D_1} &\equiv \frac{\mathcal{B}(B^+\to\bar{D}_2^{*0}\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^0\ell^+\nu)} = 0.66 \pm 0.12 \\
f_{D_0^*D\pi T/D_1D_2^*T} &\equiv \frac{\mathcal{BT}(B^+\to\bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 0.87 \pm 0.20 \\
f_{D_1'D^*\pi T/D_1D_2^*T} &\equiv \frac{\mathcal{BT}(B^+\to\bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 0.67 \pm 0.13 \\
f_{D\pi/D_0^*} &\equiv \frac{\mathcal{B}(B^+\to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_0^{*0}\ell^+\nu)} = 0.19 \pm 0.03 \\
f_{D^*\pi/D_1'} &\equiv \frac{\mathcal{B}(B^+\to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^{*0}\ell^+\nu)} = 0.06 \pm 0.01
\end{aligned} \tag{321}$$

16.2 $B \to D^{(*)}\pi\pi\ell\nu$ decay branching fraction

From PDG [12], inclusive BF is

$$\mathcal{B}(B^+ \to X_c\ell^+\nu) = 0.108 \pm 0.004 \tag{322}$$

babar measure [4]

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) &= 0.0233 \pm 0.0009 \pm 0.0009 \\
\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) &= 0.0583 \pm 0.0015 \pm 0.0030 \\
\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) &= 0.0152 \pm 0.0012 \pm 0.0010
\end{aligned} \tag{323}$$

Thus 0.108 - 0.0233 - 0.0583 - 0.0152 = 0.0112 is missing. Thus, we fix

$$\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell^+\nu) = 0.011 \pm 0.011 \tag{324}$$

As we assign 100 % uncertainty, this covers systematic uncertainty from this unmesured decay BF.

16.3 Fixing $\mathcal{B}(B \to D^{(*)}\pi\ell\nu)$

Toy MC fit revealed biases. Details of toy MC fit is described in section 17.1. Mean values of pull distributions for 1000 toy fits is given in Table 27. Average χ^2/n.d.f = 466/468. Since we don't intend to measure $\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu)$, we fix it.

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0158 \pm 0.0017 \tag{325}$$

Parameters	Mean	r.m.s.
ρ_D^2	+0.0559	1.0115
ρ^2	-0.1367	1.0057
R_1	-0.0170	1.0012
R_2	+0.1809	0.9831
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	+0.1452	1.0007
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	-0.1776	0.9985
$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu)$	+0.1815	0.9841
Uncertainty	0.0317	0.0224

Table 27: [**Test Toy MC : Mean and Standard Deviation**] Mean and standard deviation (r.m.s.) of pull distributions of toy MC. The uncertainty on mean and standard deviation are also listed in the bottom line. Pulls are calculated from the difference to nominal fit results.

16.4 Summary of the nominal fit configuration

As described in previous sections, we are trying to fit for two branching fractions and four form factors

- $\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$
- $\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$
- $B \to D\ell\nu$ decay FF slope : ρ_D^2
- $B \to D^*\ell\nu$ decay FF slope : ρ^2
- $B \to D^*\ell\nu$ decay FF ratio : R_1
- $B \to D^*\ell\nu$ decay FF ratio : R_2

by fixing

- $\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu)$

by including an fixing

- $\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell^+\nu)$

by

- imposing isospin constraints

and fixing relative branching fractions between $\mathcal{B}(B^+ \to D^{**}\ell^+\nu)$ and $\mathcal{B}$(non-resonant $B^+ \to D^{(*)}\pi\ell^+\nu$)
We fix all background coefficients to 1.
We also fix f_{+0}

17 Fit validation (Electron only)

We validate our fit as described in the following subsections.

17.1 Toy MC study

17.1.1 Method of toy MC study

To validate our fitting method, we perform toy MC study. The procedure of toy MC study is the following

1. We create fake (OnPeak - OffPeak) data from $B\bar{B}$ MC, wth input values shown in Table 28.
2. Fluctuate the entry of each bin of the fake data using a Gaussian random number based on the uncertainty of each bin.
3. Fit for the fluctuated fake OnPeak data.
4. Iterate 2 and 3 for approximately 1,000 times.
5. Plot pull distributions and get the mean and the standard deviation of each distribution.

In Tablee 28, Set b has different FF parameters than Set a, and Set c has different B decay BF values.

Parameters	Set a	Set b	Set c
ρ_D^2	1.17	1.3	1.17
ρ^2	1.191	1.3	1.191
R_1	1.429	1.6	1.429
R_2	0.827	0.6	0.827
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.0233	0.0233	0.025
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.0583	0.0583	0.056
$\mathcal{B}(B^+ \to \bar{D}^{(*)} \pi \ell^+ \nu)$	0.0158	0.0158	0.0158

Table 28: [**Parameters of fake data**] The parameter values used to create fake data. Set b has different FF slopes than Set a, Set c has different BF values.

17.1.2 Pull results.

Pulls are calculated by

$$\text{pull} = \frac{\text{fitted value} - \text{true value}}{\text{fitted uncertainty}} \tag{326}$$

The mean and standard deviation (r.m.s.) of these pull distributions are listed in Table 29. In the bottom line of the Table, the uncertainty on mean and standard deviation are listed. We did 1000 toy fits for each parameter sets. Thus, assuming the standard deviation is $\sigma = 1$, the uncertainty on mean is

$$\frac{\sigma}{\sqrt{N}} = \frac{1}{\sqrt{1000}} = 0.0316 \tag{327}$$

and the uncertainty on standard deviation is

$$\frac{\sigma}{\sqrt{2N}} = \frac{1}{\sqrt{2 \times 1000}} = 0.0224 \tag{328}$$

The χ^2 and the pull plots are shown in Figure 67 and 68 only for parameter Set a.

	Set a		Set b		Set c	
Parameters	Mean	r.m.s.	Mean	r.m.s.	Mean	r.m.s.
ρ_D^2	+0.0253	1.014	-0.0112	1.017	+0.0329	1.016
ρ^2	-0.0755	1.028	-0.0195	1.017	-0.0671	1.023
R_1	+0.0303	0.993	-0.1166	0.996	+0.0366	0.995
R_2	+0.1079	1.005	+0.0234	0.997	+0.0998	1.004
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	+0.1477	1.000	+0.1378	1.000	+0.1608	1.006
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	-0.0940	0.977	-0.0263	1.012	-0.1030	1.017
Uncertainty	0.0317	0.0224	0.0317	0.0224	0.0317	0.0224

Table 29: [**Toy MC : Mean and Standard Deviation**] Mean and standard deviation (r.m.s.) of pull distributions of toy MC. The uncertainty on mean and standard deviation are also listed in the bottom line. Pulls are calculated from the difference to input values.

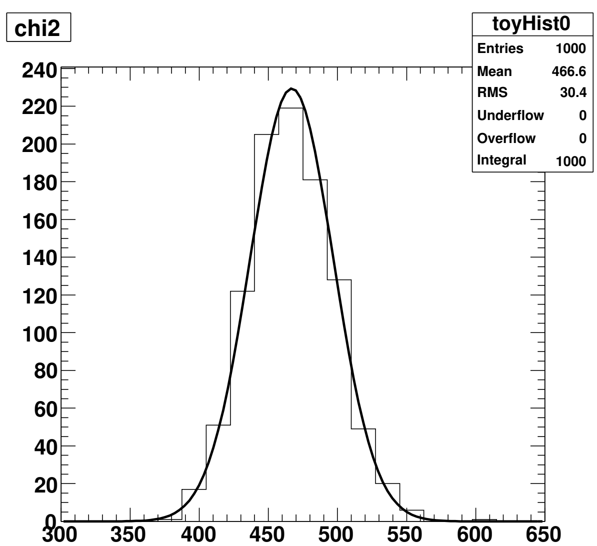


Figure 67: [**Toy MC χ^2 (Set a)**] Toy MC χ^2 plot for parameter Set a.

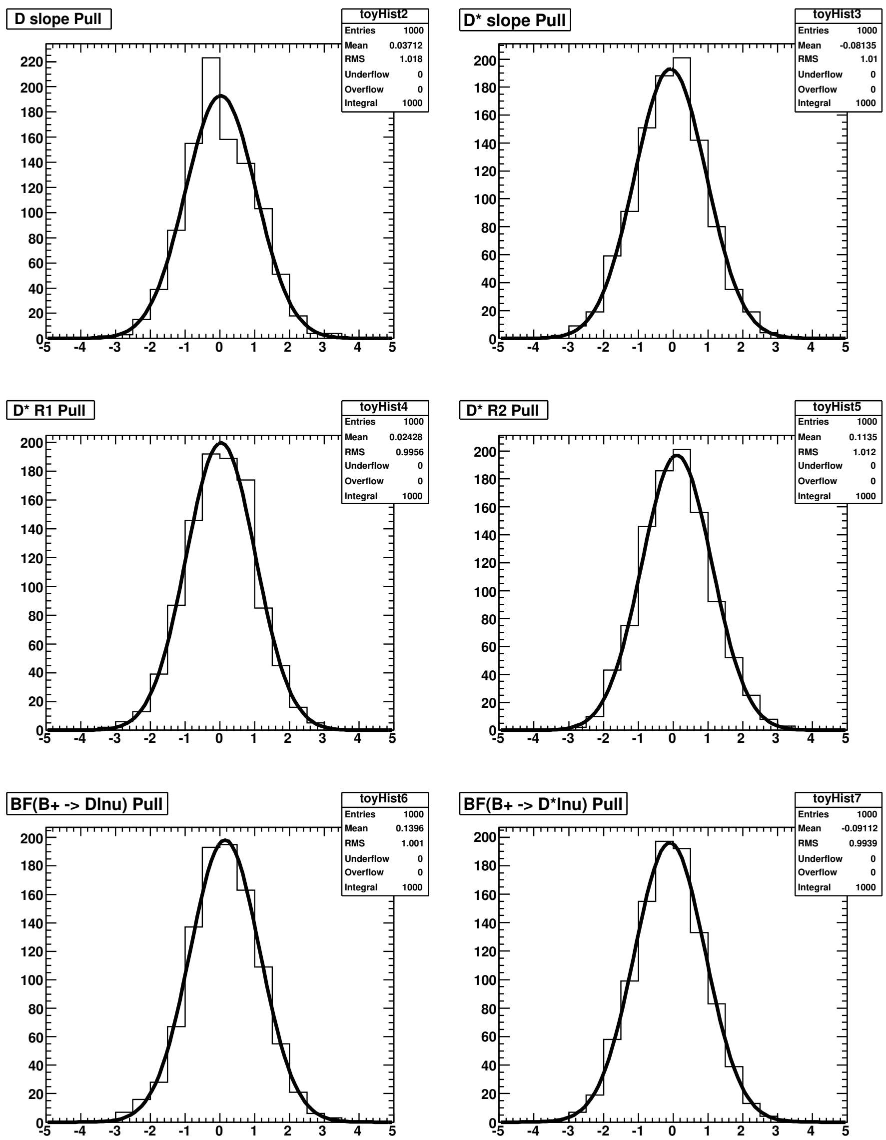


Figure 68: [**Toy MC Pull (Set a)**] Toy MC pull plots for parameter Set a.

17.2 Validation fits by splitted MC

17.2.1 Validation fit method

To validate our fitting method, we construct fake (OnPeak - OffPeak) data from $B\bar{B}$ MC. We split MC into two halves. We use one half to construct fake data. We, then, create component histograms from the other half, and fit the fake data by the histograms. We can construct fake data with any branching fractions and form factor parameters we want. We then perform fitting to see if our method produce reasonable fitting results.
We use different parameter sets to create fake data which are shown in Tables 28. The same values are listed as input values in the following tables. We split MC by four different ways

- Split A : Separate even number events and odd number events.
- Split B : Separate events using a set of random numbers.
- Split C : Separate events using another set of random numbers.
- Split D : Separate events using a different set of random numbers.

17.2.2 Validation fit results

Fit results are given in Tables 30 - 32. As an index of agreement, we calculate pulls as defined in the above Equation (326). We also caluculate overall χ^2 between input and fitted parameter values, namely

$$\chi^2_{Val} = (\Delta_1, \Delta_2, ..., \Delta_k) \begin{pmatrix} \text{Invers} \\ \text{covariance} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ . \\ . \\ . \\ \Delta_k \end{pmatrix} \tag{329}$$

where Δ_i is the differnce between fitted and input values of i-th parameter :

$$\Delta_i = \text{fitted value} - \text{true value} \tag{330}$$

	Input	Split A		Split B		Split C		Split D	
	(Set a)	fit result	pulls	fit result	pulls	fit result	pulls	fit result	pulls
ρ_D^2	1.17	1.107 ± 0.036	-1.77	1.149 ± 0.034	-0.63	1.154 ± 0.034	-0.46	1.115 ± 00035	-1.58
ρ^2	1.191	1.238 ± 0.032	+1.47	1.203 ± 0.032	+0.38	1.154 ± 0.032	-1.16	1.222 ± 0.030	+1.02
R_1	1.429	1.476 ± 0.047	+1.02	1.405 ± 0.043	-0.54	1.382 ± 0.041	-1.15	1.412 ± 0.042	-0.39
R_2	0827	0.754 ± 0.038	-1.93	0.829 ± 0.036	+0.05	0.852 ± 0.034	+0.74	0.836 ± 0.034	+0.27
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.0233	0.02304 ± 0.00024	-1.10	0.02338 ± 0.00024	+0.33	0.02316 ± 0.00024	-0.57	0.02334 ± 0.00024	+0.15
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.0583	0.05885 ± 0.00034	+1.62	0.05819 ± 0.00034	-0.31	0.05854 ± 0.00034	+0.17	0.05806 ± 0.00034	-0.71
χ^2/ndof (P-value)		466/456 (0.36)		498/458 (0.10)		437/459 (0.76)		498/457 (0.09)	
χ^2_{Val}			7.4		4.8		3.1		12.1

Table 30: [**MC vs MC (Parameter set a)**] Validation fit results using parameter set a.

	Input	Split A		Split B		Split C		Split D	
	(Set b)	fit result	pulls	fit result	pulls	fit result	pulls	fit result	pulls
ρ_D^2	1.3	1.243 ± 0.031	-1.83	1.278 ± 0.034	-0.76	1.289 ± 0.029	-0.38	1.253 ± 0.030	-1.56
ρ^2	1.3	1.363 ± 0.034	+1.84	1.320 ± 0.034	+0.60	1.267 ± 0.032	-1.00	1.328 ± 0.032	+0.87
R_1	1.6	1.674 ± 0.059	+1.26	1.587 ± 0.055	-0.23	1.546 ± 0.050	-1.07	1.580 ± 0.051	-0.40
R_2	0.6	0.501 ± 0.048	-2.09	0.589 ± 0.044	-0.24	0.628 ± 0.041	+0.69	0.610 ± 0.042	+0.25
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.0233	0.02305 ± 0.00024	-1.05	0.02336 ± 0.00024	+0.29	0.02316 ± 0.00024	-0.58	0.02333 ± 0.00024	+0.14
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.0583	0.05881 ± 0.00033	+1.55	0.05820 ± 0.00033	-0.29	0.05836 ± 0.00033	+0.19	0.05804 ± 0.00034	-0.76
χ^2/ndof (P-value)		464/456 (0.39)		504/458 (0.07)		443/459 (0.70)		498/457 (0.09)	
χ^2_{Val}			7.9		4.9		2.8		11.9

Table 31: [**MC vs MC (Parameter set b)**] Validation fit results using parameter set b.

	Input	Split A		Split B		Split C		Split D	
	(Set c)	fit result	pulls	fit result	pulls	fit result	pulls	fit result	pulls
ρ_D^2	1.17	1.110 ± 0.033	-1.80	1.149 ± 0.032	-0.68	1.156 ± 0.032	-0.46	1.119 ± 0.033	-1.56
ρ^2	1.191	1.241 ± 0.033	+1.49	1.203 ± 0.033	+0.37	1.151 ± 0.033	-1.22	1.222 ± 0.031	+0.99
R_1	1.429	1.477 ± 0.049	+0.99	1.405 ± 0.045	-0.53	1.378 ± 0.043	-1.19	1.412 ± 0.044	-0.38
R_2	0827	0.752 ± 0.039	-1.92	0.830 ± 0.037	+0.07	0.855 ± 0.035	+0.79	0.837 ± 0.035	+0.28
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.025	0.02474 ± 0.00024	-1.08	0.02508 ± 0.00024	+0.32	0.02486 ± 0.00024	-0.59	0.02503 ± 0.00024	+0.13
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.056	0.05655 ± 0.00034	+1.65	0.05589 ± 0.00034	-0.34	0.05605 ± 0.00034	+0.13	0.05576 ± 0.00034	-0.71
χ^2/ndof (P-value)		469/458 (0.35)		495/460 (0.13)		438/459 (0.75)		498/457 (0.09)	
χ^2_{Val}			7.5		5.0		3.3		11.8

Table 32: [**MC vs MC (Parameter set c)**] Validation fit results using parameter set c.

18 Cross Checks (Electron only)

We present test fit results and cross-checks.

18.1 Test fit results

Fit results are shown in Table 33 and correlations between parameters are given in Table 34.

ρ_D^2	1.240 ± 0.049 (3.9 %)
ρ^2	1.300 ± 0.052 (4.0 %)
R_1	1.530 ± 0.084 (5.5 %)
R_2	0.703 ± 0.067 (9.5 %)
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02399 ± 0.00037 (1.5 %)
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05444 ± 0.00049 (0.9 %)
χ^2/ndof (P-value)	422/468 (0.94)

Table 33: [**Nominal fit results (test fit)**] Fit results.

	ρ_D^2	ρ^2	R_1	R_2	$\mathcal{B}(D\ell\nu)$	$\mathcal{B}(D^*\ell\nu)$
ρ_D^2	1					
ρ^2	-0.216	1				
R_1	+0.001	+0.819	1			
R_2	+0.104	-0.892	-0.900	1		
$\mathcal{B}(D\ell\nu)$	+0.352	+0.170	+0.266	-0.094	1	
$\mathcal{B}(D^*\ell\nu)$	-0.427	+0.233	-0.055	-0.271	-0.585	1

Table 34: [**Correlation coefficients (test fit)**] Correlations between parameters.

Parameters	Electron result	Muon result	NR
ρ_D^2	1.240 ± 0.049	1.166 ± 0.064	-0.92
ρ^2	1.300 ± 0.052	1.337 ± 0.063	+0.46
R_1	1.530 ± 0.084	1.537 ± 0.104	+0.05
R_2	0.703 ± 0.067	0.673 ± 0.084	-0.28
$\mathcal{B}(B \to D\ell\nu)$	0.02399 ± 0.00037	0.02265 ± 0.00042	-2.40
$\mathcal{B}(B \to D^*\ell\nu)$	0.05444 ± 0.00049	0.05252 ± 0.00055	-2.62
χ^2/ndof (P-value)	422/468 (0.94)	495/465 (0.17)	13.8/6 (0.032)

Table 35: [**Electron only and muon only fits**] Fit results and normaliaed residuals (NR).

18.2 Electron vs muon

We compare the two cases of leptons, namely electrons and muons. The results are give in Table 35. The normalized residuals (NR) in the Table is defined by

$$\mathrm{NR} = \frac{(\text{Muon fitted value}) - (\text{Elctron fitted value})}{\sqrt{\sigma_e^2 + \sigma_\mu^2}} \tag{331}$$

χ^2 for NR is defined by the sum of NR^2.

Parameters	Run1-3 result	Run4 result	NR
ρ_D^2	1.174 ± 0.073	1.286 ± 0.066	+1.14
ρ^2	1.264 ± 0.070	1.328 ± 0.075	+0.62
R_1	1.488 ± 0.108	1.558 ± 0.123	+0.43
R_2	0.741 ± 0.087	0.676 ± 0.099	-0.50
$\mathcal{B}(B \to D\ell\nu)$	0.02364 ± 0.00051	0.02441 ± 0.00052	+1.05
$\mathcal{B}(B \to D^*\ell\nu)$	0.05495 ± 0.00068	0.05380 ± 0.00070	-1.18
χ^2/ndof (P-value)	427/457 (0.84)	487/458 (0.17)	4.6/6 (0.59)

Table 36: [**Run1-3 and Run4 fits**] Fit results and normaliaed residuals (NR).

18.3 Run1-3 vs Run4

We fit Run1-3 and Run4 separately. The results are give in Table 36. The normalized residuals (NR) in the Table is defined by

$$\mathrm{NR} = \frac{(\text{Run4 fitted value}) - (\text{Run1-3 fitted value})}{\sqrt{\sigma^2_{\text{Run4}} + \sigma^2_{\text{Run1-3}}}} \tag{332}$$

χ^2 for NR is defined by the sum of NR^2.

Parameters	D^0 result	D^+ result	NR
ρ_D^2	1.417 ± 0.057	0.999 ± 0.104	-3.52
ρ^2	1.280 ± 0.055	1.578 ± 0.191	+1.50
R_1	1.557 ± 0.090	1.986 ± 1.947	+0.22
R_2	0.717 ± 0.069	0.414 ± 0.314	-0.94
$\mathcal{B}(B \to D\ell\nu)$	0.02537 ± 0.00061	0.02303 ± 0.00062	-2.70
$\mathcal{B}(B \to D^*\ell\nu)$	0.05328 ± 0.00059	0.05739 ± 0.00310	+1.31
χ^2/ndof (P-value)	216/234 (0.79)	185/227 (0.98)	24.6/6 (0.00041)

Table 37: [D^0 **and** D^+ **fits**] Fit results and normaliaed residuals (NR).

18.4 D^0 vs D^+

We fit separately only for D^0 and only for D^+. The results are give in Table 37. The normalized residuals (NR) in the Table is defined by

$$\mathrm{NR} = \frac{(D^+ \text{ fitted value}) - (D^+ \text{ fitted value})}{\sqrt{\sigma^2_{D^+} + \sigma^2_{D^+}}} \tag{333}$$

χ^2 for NR is defined by the sum of NR^2.

	Fit result	Difference
ρ_D^2	1.261 ± 0.046	1.66 %
ρ^2	1.214 ± 0.025	-6.58 %
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02410 ± 0.00033	+0.44 %
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05380 ± 0.00038	-1.18 %
χ^2/ndof (P-value)	427/479 (0.92)	

Table 38: [**Fix** R_1 **and** R_2] Fit results.

18.5 Fix R_1 and R_2

We fixed $R_1 = 1.429$ and $R_2 = 0.827$.

	Fit result	NR
ρ_D^2	1.245 ± 0.050	-0.10
ρ^2	1.321 ± 0.053	+0.41
R_1	1.569 ± 0.089	+0.44
R_2	0.689 ± 0.069	-0.21
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02486 ± 0.00059	+1.48
$\mathcal{B}(B^0 \to D^- \ell^+ \nu)$	0.02193 ± 0.00046	n/a
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05326 ± 0.00125	-0.95
$\mathcal{B}(B^0 \to D^{*-} \ell^+ \nu)$	0.05148 ± 0.00172	n/a
χ^2/ndof (P-value)	418/466 (0.94)	3.5/6

Table 39: [**No ispspin constraints on** $B \to D^{(*)}\ell\nu$ **decays**] Fit results.

18.6 No ispspin constraints on $B \to D^{(*)}\ell\nu$ decays

We remobed ispspin constraints on $B \to D^{(*)}\ell\nu$ decays.

	Fit result	NR
ρ_D^2	1.240 ± 0.051	+0.00
ρ^2	1.300 ± 0.052	+0.00
R_1	1.530 ± 0.084	+0.00
R_2	0.703 ± 0.067	+0.00
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02399 ± 0.00041	-0.01
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05445 ± 0.00061	+0.01
$\mathcal{B}(D^{*+} \to D^0 \pi^+)$	0.6769 ± 0.0077	-0.01
χ^2/ndof (P-value)	358/349 (0.364)	

Table 40: [**Float** $\mathcal{B}(D^{*+} \to D^0\pi^+)$] Fit results.

18.7 Float $\mathcal{B}(D^{*+} \to D^0\pi^+)$

We changed the upper cut of $\cos\theta_{B-Dl}$ to 1.1 to avoid the sharpe edge at $\cos\theta_{B-Dl} = 1$

18.8 Minimum entry per bin

We change the requirement of minimum entries per bin to 25, 50, 75 and 100. In nominal fit it is set to 10. Results are shown in Table 41-44. No systematic uncertainty is assigned based on this.
When we change thie requirement, the number of bins goes into χ^2 changes as follows

- Minimum entry ber bin > 10 : number of bins goes into $\chi^2 = 475$
- Minimum entry ber bin > 25 : number of bins goes into $\chi^2 = 464$
- Minimum entry ber bin > 50 : number of bins goes into $\chi^2 = 449$
- Minimum entry ber bin > 75 : number of bins goes into $\chi^2 = 446$
- Minimum entry ber bin > 100 : number of bins goes into $\chi^2 = 439$

	Fit result	NR
ρ_D^2	1.240 ± 0.049	+0.01
ρ^2	1.299 ± 0.052	-0.01
R_1	1.529 ± 0.084	-0.01
R_2	0.704 ± 0.067	+0.01
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02399 ± 0.00037	-0.01
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05444 ± 0.00049	+0.00
χ^2/ndof (P-value)	415/457 (0.92)	

Table 41: [**Effect of minimum candidates per bin** > **25**] Fit results.

	Fit result	NR
ρ_D^2	1.238 ± 0.049	-0.04
ρ^2	1.292 ± 0.052	-0.14
R_1	1.519 ± 0.083	-0.14
R_2	0.712 ± 0.067	+0.13
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02398 ± 0.00037	-0.04
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05445 ± 0.00049	+0.01
χ^2/ndof (P-value)	402/442 (0.92)	

Table 42: [**Effect of minimum candidates per bin** > **50**] Fit results.

	Fit result	NR
ρ_D^2	1.238 ± 0.049	-0.05
ρ^2	1.294 ± 0.0520	-0.11
R_1	1.520 ± 0.083	-0.12
R_2	0.710 ± 0.067	+0.10
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02398 ± 0.00037	-0.03
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05446 ± 0.00049	+0.04
χ^2/ndof (P-value)	401/439 (0.90)	

Table 43: [**Effect of minimum candidates per bin** > **75**] Fit results.

	Fit result	NR
ρ_D^2	1.243 ± 0.049	+0.06
ρ^2	1.287 ± 0.053	-0.24
R_1	1.515 ± 0.083	-0.18
R_2	0.716 ± 0.066	+0.19
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02396 ± 0.00037	-0.07
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05444 ± 0.00049	-0.02
χ^2/ndof (P-value)	395/432 (0.90)	

Table 44: [**Effect of minimum candidates per bin $>$ 100**] Fit results.

18.9 Binning

We change binning and see how fit results change. Nominal binnings are (section 13.1).

- lepton momentum in the CM frame p_ℓ^*
  (10 bins (GeV)) : 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.35

- D momentum in the CM frame p_D^*
  (9 bins (GeV)) : 0.8, 1.1, 1.35, 1.5, 1.65, 1.8, 1.95, 2.1, 2.25, 2.45

- $\cos\theta_{B-Dl}$ (see below)
  (3 bins) : -2, -1, 0, 1.1

First, we chcnage lepton momentum binning

- p_ℓ^* Binning 1
  (10 bins (GeV)) : 1.2, 1.35, 1.45, 1.55, 1.65, 1.75, 1.85, 1.95, 2.05, 2.15, 2.35)

- p_ℓ^* Binning 2
  (5 bins (GeV)) : 1.2, 1.4, 1.6, 1.8, 2.0, 2.35

Results are given in Table 45 and 46. No systematic uncertainty is assigned based on this.
Next, we change D momentum binning

- p_D^* Binning 1
  (9 bins (GeV)) : 0.8, 1.05, 1.3, 1.45, 1.6, 1.75, 1.9, 2.05, 2.2, 2.45

- p_D^* Binning 2
  (4 bins (GeV)) : 0.8, 1.2, 1.6, 2.0, 2.45

Results are given in Table 47 and 48. No systematic uncertainty is assigned based on this.
Finally, we change $\cos\theta_{B-Dl}$ bining

- $\cos\theta_{B-Dl}$ Binning 1
  (3 bins) : -2, -0.5, 0.5, 1.1

- $\cos\theta_{B-Dl}$ Binning 2
  (2 bins) : -2, 0, 1.1

Results are given in Table 49 and 50. No systematic uncertainty is assigned based on this.

	Fit result	NR
ρ_D^2	1.230 ± 0.049	-0.20
ρ^2	1.307 ± 0.052	+0.14
R_1	1.526 ± 0.082	-0.05
R_2	0.693 ± 0.067	-0.16
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	0.02396 ± 0.00037	-0.09
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	0.05465 ± 0.00049	+0.41
χ^2/ndof (P-value)	469/453 (0.29)	

Table 45: [**Effect of p_ℓ binning 1**] Fit results.

	Fit result	NR
ρ_D^2	1.241 ± 0.049	+0.02
ρ^2	1.318 ± 0.054	+0.34
R_1	1.539 ± 0.086	+0.11
R_2	0.678 ± 0.070	-0.36
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	0.02397 ± 0.00037	-0.04
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	0.05462 ± 0.00049	+0.36
χ^2/ndof (P-value)	237/241 (0.57)	

Table 46: [**Effect of p_ℓ binning 2**] Fit results.

	Fit result	NR
ρ_D^2	1.224 ± 0.049	-0.33
ρ^2	1.326 ± 0.051	+0.52
R_1	1.548 ± 0.083	+0.22
R_2	0.679 ± 0.066	-0.37
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	0.02399 ± 0.00037	+0.01
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	0.05457 ± 0.00049	+0.27
χ^2/ndof (P-value)	467/483 (0.69)	

Table 47: [**Effect of p_D binning 1**] Fit results.

	Fit result	NR
ρ_D^2	1.246 ± 0.051	+0.12
ρ^2	1.288 ± 0.057	-0.21
R_1	1.499 ± 0.088	-0.35
R_2	0.732 ± 0.073	+0.39
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02410 ± 0.00037	+0.30
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05434 ± 0.00050	-0.22
χ^2/ndof (P-value)	252/220 (0.07)	

Table 48: [**Effect of** p_D **binning 2**] Fit results.

	Fit result	NR
ρ_D^2	1.207 ± 0.051	-0.65
ρ^2	1.290 ± 0.051	-0.20
R_1	1.516 ± 0.077	-0.18
R_2	0.707 ± 0.063	+0.06
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02367 ± 0.00037	-0.85
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05482 ± 0.00047	+0.79
χ^2/ndof (P-value)	500/498 (0.47)	

Table 49: [**Effect of** $\cos\theta_{B-Dl}$ **binning 1**] Fit results.

	Fit result	NR
ρ_D^2	1.214 ± 0.052	-0.50
ρ^2	1.316 ± 0.053	+0.32
R_1	1.574 ± 0.087	+0.50
R_2	0.674 ± 0.070	-0.42
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02381 ± 0.00039	-0.46
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05453 ± 0.00050	+0.18
χ^2/ndof (P-value)	314/336 (0.81)	

Table 50: [**Effect of** $\cos\theta_{B-Dl}$ **binning 2**] Fit results.

	Fit result	NR
ρ_D^2	1.201 ± 0.055	-0.71
ρ^2	1.320 ± 0.055	+0.37
R_1	1.562 ± 0.091	+0.35
R_2	0.664 ± 0.073	-0.54
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	0.02392 ± 0.00040	-0.17
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	0.05465 ± 0.00052	+0.40
χ^2/ndof (P-value)	446/468 (0.76)	1.01

Table 51: [D **mass peak region : wider**] Fit results.

	Fit result	NR
ρ_D^2	1.230 ± 0.048	-0.20
ρ^2	1.310 ± 0.052	+0.20
R_1	1.553 ± 0.084	+0.28
R_2	0.670 ± 0.068	-0.49
$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)$	0.02389 ± 0.00035	-0.29
$\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)$	0.05448 ± 0.00048	+0.07
χ^2/ndof (P-value)	462/468 (0.57)	1.21

Table 52: [D **mass peak region : narrower**] Fit results.

18.10 D mass peak region

Wider or narrower D mass peak region by 2 MeV at both ends of the region.

19 Effect of lepton mass

In the usual decay rate formula of $B \to D^{(*)}\ell\nu$, leptpn masses are set to zero; however, muon mass effect may not be neglected[54][55][56]. In case of $B \to D^{(*)}\mu\nu$ decays, an additional phase space factor

$$f_{ps} \equiv \left(1 - \frac{m_\ell^2}{q^2}\right)^2 \tag{334}$$

need to be inserted in the decay rate formula. Since

$$q^2 = m_B^2 + m_D^2 - 2m_B m_D w \tag{335}$$

the phase space factor can be

$$f_{ps} = \left(1 - \frac{m_\ell^2}{m_B^2 + m_D^2 - 2m_B m_D w}\right)^2 = \left(1 - \frac{1}{1 + r^2 - 2rw}\frac{m_\ell^2}{m_B^2}\right)^2 \tag{336}$$

Moreover, in case of $B \to D\mu\nu$ decays, form factor contribution to the differential decay rate is given by (see section 10),

$$\mathcal{J}_D(w) = [(1+r)h_+(w)]^2 \tag{337}$$

If we include lepton mass [55],

$$\mathcal{J}_D(w) \propto |f_+(w)|^2 + \frac{m_\ell^2}{2q^2}\left(|f_+(w)|^2 + 3\frac{|(m_B^2 - m_D^2)f_+(w) + q^2 f_-(w)|^2}{4m_B^2 m_D^2(w^2-1)}\right) \tag{338}$$

where

$$Rf_+ = h_+ - \frac{1-r}{1+r}h_- \tag{339}$$

and

$$R^{-1}f_- = \left[\left(\frac{w+1}{2} - \frac{1}{R}\right)h_+ + \left(\frac{w-1}{2}\frac{1+r}{1-r}\right)h_-\right]\frac{m_B^2 - m_D^2}{q^2} \tag{340}$$

In hevy quark limit we can set $h_-(w) = 0$. Thus,

$$f_- = \frac{R^2(m_B^2 - m_D^2)}{q^2}\left(\frac{w+1}{2} - \frac{1}{R}\right)f_+ \tag{341}$$

Then

$$\mathcal{J}_D(w) = (1+r)^2|h_+(w)|^2\left[1 + \mathcal{K}_D(w)\frac{m_\ell^2}{m_B^2}\right] \tag{342}$$

where

$$\mathcal{K}_D(w) \equiv \left[1 + 3\left(\frac{1-r}{1+r}\right)^2\left(\frac{w+1}{w-1}\right)\right]\frac{1}{2(1+r^2-2rw)} \tag{343}$$

Hence, we need an additional weight for $B \to D\mu\nu$ decays

$$W_D = \left(1 - \frac{1}{1+r^2-2rw}\frac{m_\ell^2}{m_B^2}\right)^2 \left[1 + \mathcal{K}_D(w)\frac{m_\ell^2}{m_B^2}\right] \tag{344}$$

In case of $B \to D^*\mu\nu$ decays

$$\mathcal{J}_{WD}(R_1, R_2, \rho^2) = (w+1)^2[h_{A_1}]^2(\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2) \tag{345}$$

need to be replaced by [55]

$$\mathcal{J}_{WD}(R_1, R_2, \rho^2) = (w+1)^2[h_{A_1}]^2 \left[(\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2) + (\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2 + \frac{3}{2}\tilde{h}_t^2)\frac{m_\ell^2}{2q^2}\right] \tag{346}$$

where

$$\tilde{h}_t(w) \equiv (1-r)\tilde{H}_t(w) \tag{347}$$

where $\widetilde{H}_t(w)$ is defined by

$$H_t(w) = -m_B \frac{R^*(1-r^2)(w+1)}{2\sqrt{1-2wr+r^2}} h_{A_1}(w)\widetilde{H}_t(w) \tag{348}$$

and H_t is given by

$$H_t(w) = \frac{\sqrt{(q^2-m_B^2-m_D^2)^2 - 4m_B^2m_D^2}}{m_D\sqrt{q^2}} \left(\frac{2q^2}{m_B+m_D}A_2(w) - 2m_D A_0(w)\right) \tag{349}$$

where

$$A_2(w) = \frac{R_2(w)}{R}h_{A_1} \tag{350}$$

and if we define a form facrot ratio

$$R_3(w) \equiv \frac{h_{A_3} - rh_{A_2}}{h_{A_1}} \tag{351}$$

then,

$$A_0(w) = \frac{1}{(1+r)R}\left[(w+1)h_{A_1} + (r-w)R_3(w)h_{A_1}\right] \tag{352}$$

Thus, by putting everything together, we get

$$\tilde{h}_t(w) = \sqrt{w^2-1}\left(1 + \frac{r-w}{w+1}R_3 - \frac{1-2rw+r^2}{r(w+1)}R_2\right) \tag{353}$$

Then

$$\mathcal{J}_{WD}(R_1, R_2, \rho^2) = (w+1)^2[h_{A_1}]^2(\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2)\left[1 + \mathcal{K}_{D^*}(w)\frac{m_\ell^2}{m_B^2}\right] \tag{354}$$

	Result	Difference from Nominal fit
ρ_D^2	1.132 ± 0.066 (5.9 %)	-3.25 %
ρ^2	1.326 ± 0.064 (4.8 %)	-3.35 %
R_1	1.527 ± 0.104 (6.8 %)	-4.17 %
R_2	0.684 ± 0.084 (12.3 %)	+7.94 %
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.02282 ± 0.00042 (1.9 %)	-0.24 %
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.05272 ± 0.00056 (1.1 %)	+0.54 %
χ^2/ndof (P-value)	492/464 (0.17)	

Table 53: [**Muon mass effect**] Fit results.

where

$$\mathcal{K}_{D^*}(w) \equiv \left[1 + \frac{3}{2} \frac{\tilde{h}_t^2}{\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2}\right] \frac{1}{2(1 + r^2 - 2rw)} \tag{355}$$

Hence, we need an additional weight for $B \to D^* \mu\nu$ decays

$$W_{D^*} = \left(1 - \frac{1}{1 + r^2 - 2rw} \frac{m_\ell^2}{m_B^2}\right)^2 \left[1 + \mathcal{K}_{D^*}(w) \frac{m_\ell^2}{m_B^2}\right] \tag{356}$$

Since the contribution of $\mathcal{K}_{D^*}(w)$ term is small, we approximate $R_3(w) \simeq 1$ in practical calculation. We apply these weights to $B \to D^{(*)} \mu\nu$ decays.
Fit results are given in Table 53. Plots to show the effect of the weights are given in Fifure 69-71 for $B \to D\mu\nu$ decays and in Fifure 72-74 for $B \to D^* \mu\nu$ decays.

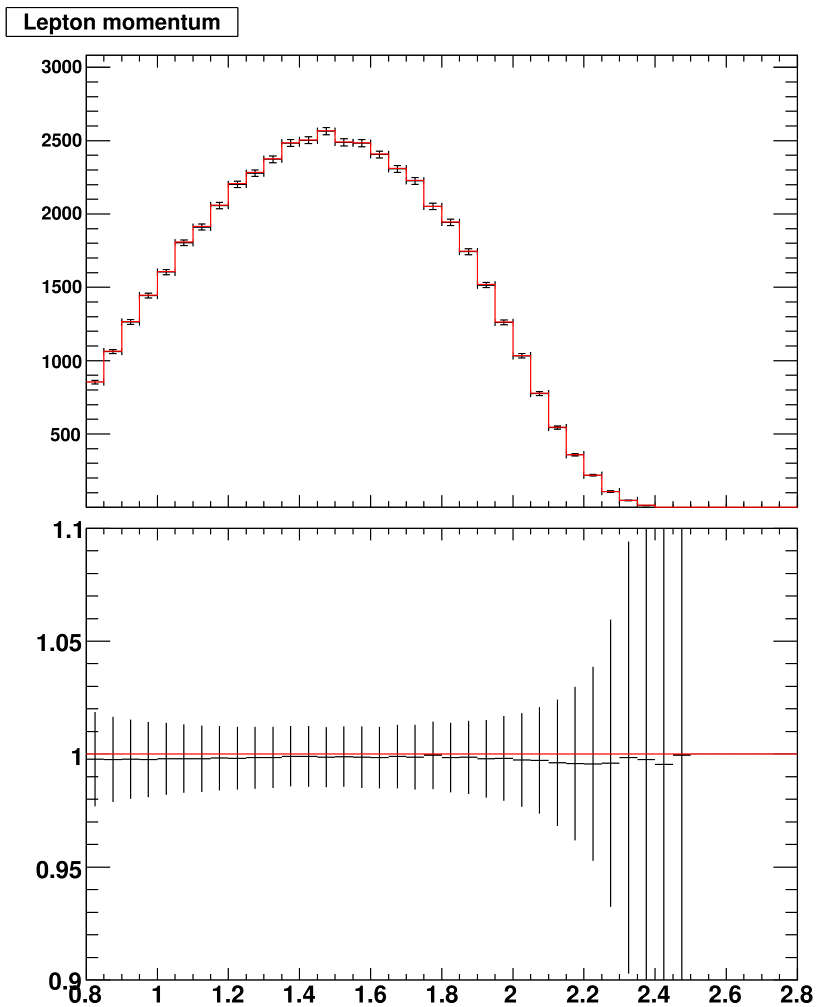


Figure 69: [**Muon mass effect ($B \to D\mu\nu$) :** p_ℓ] p_ℓ plot. Black points are corrected for muon mass effect and red histgram is original. Bottom plot ahows ratio = corrected / original.

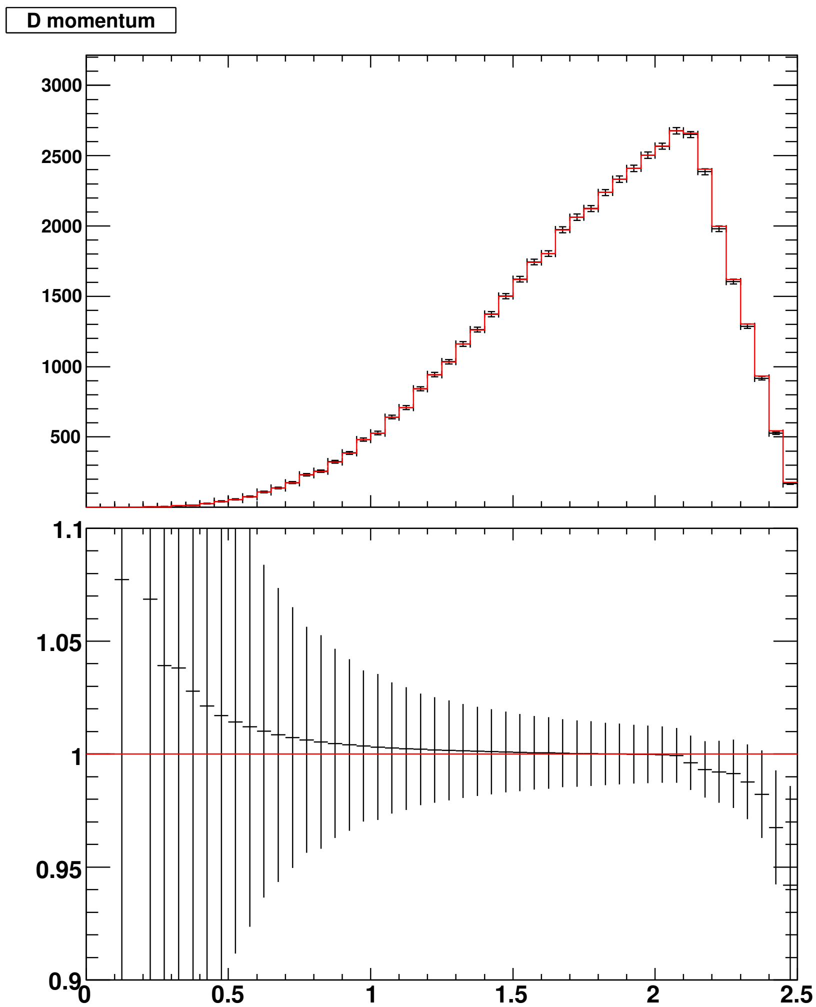


Figure 70: [**Muon mass effect ($B \to D\mu\nu$) :** p_D] p_D plot. Black points are corrected for muon mass effect and red histgram is original. Bottom plot ahows ratio = corrected / original.

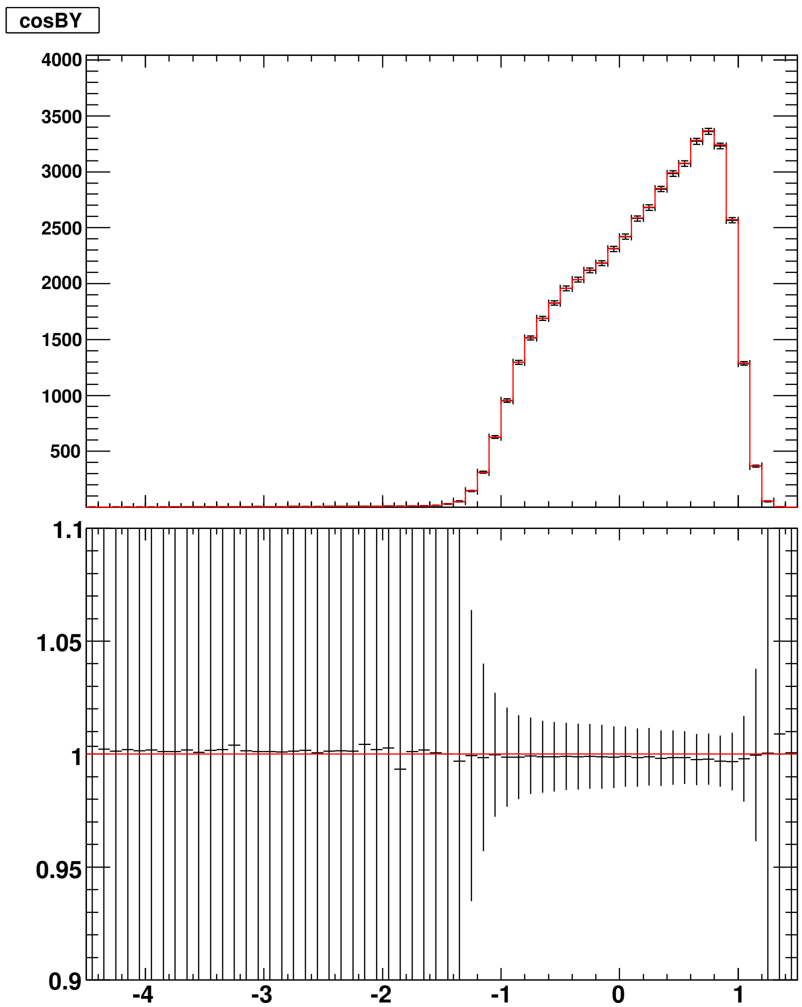


Figure 71: [**Muon mass effect** $(B \to D\mu\nu)$ **:** $\cos\theta_{B-Dl}$] $\cos\theta_{B-Dl}$ plot. Black points are corrected for muon mass effect and red histgram is original. Bottom plot ahows ratio = corrected / original.

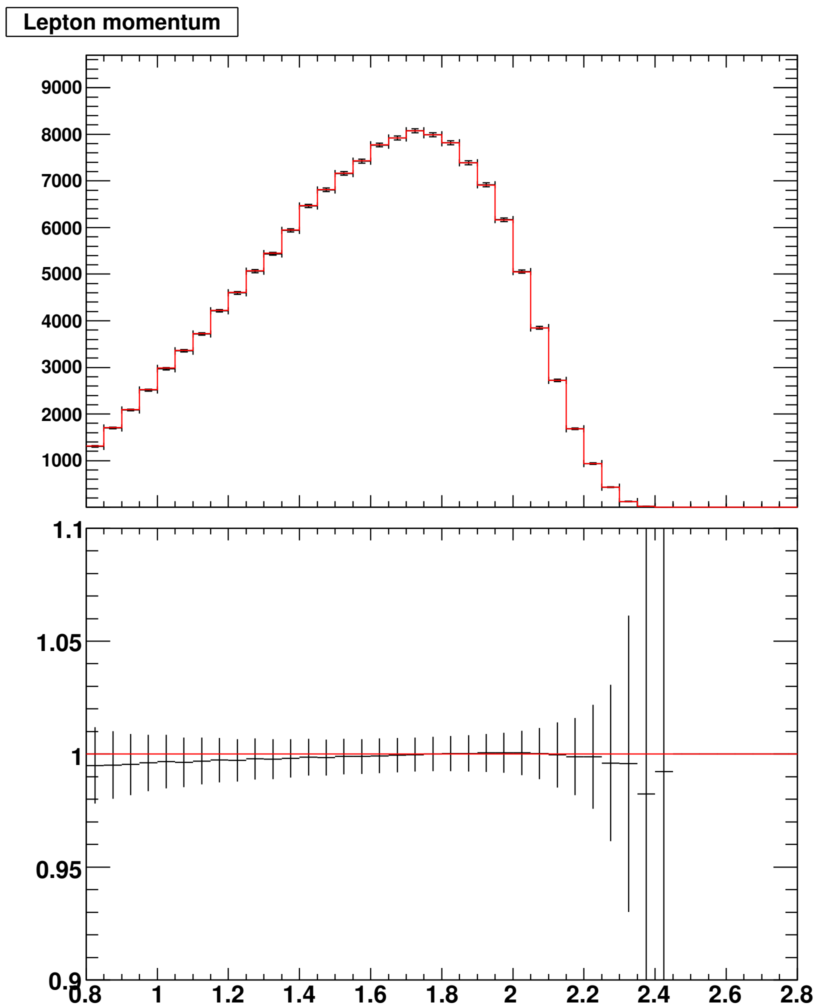


Figure 72: [**Muon mass effect** $(B \to D^*\mu\nu)$ **:** p_ℓ] p_ℓ plot. Black points are corrected for muon mass effect and red histgram is original. Bottom plot ahows ratio = corrected / original.

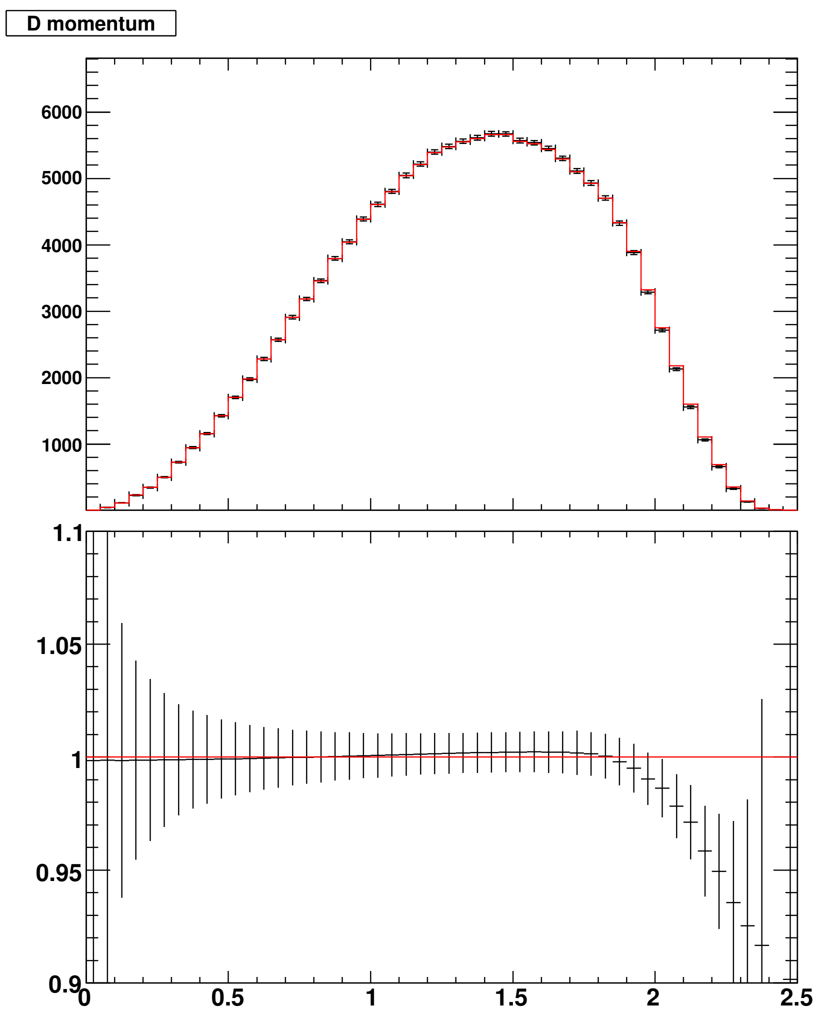


Figure 73: [**Muon mass effect ($B \to D^{*}\mu\nu$) :** p_D] p_D plot. Black points are corrected for muon mass effect and red histgram is original. Bottom plot ahows ratio = corrected / original.

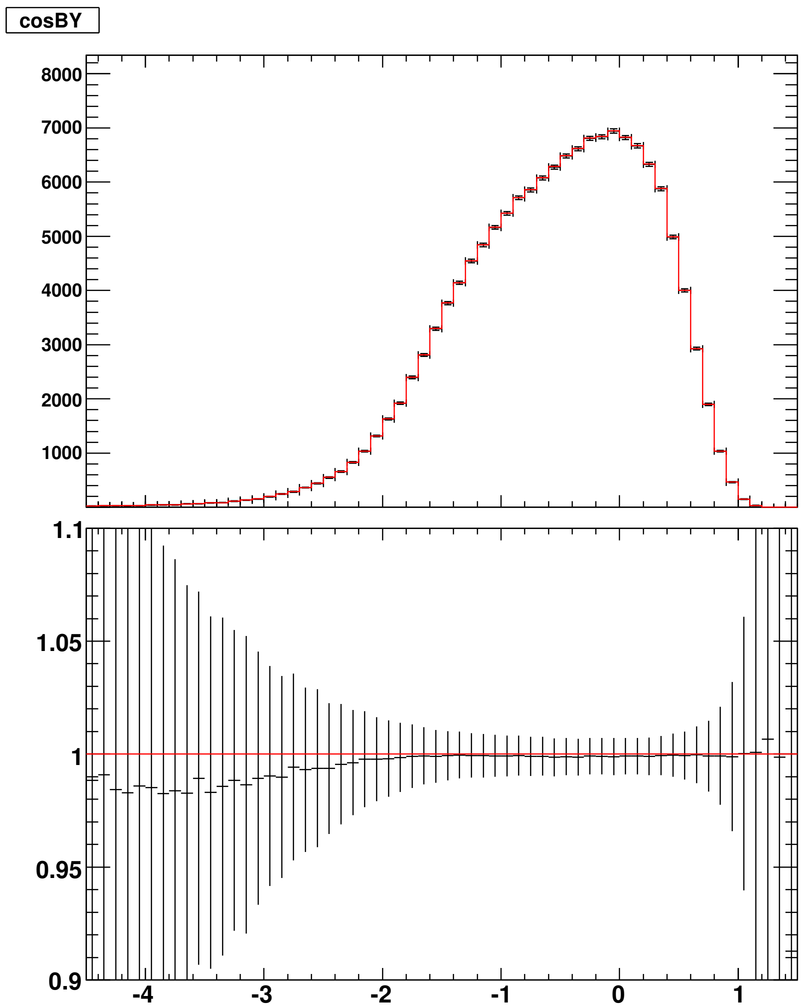


Figure 74: [**Muon mass effect** $(B \rightarrow D^*\mu\nu)$ **:** $\cos\theta_{B-Dl}$] $\cos\theta_{B-Dl}$ plot. Black points are corrected for muon mass effect and red histgram is original. Bottom plot ahows ratio = corrected / original.

20 Calculation of $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$

20.1 $\mathcal{G}(1)|V_{cb}|$

Differential decay rate of $B \to D\ell\nu$ decays are given by

$$\frac{d\Gamma(B \to D\ell\nu)}{dw} = \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3(w^2-1)^{3/2}\, \mathcal{J}_D(w) \tag{357}$$

where

$$\mathcal{J}_D(w) = [(1+r)h_+(w) - (1-r)h_-(w)]^2 \tag{358}$$

and

$$r = \frac{m_D}{m_B} \tag{359}$$

In HQET model, we can set

$$\begin{aligned} h_+(w) &= h_+(1)[1 - 8\rho_D^2 z + (51\rho_D^2 - 10)z^2 - (252\rho_D^2 - 84)z^3] \\ h_-(w) &= 0 \end{aligned} \tag{360}$$

where

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}} \tag{361}$$

Thus, differential decay rate is

$$\begin{aligned} &\frac{d\Gamma(B \to D\ell\nu)}{dw} \\ &= \frac{G_F^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{3/2}\,(1+r)^2[1 - 8\rho_D^2 z + (51\rho_D^2 - 10)z^2 - (252\rho_D^2 - 84)z^3]^2 \\ &\quad \times (\mathcal{G}(1)|V_{cb}|)^2 \end{aligned} \tag{362}$$

Note that $\mathcal{G}(1) = h_+(1)$. By integrating over w, we get

$$\begin{aligned} &\frac{\mathcal{B}(B \to D\ell\nu)}{\tau_B} \\ &= \frac{G_F^2 m_B^5}{48\pi^3} r^3 (1+r)^2 \int (w^2-1)^{3/2}[1 - 8\rho_D^2 z + (51\rho_D^2 - 10)z^2 - (252\rho_D^2 - 84)z^3]^2 dw \\ &\quad \times (\mathcal{G}(1)|V_{cb}|)^2 \end{aligned} \tag{363}$$

where τ_B the lifetime of the B meson. If we define

$$C^D \equiv \sqrt{\frac{48\pi^3}{G_F^2 m_B^5 \tau_B r^3 (1+r)^2}} \tag{364}$$

$$I_1^D \equiv \int (w^2-1)^{3/2}(1 - 10z^2 + 84z^3)^2 dw \tag{365}$$

$$I_2^D \equiv 2\int (w^2-1)^{3/2}(1 - 10z^2 + 84z^3)(8z - 51z^2 + 252z^3) dw \tag{366}$$

$$I_3^D \equiv \int (w^2-1)^{3/2}(8z - 51z^2 + 252z^3)^2 dw \tag{367}$$

then

$$\mathcal{G}(1)|V_{cb}| = C^D \mathcal{B}(B \to D\ell\nu)^{1/2}[I_1^D - \rho_D^2 I_2^D + (\rho_D^2)^2 I_3^D]^{-1/2} \tag{368}$$

Thus, using fit results $\mathcal{B}(B \to D\ell\nu)$ and ρ_D^2, we can calculate $\mathcal{G}(1)|V_{cb}|$. We use the following numerical values

$$\begin{aligned}
\tau_{B^+} &= 1.638 \times 10^{-12}/6.58211915 \times 10^{-25} \;\; \text{GeV}^{-1} \\
m_{B^+} &= 5.2790 \;\; \text{GeV} \\
m_{D^0} &= 1.8645 \;\; \text{GeV} \\
G_F &= 1.166371 \times 10^{-5} \;\; \text{GeV}^{-2}
\end{aligned} \tag{369}$$

20.2 $\mathcal{F}(1)|V_{cb}|$

Differential decay rate of $B \to D^*\ell\nu$ decays are given by

$$\frac{d\Gamma(B \to D^*\ell\nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2} \mathcal{J}_{WD}(R_1, R_2, \rho^2) \tag{370}$$

where

$$\mathcal{J}_{WD}(R_1, R_2, \rho^2) = (w+1)^2 [h_{A_1}]^2 (\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2) \tag{371}$$

Here, the form factors are given by

$$\begin{aligned}
&\tilde{h}_\pm(w) \equiv (1-r)\widetilde{H}_\pm(w) = \sqrt{1-2wr+r^2}\left(1 \mp \sqrt{\tfrac{w-1}{w+1}} R_1(w)\right) \\
&\tilde{h}_0(w) \equiv (1-r)\widetilde{H}_0(w) = (1-r) + (w-1)\left(1 - R_2(w)\right) = (w-r) - (w-1)R_2(w) \\
&h_{A_1}(w) = h_{A_1}(1)\left[1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3\right]
\end{aligned} \tag{372}$$

where

$$\begin{aligned}
R_1(w) &= R_1 - 0.12(w-1) + 0.05(w-1)^2 \\
R_2(w) &= R_2 + 0.11(w-1) - 0.06(w-1)^2 \\
r &= \tfrac{m_{D^*}}{m_B} \\
z &= \tfrac{\sqrt{w+1}-\sqrt{2}}{\sqrt{w+1}+\sqrt{2}}
\end{aligned} \tag{373}$$

Thus, the differential decay rate is

$$\begin{aligned}
&\frac{d\Gamma(B \to D^*\ell\nu)}{dw} \\
&= \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2} (w+1)^2 \\
&\quad \times [\mathcal{F}(1)]^2 [1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3]^2 (\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2)
\end{aligned} \tag{374}$$

Note that $\mathcal{F}(1) = h_{A_1}(1)$. By integrating over w, we get

$$\begin{aligned}
\frac{\mathcal{B}(B \to D^*\ell\nu)}{\tau_B} = \frac{G_F^2 m_B^5}{48\pi^3} r^3 \left(\mathcal{F}(1)|V_{cb}|\right)^2 \\
\int (w^2-1)^{1/2}(w+1)^2(\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2) \\
\times [1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3]^2 dw
\end{aligned} \tag{375}$$

where τ_B the lifetime of the B meson. Let us simplify this equation. By defining

$$\begin{aligned} k_1 &\equiv -0.12(w-1) + 0.05(w-1)^2 \\ k_2 &\equiv 0.11(w-1) - 0.06(w-1)^2 \end{aligned} \tag{376}$$

we get

$$\begin{aligned} &\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2 \\ &\quad = (1-2wr+r^2)\left[1 - 2\sqrt{\frac{w-1}{w+1}}(R_1+k_1) + \frac{w-1}{w+1}(R_1+k_1)^2\right] \\ &\qquad +(1-2wr+r^2)\left[1 + 2\sqrt{\frac{w-1}{w+1}}(R_1+k_1) + \frac{w-1}{w+1}(R_1+k_1)^2\right] \\ &\qquad +(w-r)^2 - 2(w-r)(w-1)(R_2+k_2) + (w-1)^2(R_2+k_2)^2 \\ &\quad = (2+w^2-6wr+3r^2) + 2(1-2wr+r^2)\frac{w-1}{w+1}(R_1^2+2k_1R_1+k_1^2) \\ &\qquad -2(w-r)(w-1)(R_2+k_2) + (w-1)^2(R_2^2+2k_2R_2+k_2^2) \end{aligned} \tag{377}$$

Thus, if we define

$$\begin{aligned} A &\equiv (2+w^2-6wr+3r^2) + 2(1-2wr+r^2)\frac{w-1}{w+1}k_1^2 \\ &\qquad -2(w-r)(w-1)k_2 + (w-1)^2k_2^2 \end{aligned} \tag{378}$$

$$B_1 \equiv 4(1-2wr+r^2)\frac{w-1}{w+1}k_1 \tag{379}$$

$$D_1 \equiv 2(1-2wr+r^2)\frac{w-1}{w+1} \tag{380}$$

$$B_2 \equiv -2(w-r)(w-1) + 2k_2(w-1)^2 \tag{381}$$

$$D_2 \equiv (w-1)^2 \tag{382}$$

then

$$\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2 = A + B_1R_1 + D_1R_1^2 + B_2R_2 + D_2R_2^2 \tag{383}$$

If we further define

$$\begin{aligned} E_1 &\equiv (w^2-1)^{1/2}(w+1)^2(1-15z^2+91z^3)^2 \\ E_2 &\equiv 2(w^2-1)^{1/2}(w+1)^2(1-15z^2+91z^3)(8z-53z^2+231z^3) \\ E_3 &\equiv (w^2-1)^{1/2}(w+1)^2(8z-53z^2+231z^3)^2 \end{aligned} \tag{384}$$

and

$$I_1 \equiv \int AE_1 dw \tag{385}$$

$$I_2 \equiv \int AE_2 dw \tag{386}$$

$$I_3 \equiv \int AE_3 dw \tag{387}$$

$$I_4 \equiv \int B_1 E_1 dw \tag{388}$$

$$I_5 \equiv \int B_1 E_2 dw \tag{389}$$

$$I_6 \equiv \int B_1 E_3 dw \tag{390}$$

$$I_7 \equiv \int C_1 E_1 dw \tag{391}$$

$$I_8 \equiv \int C_1 E_2 dw \tag{392}$$

$$I_9 \equiv \int C_1 E_3 dw \tag{393}$$

$$I_{10} \equiv \int B_2 E_1 dw \tag{394}$$

$$I_{11} \equiv \int B_2 E_2 dw \tag{395}$$

$$I_{12} \equiv \int B_2 E_3 dw \tag{396}$$

$$I_{13} \equiv \int C_2 E_1 dw \tag{397}$$

$$I_{14} \equiv \int C_2 E_2 dw \tag{398}$$

$$I_{15} \equiv \int C_2 E_3 dw \tag{399}$$

and also

$$C \equiv \sqrt{\frac{48\pi^3}{G_F^2 m_B^5 \tau_B r^3}} \tag{400}$$

$$\begin{aligned}
\mathcal{F}(1)|V_{cb}| = C\mathcal{B}(B \to D^*\ell\nu)^{1/2}[&I_1 - I_2\rho^2 + I_3(\rho^2)^2 \\
&+I_4 R_1 - I_5 R_1 \rho^2 + I_6 R_1 (\rho^2)^2 \\
&+I_7 R_1^2 - I_8 R_1^2 \rho^2 + I_9 R_1^2 (\rho^2)^2 \\
&+I_{10} R_2 - I_{11} R_2 \rho^2 + I_{12} R_2 (\rho^2)^2 \\
&+I_{13} R_2^2 - I_{14} R_2^2 \rho^2 + I_{15} R_2^2 (\rho^2)^2]^{-1/2}
\end{aligned} \tag{401}$$

Thus, using fit results $\mathcal{B}(B \to D^*\ell\nu)$, R_1, R_2 and ρ^2, we can calculate $\mathcal{F}(1)|V_{cb}|$. We use the following numerical values

$$\begin{aligned}
\tau_{B^+} &= 1.638 \times 10^{-12}/6.58211915 \times 10^{-25} \quad \mathrm{GeV}^{-1} \\
m_{B^+} &= 5.2790 \quad \mathrm{GeV} \\
m_{D^{*0}} &= 2.0067 \quad \mathrm{GeV} \\
G_F &= 1.166371 \times 10^{-5} \quad \mathrm{GeV}^{-2}
\end{aligned} \tag{402}$$

20.3 Statistical uncertainties

To calculate statistical uncertainties and correlation, for simplicity, we define

$$\begin{aligned}
y_1 &\equiv \mathcal{G}(1)|V_{cb}| \\
y_2 &\equiv \mathcal{F}(1)|V_{cb}| \\
y_3 &\equiv \rho^2 \\
y_4 &\equiv \rho_D^2 \\
y_5 &\equiv R_1 \\
y_6 &\equiv R_2 \\
x_1 &\equiv \rho_D^2 \\
x_2 &\equiv \rho^2 \\
x_3 &\equiv R_1 \\
x_4 &\equiv R_2 \\
x_5 &\equiv \mathcal{B}(B \to D\ell\nu) \\
x_6 &\equiv \mathcal{B}(B \to D^*\ell\nu)
\end{aligned} \tag{403}$$

then, Equation (368) becomes

$$y_1 = C^D x_5^{1/2} (I_1^D - I_2^D x_1 + I_3^D x_1^2)^{-1/2} \tag{404}$$

and Equation (401) becomes

$$\begin{aligned}
y_2 = C x_6^{1/2} [&I_1 - I_2 x_2 + I_3 x_2^2 \\
&+ I_4 x_3 - I_5 x_3 x_2 + I_6 x_3 x_2^2 \\
&+ I_7 x_3^2 - I_8 x_3^2 x_2 + I_9 x_3^2 x_2^2 \\
&+ I_{10} x_4 - I_{11} x_4 x_2 + I_{12} x_4 x_2^2 \\
&+ I_{13} x_4^2 - I_{14} x_4^2 x_2 + I_{15} x_4^2 x_2^2]^{-1/2}
\end{aligned} \tag{405}$$

The error matrix M(y) of y_1 - y_6 is determined by

$$M(y) = \left(\frac{\partial y_i}{\partial x_j}\right) M(x) \left(\frac{\partial y_i}{\partial x_j}\right) \tag{406}$$

where $M(x)$ the error matrix of x_1 - x_6 and

$$\frac{\partial y_1}{\partial x_1} = \frac{C}{2} x_5^{1/2} (I_2^D - 2I_3^D x_1)(I_1^D - I_2^D x_1 + I_3^D x_1^2)^{-3/2} \tag{407}$$

$$\frac{\partial y_1}{\partial x_5} = \frac{C}{2} x_5^{-1/2} (I_1^D - I_2^D x_1 + I_3^D x_1^2)^{-1/2} \tag{408}$$

$$\begin{aligned}\frac{\partial y_2}{\partial x_2} = -\frac{C}{2} x_6^{1/2} (&-I_2 + 2I_3 x_2 - I_5 x_3 + 2I_6 x_3 x_2 - I_8 x_3^2 + 2I_9 x_3^2 x_2 \\ &-I_{11} x_6 + 2I_{12} x_6 x_2 - I_{14} x_6^2 + 2I_{15} x_6^2 x_2) F^{-3/2}\end{aligned} \tag{409}$$

$$\frac{\partial y_2}{\partial x_3} = -\frac{C}{2} x_6^{1/2} (I_4 - I_5 x_2 + I_6 x_2^2 + 2I_7 x_3 - 2I_8 x_3 x_2 + 2I_9 x_3 x_2^2) F^{-3/2} \tag{410}$$

$$\frac{\partial y_2}{\partial x_4} = -\frac{C}{2} x_6^{1/2} (I_{10} - I_{11} x_2 + I_{12} x_2^2 + 2I_{13} x_4 - 2I_{14} x_4 x_2 + 2I_{15} x_4 x_2^2) F^{-3/2} \tag{411}$$

$$\frac{\partial y_2}{\partial x_6} = \frac{C}{2} x_6^{-1/2} F^{-1/2} \tag{412}$$

$$\frac{\partial y_3}{\partial x_1} = 1 \tag{413}$$

$$\frac{\partial y_4}{\partial x_2} = 1 \tag{414}$$

$$\frac{\partial y_5}{\partial x_3} = 1 \tag{415}$$

$$\frac{\partial y_6}{\partial x_4} = 1 \tag{416}$$

where

$$\begin{aligned}F = I_1 &- I_2 x_2 + I_3 x_2^2 + I_4 x_3 - I_5 x_3 x_2 + I_6 x_3 x_2^2 + I_7 x_3^2 - I_8 x_3^2 x_2 + I_9 x_3^2 x_2^2 \\ &+ I_{10} x_4 - I_{11} x_4 x_2 + I_{12} x_4 x_2^2 + I_{13} x_4^2 - I_{14} x_4^2 x_2 + I_{15} x_4^2 x_2^2\end{aligned} \tag{417}$$

and all other partial derivatives are zero.
The correlation coefficient, for example, between $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$ can be calculated by

$$\frac{\mathrm{cov}(y_1, y_2)}{\sigma_{y_1} \sigma_{y_2}} \tag{418}$$

20.4 Extraction of $|V_{cb}|$

Our $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$ agrees with world average [24]. To extract $|V_{cb}|$, we use [35]

$$\mathcal{G}(1) = 1.074 \pm 0.018 \pm 0.016 \tag{419}$$

and [37]

$$\mathcal{F}(1) = 0.921 \pm 0.013 \pm 0.020 \tag{420}$$

These need to be multiplied by the QED correction of 1.007 [58].

21 Systematics of Radiative Corrections (PHOTOS)

In our MC, the module to take care of radiative correction is PHOTOS [46]. In this section, the method to evaluate the systematic uncertainty due to radative corrections.

21.1 Effect of PHOTOS

We study the effect of radiative corrections in generator level by comparing tow MC : one is generated with PHOTOS on and the other is generated with PHOTOS off.

1. We generate full MC with PHOTOS on and off.
2. We produce 3D histograms as we do in our analysis with the above genarated MC.
3. We divide the above histograms, bin by bin, to get on/off ratio histograms.

Thse are plotted in Figure 75 and 76 for electron and in Figure 77 and 78 for muon for $B^+ \to \bar{D}^0 \ell^+ \nu$ mode.

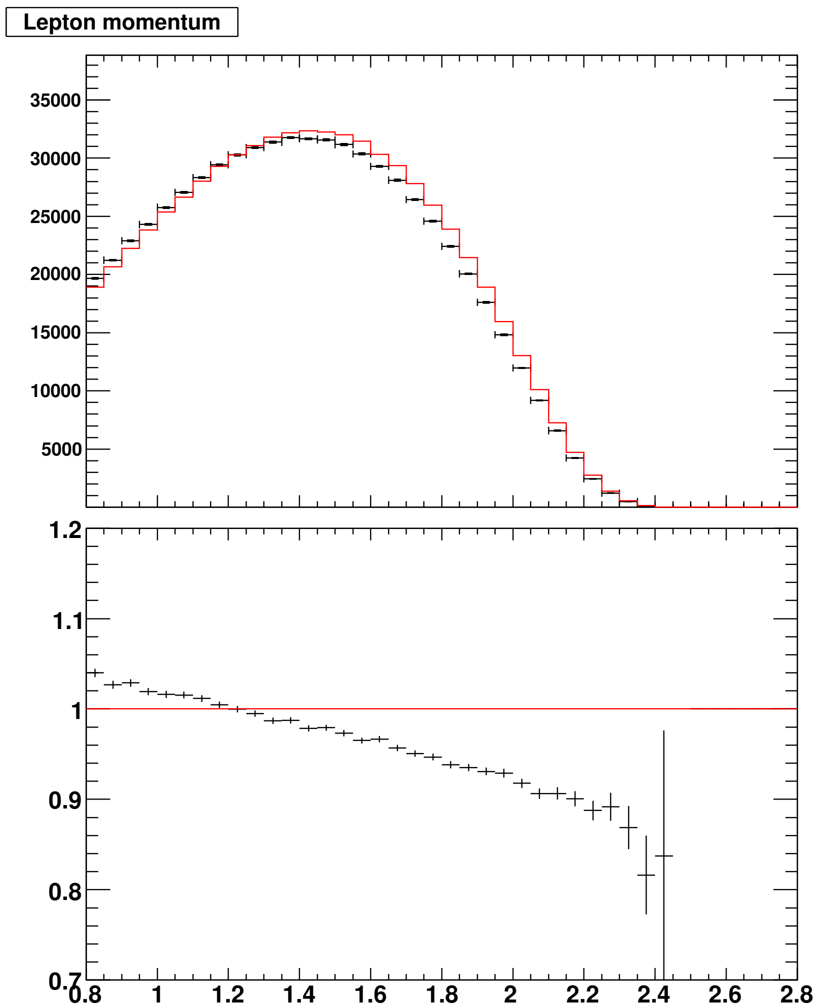


Figure 75: [**PHOTOS on/off (Electron, p_ℓ)**] Lepton momentum plot with PHOTOS on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

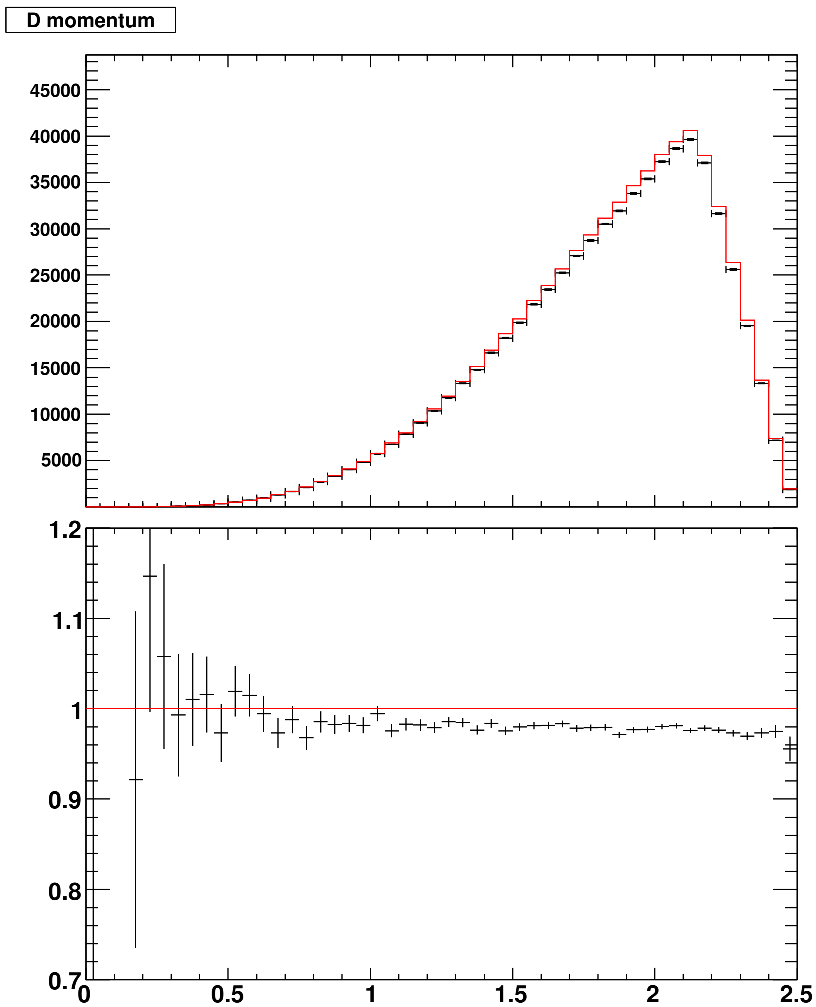


Figure 76: [**PHOTOS on/off (Electron, p_D)**] D momentum plot with PHOTOS on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

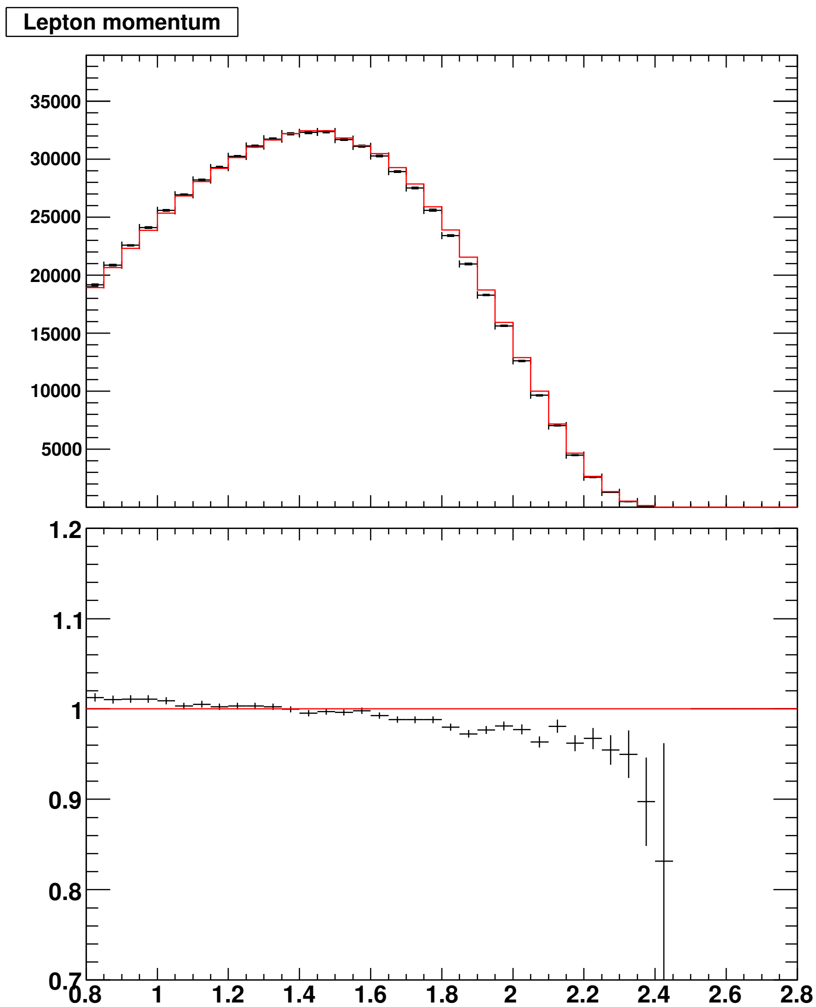


Figure 77: [**PHOTOS on/off (Muon, p_ℓ)**] Lepton momentum plot with PHOTOS on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

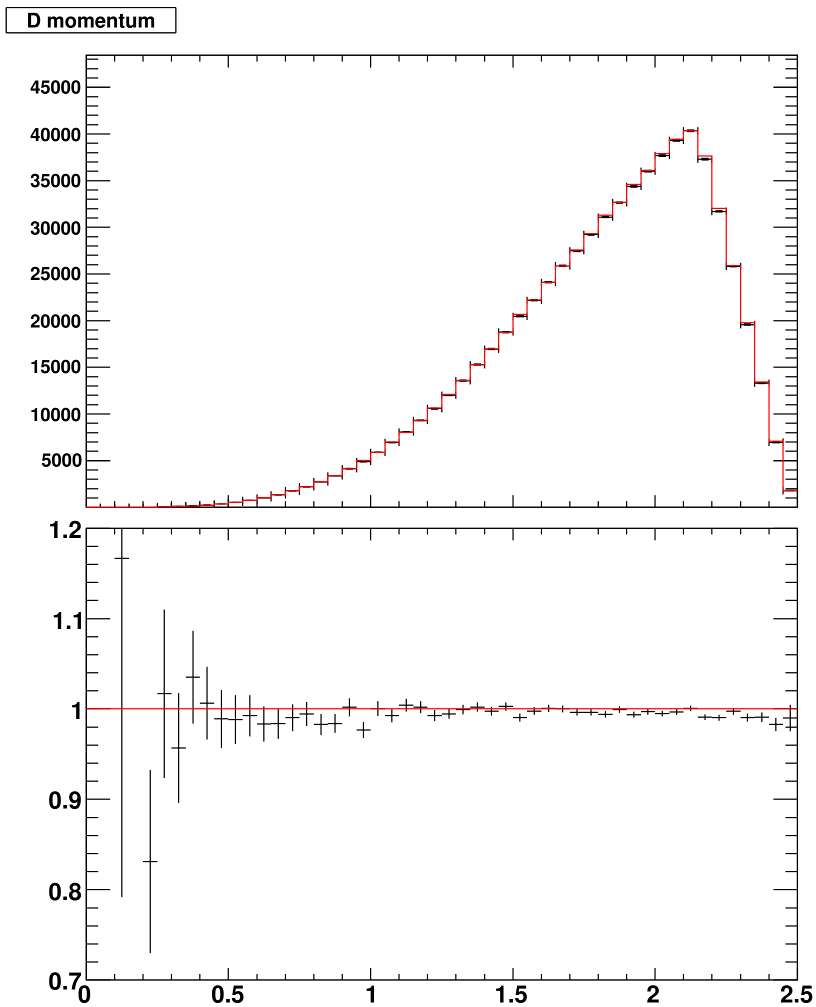


Figure 78: [**PHOTOS on/off (Muon, p_D)**] D momentum plot with PHOTOS on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

21.2 BLOR

For semileptonic K decays, there is a successful program, KLOR, to calculate radiative correction [47]. There is a dedicated study to modify KLOR to apply for semileptonic B decays [48]. The module is called BLOR. The comparison of BLOR on and off are shown in Figure 79 and 80 for electron and in Figure 81 and 82 for muon and for $B^+ \to \bar{D}^0 \ell^+ \nu$ mode. These are almost identical to PHOTOS on and off comparison. These indicate that BLOR is doing a good job to calculate radiative correctiond of semileptonic B decays.

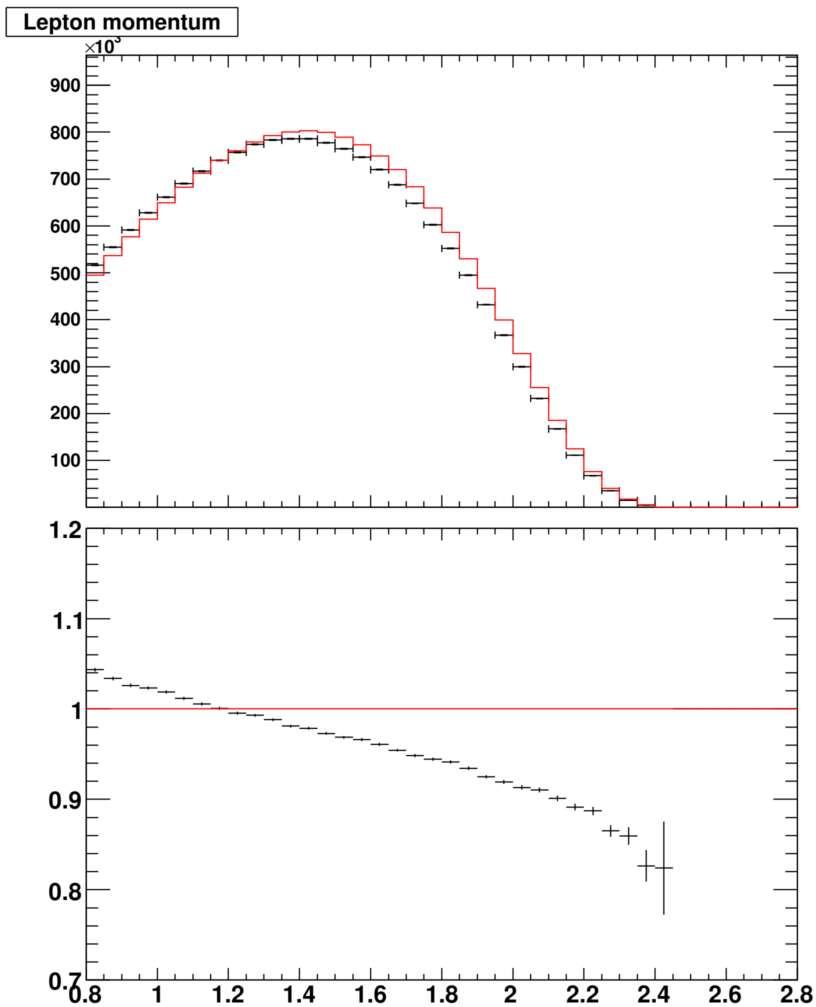


Figure 79: [**BLOR on/off (Electron, p_ℓ)**] Lepton momentum plot with BLOR on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

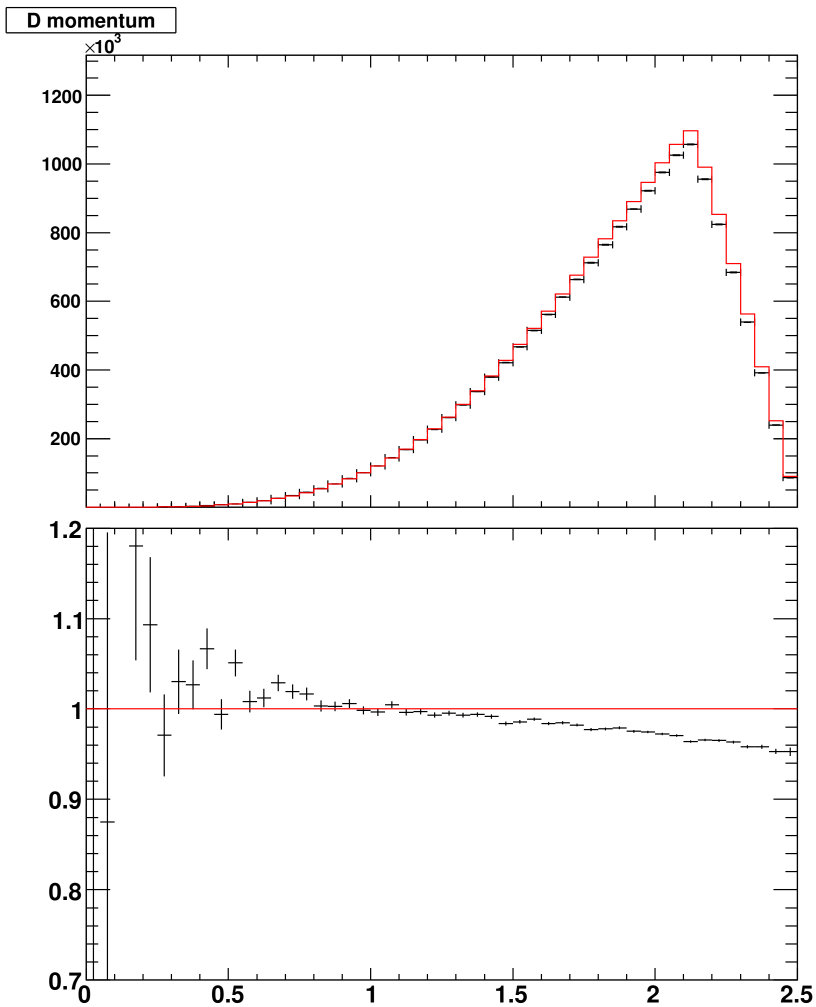


Figure 80: [**BLOR on/off (Electron, p_D)**] D momentum plot with BLOR on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

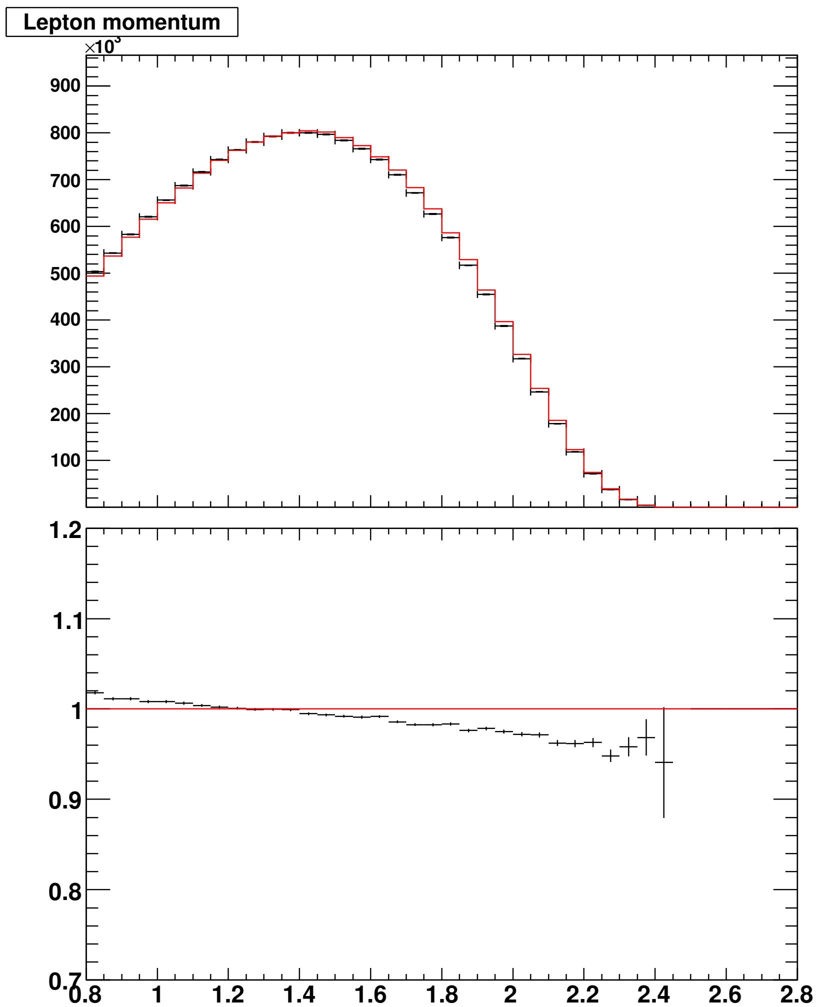


Figure 81: [**BLOR on/off (Muon, p_ℓ)**] Lepton momentum plot with BLOR on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

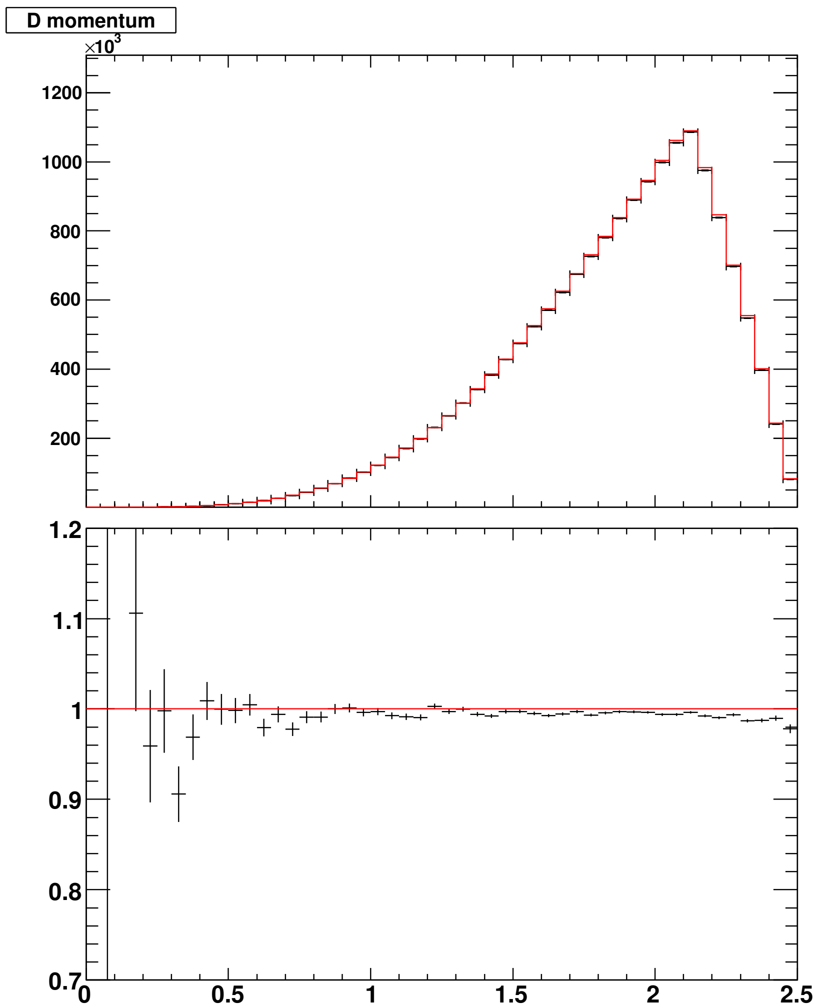


Figure 82: [**BLOR on/off (Muon, p_D)**] D momentum plot with BLOR on and off. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

21.3 Method to estimate systematic uncertainty

We multiply the corresponding histograms which we use for the nominal fitting by the PHOTOS off/on ratio to ger PHOTOS off reconstructed level histograms for systematic fitting. Traditionally, around 25 % of the differences of fit results when we tunrn off PHOTS was taken as systematic uncertainty. In this version we still use the same method.
The following is a plan for the future version :
If we can replace PHOTOS by BLOR, fit difference can be a goood mesure of the uncertainty of PHOTOS. Comparison of PHOTOS and BLOR are plotted in Figure 83 and 84 for electron and in Figure 85 and 86 for muon only for $B^+ \to \bar{D}^0 \ell^+ \nu$ mode. Right now, only $B \to D\ell\nu$ mode is avairable from BLOR. If we can get $B \to D^*\ell\nu$ mode as well, we can use this method. As a trial, we compare the following two fit

- Switch off PHOTOS only for $B \to D\ell\nu$ mode, and take 25 % of difference.
- Use KLOR only for $B \to D\ell\nu$ mode, and take 100 % of difference.

Results are given in Table 54 for electron and in Table 55 for muon.
We cannot draw a reasonable conclusion from this result. We need to wait for $B \to D^*\ell\nu$ mode being avairable.

	25 % of PHOTOS off	BLOR
ρ_D^2	-2.13 %	-2.91 %
ρ^2	-0.06 %	-0.05 %
R_1	-0.14 %	0.19 %
R_2	-3.20 %	-0.04 %
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	-2.55 %	-0.29 %
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	1.15 %	0.11 %

Table 54: [**PHOTOS off vs BLOR systematics (Electron)**] Systematic uncertainties.

	25 % of PHOTOS off	BLOR
ρ_D^2	-0.49 %	0.02 %
ρ^2	-0.20 %	0.10 %
R_1	-0.02 %	0.25 %
R_2	-1.23 %	0.49 %
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	-0.90 %	0.26 %
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.41 %	-0.21 %

Table 55: [**PHOTOS off vs BLOR fit systematics (Muon)**] Systematic uncertainties.

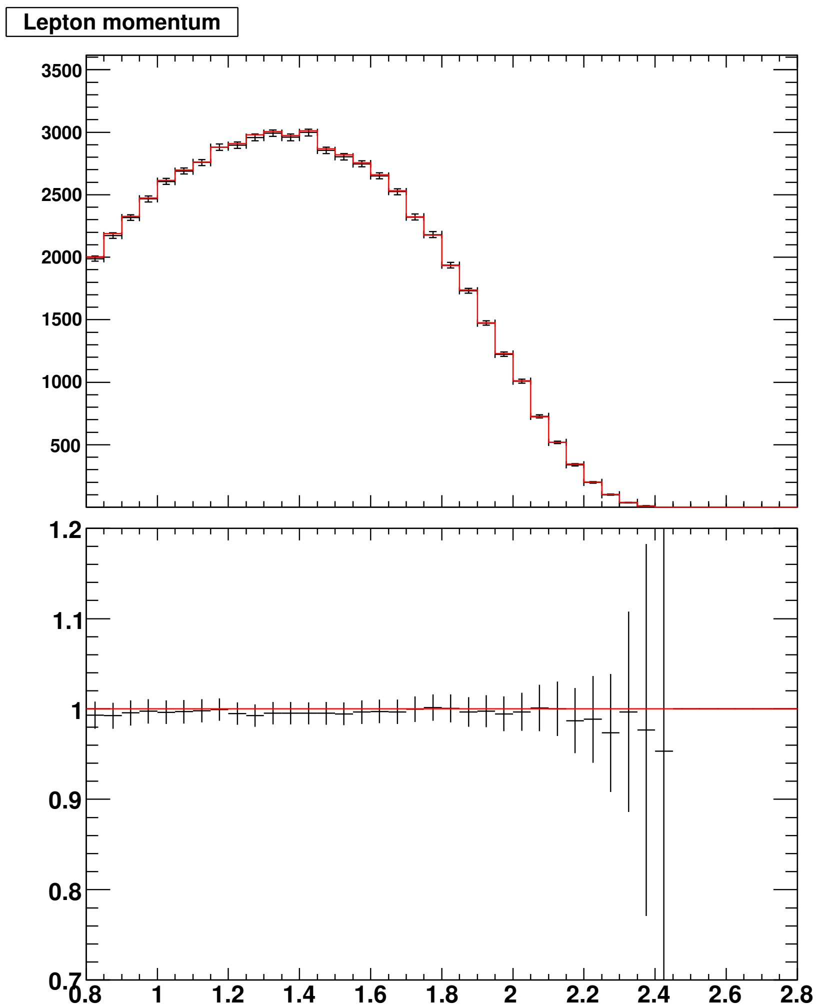


Figure 83: [**BLOR / PHOTOS (Electron, p_ℓ)**] Lepton momentum plot with BLOR and PHOTOS. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

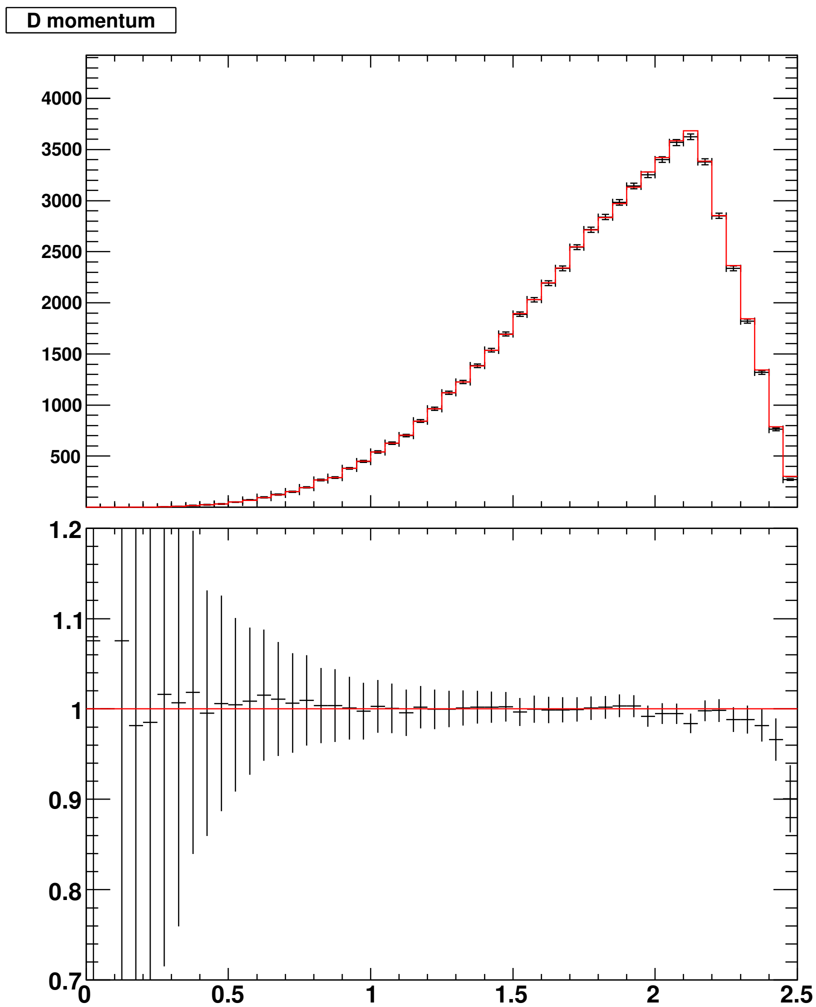


Figure 84: [**BLOR / PHOTOS (Electron, p_D)**] D momentum plot with BLOR and PHOTOS. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

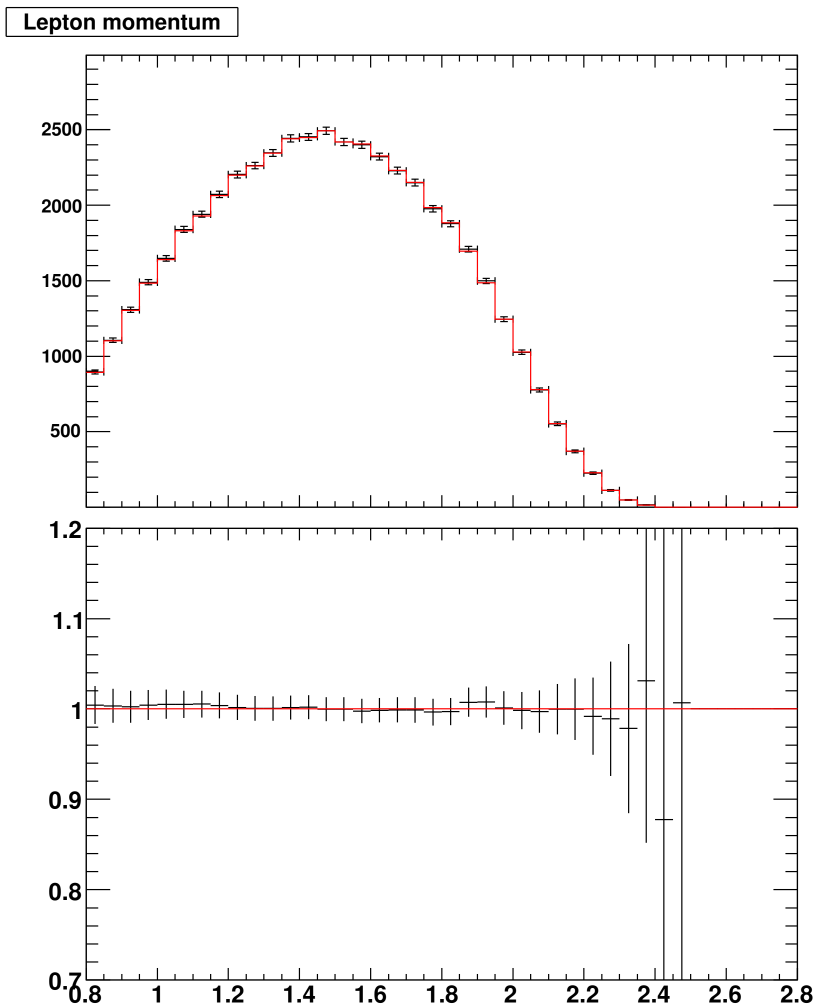


Figure 85: [**BLOR / PHOTOS (Muon, p_ℓ)**] Lepton momentum plot with BLOR and PHOTOS. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

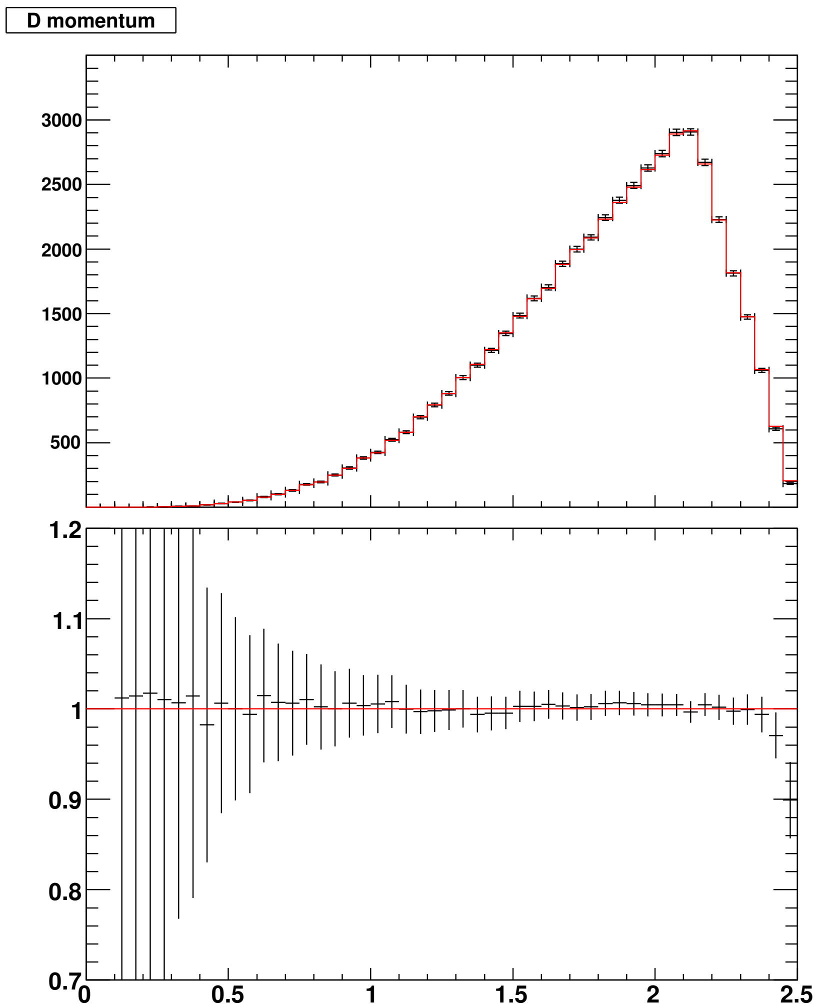


Figure 86: [**BLOR / PHOTOS (Muon, p_D)**] D momentum plot with BLOR and PHOTOS. Black poins are POTOS on and red histogra is PHTOS off. The bottom plot shows the ration on/off.

22 New BF for Nominal fit configuration

We include new results for PDG2008 and calculate new input BFs.

22.1 $B \to D^{(*)}\pi\ell\nu$ branching fraction

From PDG [12]

$$\begin{aligned}
\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) &= 0.0053 \pm 0.0009 \pm 0.0005 \\
\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) &= 0.0064 \pm 0.0015 \pm 0.0002 \\
\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) &= 0.0032 \pm 0.0009 \pm 0.0003 \\
\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) &= 0.0065 \pm 0.0015 \pm 0.0005
\end{aligned} \tag{421}$$

We assume isospin symmetry

$$\begin{aligned}
\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) &= 2\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu) \\
\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) &= 2\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)
\end{aligned} \tag{422}$$

Babar measure [4]

$$\begin{aligned}
\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) &= 0.0152 \pm 0.0012 \pm 0.0010 \\
\mathcal{B}(B^0 \to \bar{D}^{(*)}\pi\ell^+\nu) &= 0.0137 \pm 0.0017 \pm 0.0010
\end{aligned} \tag{423}$$

We take isospin average over B^+ and B^0 decay branching fractions and also take an averge over two measurements assuming maximum correlation between systematic uncertainties.

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0151 \pm 0.0015 \tag{424}$$

22.2 $B \to D^{**}\ell\nu$ decay branching fractions

From HFAG [24]

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\mathcal{B}(D_1^0 \to D^{*+}\pi^-) &= 0.0024 \pm 0.0004 \\
\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-) &= 0.0009 \pm 0.00025
\end{aligned} \tag{425}$$

Using $f_{D_2^*} = 1.7$, we get

$$\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^{(*)+}\pi^-) = 0.00256 \pm 0.00072 \tag{426}$$

Babar measure [15]

$$\begin{aligned}
\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\mathcal{B}(D_1^0 \to D^{*+}\pi^-) &= 0.00297 \pm 0.00017 \pm 0.00017 \\
\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^{(*)+}\pi^-) &= 0.00229 \pm 0.00023 \pm 0.00021 \\
\mathcal{B}(B^0 \to D_1^-\ell^+\nu)\mathcal{B}(D_1^- \to D^{*0}\pi^-) &= 0.00278 \pm 0.00024 \pm 0.00025 \\
\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*-} \to D^{(*)0}\pi^-) &= 0.00177 \pm 0.00026 \pm 0.00011
\end{aligned} \tag{427}$$

We take isospin average of these results assuming maximum correlation

$$\begin{aligned} \mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu) &= 0.00297 \pm 0.00031 \\ \mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu) &= 0.00209 \pm 0.00030 \end{aligned} \tag{428}$$

Babar also measure by assuming isospin symmetry [45]

$$\begin{aligned} \mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu)\mathcal{B}(D_1^0 \to D^{*+}\pi^-) &= 0.0029 \pm 0.0003 \pm 0.0003 \\ \mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu)\mathcal{B}(D_2^{*0} \to D^{+}\pi^-) &= 0.0012 \pm 0.0002 \pm 0.0001 \\ \mathcal{B}(B^+ \to \bar{D}_1^{\prime 0} \ell^+ \nu)\mathcal{B}(D_1^{\prime 0} \to D^{*+}\pi^-) &= 0.0030 \pm 0.0003 \pm 0.0004 \\ \mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+ \nu)\mathcal{B}(D_0^{*0} \to D^{+}\pi^-) &= 0.0032 \pm 0.0004 \pm 0.0004 \end{aligned} \tag{429}$$

Using $f_{D_2^*} = 1.7$, we get

$$\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu)\mathcal{B}(D_2^{*0} \to D^{(*)+}\pi^-) = 0.00192 \pm 0.00036 \tag{430}$$

Combining these three results we obtain

$$\begin{aligned} \mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu)\mathcal{B}(D_1^0 \to D^{*+}\pi^-) &= 0.0028 \pm 0.0003 \\ \mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu)\mathcal{B}(D_2^{*0} \to D^{(*)+}\pi^-) &= 0.0021 \pm 0.0003 \\ \mathcal{B}(B^+ \to \bar{D}_1^{\prime 0} \ell^+ \nu)\mathcal{B}(D_1^{\prime 0} \to D^{*+}\pi^-) &= 0.0030 \pm 0.0005 \\ \mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+ \nu)\mathcal{B}(D_0^{*0} \to D^{+}\pi^-) &= 0.0032 \pm 0.0006 \end{aligned} \tag{431}$$

Using

$$\begin{aligned} &\mathcal{B}(D_1^0 \to D^{*+}\pi^-) = \mathcal{B}(D_2^{*0} \to D^{(*)+}\pi^-) \\ &= \mathcal{B}(D_1^{\prime 0} \to D^{*+}\pi^-) = \mathcal{B}(D_0^{*0} \to D^{+}\pi^-) \\ &= 0.6667 \end{aligned} \tag{432}$$

we get

$$\begin{aligned} \mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu) &= 0.00419 \pm 0.00044 \\ \mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu) &= 0.00309 \pm 0.00050 \\ \mathcal{B}(B^+ \to \bar{D}_1^{\prime 0} \ell^+ \nu) &= 0.0045 \pm 0.00075 \\ \mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+ \nu) &= 0.0048 \pm 0.00085 \end{aligned} \tag{433}$$

By summing these 4 BF

$$\mathcal{B}(B^+ \to \bar{D}^{**0} \ell^+ \nu) = 0.0166 \pm 0.0022 \tag{434}$$

22.3 Non-resonant $B \to D^{(*)}\pi\ell\nu$ decay branching fractions

Comparing the sum of $B^+ \to D^{**0}\ell^+\nu$ BF with $B \to D^{(*)}\pi\ell\nu$ BF, there is only tiny space for non-resonant $\mathcal{B}(B^+ \to \bar{D}\pi\ell^+\nu)$.

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) - \mathcal{B}(B^+ \to \bar{D}^{**0}\ell^+\nu) = -0.0015 \pm 0.0040 \tag{435}$$

SP10 MC uses

$$\begin{aligned}
&\mathcal{B}(\text{non-resonant } B^+ \to D^-\pi^+\ell^+\nu) = 0.0040 \pm 0.0024\\
&\mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^0\pi^0\ell^+\nu) = 0.0020 \pm 0.0012\\
&\mathcal{B}(\text{non-resonant } B^+ \to D^{*-}\pi^+\ell^+\nu) = 0.0012 \pm 0.0008\\
&\mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) = 0.0006 \pm 0.0004
\end{aligned} \tag{436}$$

However these are too big to fit in the tiny space. Thus we shrink them keeping same D^*/D ratio

$$\begin{aligned}
&\mathcal{B}(\text{non-resonant } B^+ \to D^-\pi^+\ell^+\nu) = 0.0010 \pm 0.0010\\
&\mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^0\pi^0\ell^+\nu) = 0.0005 \pm 0.0005\\
&\mathcal{B}(\text{non-resonant } B^+ \to D^{*-}\pi^+\ell^+\nu) = 0.0003 \pm 0.0003\\
&\mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) = 0.00015 \pm 0.00015
\end{aligned} \tag{437}$$

22.4 Branching fraction ratios

Summing over all BF in Equation (433) and (437), we get

$$\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0191 \pm 0.0047 \tag{438}$$

This is consistent with Equation(424).
We calculate the branching fraction ratios by assuming the errors of above branching fractions are 100 % correlated.

$$\begin{aligned}
&f_{D_2^*/D_1} \equiv \tfrac{\mathcal{B}(B^+\to\bar{D}_2^{*0}\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^0\ell^+\nu)} = 0.74 \pm 0.20\\
&f_{D_0^* D\pi T/D_1 D_2^* T} \equiv \tfrac{\mathcal{BT}(B^+\to\bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 0.87 \pm 0.43\\
&f_{D_1' D^*\pi T/D_1 D_2^* T} \equiv \tfrac{\mathcal{BT}(B^+\to\bar{D}_1^{\prime 0}/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+\to\bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 0.68 \pm 0.25\\
&f_{D\pi/D_0^*} \equiv \tfrac{\mathcal{B}(B^+\to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_0^{*0}\ell^+\nu)} = 0.21 \pm 0.21\\
&f_{D^*\pi/D_1'} \equiv \tfrac{\mathcal{B}(B^+\to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+\to\bar{D}_1^{*0}\ell^+\nu)} = 0.07 \pm 0.07
\end{aligned} \tag{439}$$

22.5 $B \to D^{(*)}\pi\pi\ell\nu$ decay branching fraction

From PDG [12], inclusive BF is

$$\mathcal{B}(B^+ \to X_c\ell^+\nu) = 0.108 \pm 0.004 \tag{440}$$

babar measure [4]

$$\begin{aligned}
&\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) = 0.0233 \pm 0.0009 \pm 0.0009\\
&\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) = 0.0583 \pm 0.0015 \pm 0.0030\\
&\mathcal{B}(B^+ \to D^{(*)}\pi\ell^+\nu) = 0.0152 \pm 0.0012 \pm 0.0010
\end{aligned} \tag{441}$$

Thus 0.108 - 0.0233 - 0.0583 - 0.0152 = 0.0112 is missing. Thus, we fix

$$\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell^+\nu) = 0.011 \pm 0.011 \tag{442}$$

As we assign 100 % uncertainty, this covers systematic uncertainty from this unmesured decay BF.

22.6 f_{+0} and others

From PDG [12]

$$f_{+0} = 1.065 \pm 0.026 \tag{443}$$

We also fix R_1 and R_2 to Babar mesurements [40].

23 Fit Results

Using new BF values in section 22, we performed fits. We fit electron and muon smaples separately and combine them after calculating systematic uncertainties.

23.1 Fit results

Fit results are given in Table 56 including systematic uncertainties and combined results. Detailes of Systematic uncertainties are given in section 24 and method to combine electron and muon results are explained in section 25. Statistical correlations between fitted variables are given in Table 57. Plots are shown in Figures 87 - 90

23.2 Toy MC study

We perform toy MC fits with parameters of nominal fit results to determine the size of fit bias. Results are given in Table 58. Since bias is very small, we do not correct for the bias.

	Electron	Muon	Combined
ρ_D^2	$1.26 \pm 0.05 \pm 0.07$	$1.15 \pm 0.06 \pm 0.09$	$1.22 \pm 0.04 \pm 0.07$
ρ^2	$1.22 \pm 0.02 \pm 0.07$	$1.23 \pm 0.03 \pm 0.07$	$1.21 \pm 0.02 \pm 0.07$
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$ (%)	$2.41 \pm 0.03 \pm 0.14$	$2.29 \pm 0.04 \pm 0.16$	$2.36 \pm 0.03 \pm 0.12$
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$ (%)	$5.42 \pm 0.03 \pm 0.22$	$5.23 \pm 0.04 \pm 0.37$	$5.37 \pm 0.02 \pm 0.21$
χ^2/ndof (P-value)	424/470 (0.94)	496/466 (0.16)	2.1/4 (0.72)

Table 56: [R_1 **and** R_2 **fixed fit results**] Fit results.

	Electron				Muon			
	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$
ρ_D^2	1				1			
ρ^2	-0.304	1			-0.308	1		
$\mathcal{B}(D^0 \ell \nu)$	0.303	0.177	1		0.274	0.195	1	
$\mathcal{B}(D^{*0} \ell \nu)$	-0.386	0.077	-0.527	1	-0.395	0.073	-0.519	1

Table 57: [**Statistical Correlation coefficients (**R_1 **and** R_2 **fixed fit)**] Statistical Correlation coefficients.

Parameters	Elecrron		Muon	
	Mean	r.m.s.	Mean	r.m.s.
ρ_D^2	-0.0135	1.0078	-0.0038	0.9978
ρ^2	-0.0513	1.0035	-0.0510	0.9987
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.0660	1.0187	0.05899	1.0164
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.0953	1.0158	0.08900	1.0076
Uncertainty	0.0158	0.0112	0.0158	0.0112

Table 58: [**Toy MC :** R_1 **and** R_2 **fixed fit**] Mean and standard deviation (r.m.s.) of pull distributions of toy MC. The uncertainty on mean and standard deviation are also listed in the bottom line. Pulls are calculated from the difference to input values.

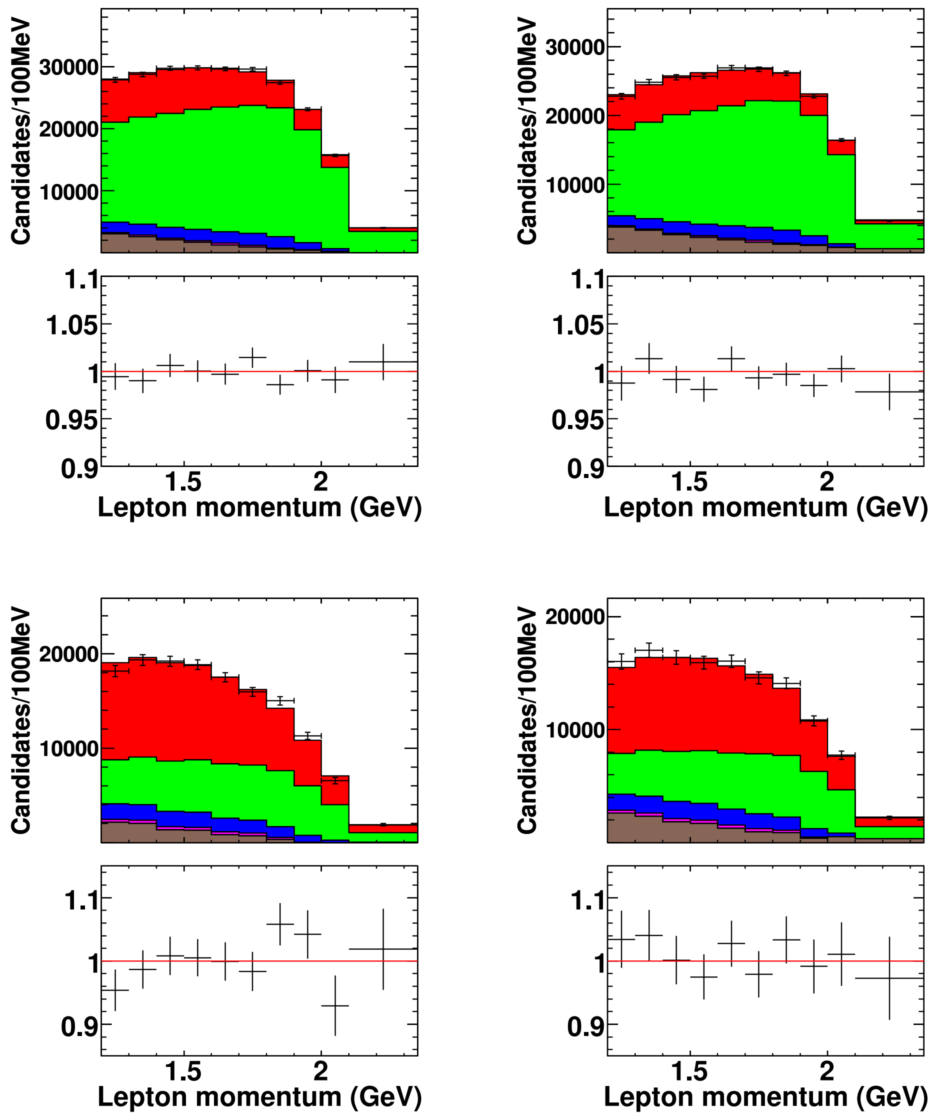


Figure 87: [p_ℓ **(Nominal fit)**] Data and fit results onto the lepton momentum. The left column is for the electron sample and the right column is for the muon sample. The top row is $D^0\ell$ and the bottom row is $D^+\ell$. Black points are OnPeak - OffPeak data. The red histogram is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{(*)}\pi\ell\nu$, magenta is $B \to D^{(*)}\pi\pi\ell\nu$, and brown is background.

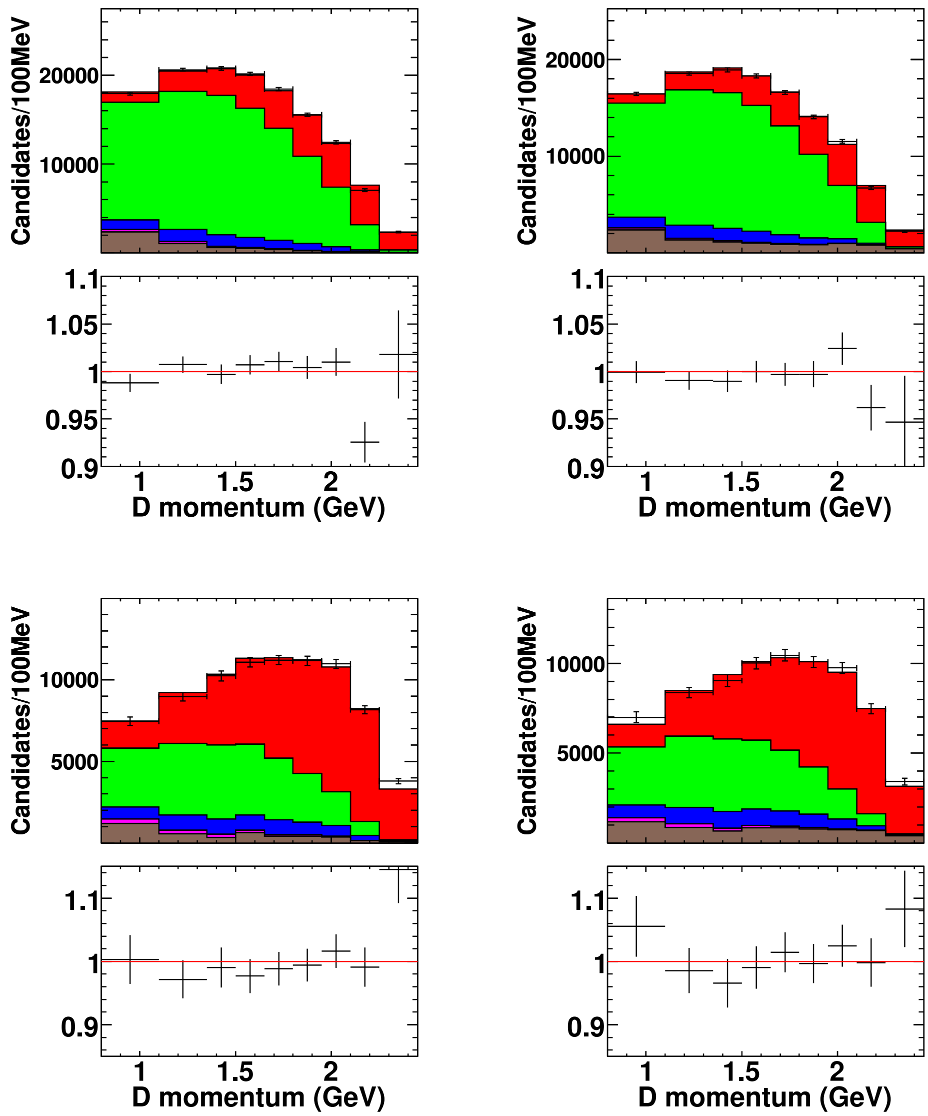


Figure 88: [p_D **(Nominal fit)**] Data and fit results onto the D momentum. The left column is for the electron sample and the right column is for the muon sample. The top row is $D^0\ell$ and the bottom row is $D^+\ell$. Black points are OnPeak - OffPeak data. The red histogram is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{(*)}\pi\ell\nu$, magenta is $B \to D^{(*)}\pi\pi\ell\nu$, and brown is background.

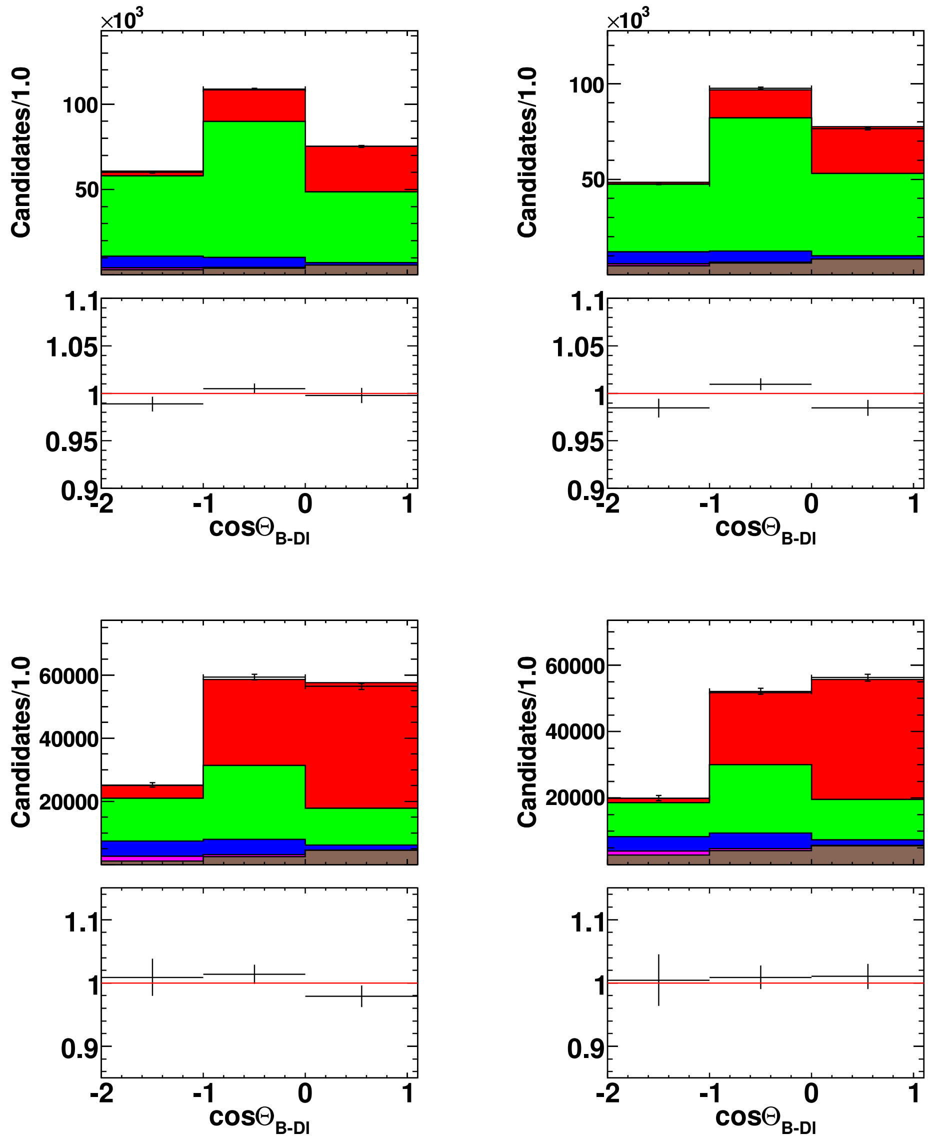


Figure 89: [$\cos\theta_{B-Dl}$ **(Nominal fit)**] Data and fit results onto the $\cos\theta_{B-Dl}$. The left column is for the electron sample and the right column is for the muon sample. The top row is $D^0\ell$ and the bottom row is $D^+\ell$. Black points are OnPeak - OffPeak data. The red histogram is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{(*)}\pi\ell\nu$, magenta is $B \to D^{(*)}\pi\pi\ell\nu$, and brown is background.

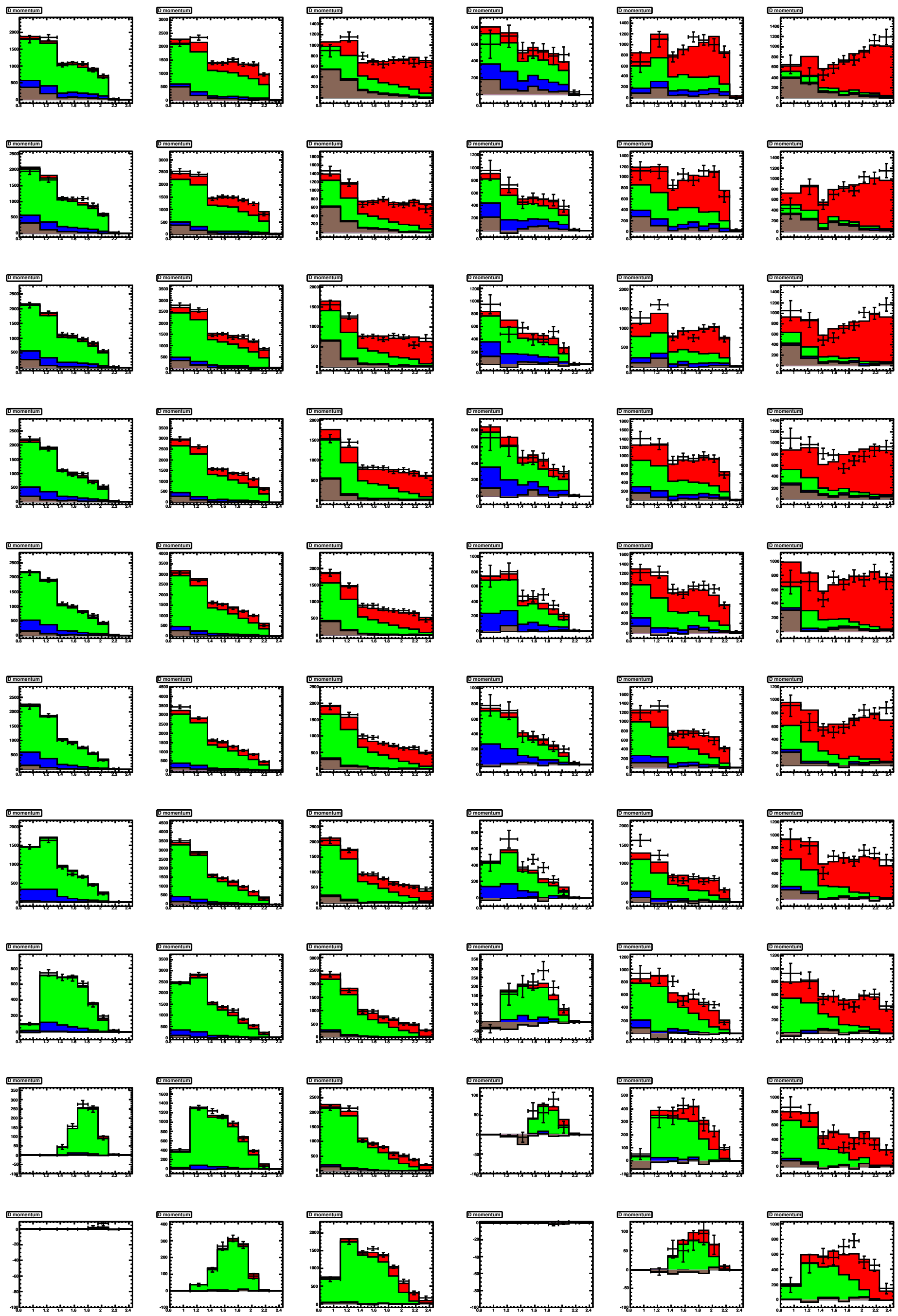


Figure 90: [**All bin plots (Nominal fit)**] Data and fit results showing all bins for the electron sample. The left three columns are for the D^0e sample and the right three columns are for the D^+e sample. The tree columns correspond to the three bins of $\cos\theta_{B-Dl}$. The ten rows correspond to the ten bins of p_ℓ. The eight bins in each plot correspond to the eight bins of p_D. Black points are OnPeak - OffPeak data. The red histogram is $B \to D\ell\nu$, green is $B \to D^*\ell\nu$, blue is $B \to D^{(*)}\pi\ell\nu$ and $B \to D^{(*)}\pi\pi\ell\nu$, and brown is background. Note that the y-axis range varies plot by plot.

24 Systematic Uncertainties

Systematic uncertainties are shown in Tables 59 and 60. Numbers are negative when the corresponding fitted values decrease as a input parameter increases. Details are given in the following subsections.

24.1 Form factors

24.1.1 R_1 and R_2

We use R_1 and R_2 as input

$$\begin{aligned} &R_1 = 1.429, \quad \sigma_1 = \sqrt{0.061^2 + 0.044^2} \\ &R_1 = 0.827, \quad \sigma_2 = \sqrt{0.038^2 + 0.022^2} \\ &\text{correlation : } \text{corr}_{12} = -0.84 \end{aligned} \tag{444}$$

If we define

$$\begin{aligned} &v_1 \equiv \sigma_1^2 = 0.061^2 + 0.044^2 \\ &v_2 \equiv \sigma_2^2 = 0.038^2 + 0.022^2 \\ &\text{covariance : } c \equiv \sigma_1\sigma_2\text{corr}_{12} = -0.84 \times \sqrt{(0.061^2 + 0.044^2)(0.038^2 + 0.022^2)} \end{aligned} \tag{445}$$

The error matrix is given by

$$M = \begin{pmatrix} v_1 & c \\ c & v_2 \end{pmatrix} \tag{446}$$

We transfrom varibles ($x_1 = R_1 \pm \sigma_1, x_2 = R_2 \pm \sigma_2$) to (y_1, y_2) to diagonalize the error matrix. Then we can vary y_1 and y_2 independently to estimate the systematic uncertainty due to R_1 and R_2.
First we need to find eigen values λ of the error matrix.

$$(v_1 - \lambda)(v_2 - \lambda) - c^2 = 0 \tag{447}$$

This gives

$$\lambda = \frac{v_1 + v_2}{2} \pm \sqrt{\left(\frac{v_1 - v_2}{2}\right)^2 + c^2} = 0.00714, 0.00045 \tag{448}$$

Then eigen vectors (a, b) are give by solving following two equations

$$\begin{aligned} &v_1 a + cb = \lambda a \\ &a^2 + b^2 = 1 \end{aligned} \tag{449}$$

These give

$$a = \frac{c}{\sqrt{(\lambda - v_1)^2 + c^2}}, \quad b = \frac{\lambda - v_1}{\sqrt{(\lambda - v_1)^2 + c^2}}, \tag{450}$$

The variable change is given by

$$\begin{aligned} y_1 &= a_1 x_1 + a_2 x_2 \\ y_2 &= b_1 x_1 + b_2 x_2 \end{aligned} \tag{451}$$

where (a_1, b_1) and (a_2, b_2) correspond to two eigenvalues given by Equation (448). The uncertainty of y is given by $\sqrt{\lambda}$.
Now, we need to inverse the variation of y_1 and y_2 into the variations of x_1 and x_2. Variation of y_1 gives

$$\begin{aligned} \pm\sqrt{\lambda} &= a_1 \delta R_1 + a_2 \delta R_2 \\ 0 &= b_1 \delta R_1 + b_2 \delta R_1 \end{aligned} \tag{452}$$

These give

$$\delta R_1 = \pm b_2 \sqrt{\lambda_1} = \pm 0.075, \quad \delta R_2 = \mp b_1 \sqrt{\lambda_1} = \mp 0.040, \tag{453}$$

Similarly,vVariation of y_1 gives

$$\delta R_1 = \pm a_2 \sqrt{\lambda_2} = \pm 0.010, \quad \delta R_2 = \mp a_1 \sqrt{\lambda_2} = \mp 0.019, \tag{454}$$

24.1.2 $B \to D^{**}\ell\nu$ decay form factors

There are two sources of systematic uncertainty of D^{**} form factors :

- One is the FF slope $\tilde{\tau}' = -1.5$
- The other is the Approximation B$_1$ ($\tau_1 = 0$ and $\tau_2 = 0$).

To estimate systematic uncertainties,

- We change slope to -2.0 and -1.0, and take larger deviation.
- We use the Approximation B$_2$ ($\tau_1 \neq 0$ and $\tau_2 \neq 0$).

The slope range is the one recommended in LLSW paper [41]

24.2 $B \to D^{**}/D^{(*)}\pi\ell\nu$ decay branching fractions

24.2.1 $\mathcal{B}(B^+ \to D^{(*)}\pi\ell\nu)$

As described in section 22, we use

- $\mathcal{B}(B^+ \to D^{(*)}\pi\ell\nu) = 0.0151 \pm 0.0015$

in our fit. We change this BF to ± one standard deviation and take larger deviation as systematic uncertainty.

24.2.2 BF ratios

We have five BF ratios as input (see section 22):

- $f_{D_2^*/D_1} \equiv \frac{\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)} = 0.44 \pm 0.13$
- $f_{D_0^* D\pi T/D_1 D_2^* T} \equiv \frac{\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 1.40 \pm 0.70$
- $f_{D_1' D^*\pi T/D_1 D_2^* T} \equiv \frac{\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)} = 1.08 \pm 0.42$
- $f_{D\pi/D_0^*} \equiv \frac{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)} = 0.19 \pm 0.19$
- $f_{D^*\pi/D_1'} \equiv \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_1^{*0}\ell^+\nu)} = 0.06 \pm 0.006$

We change these ratios to $\pm$ one standard deviation and take larger deviations as systematic uncertainties.

24.2.3 Non-resonant $B \to D^{(*)}\pi\ell\nu$ decay BF

Non-resonant $B \to D^{(*)}\pi\ell\nu$ decay BFs are not well measured. Thus we assign 100 % uncertainty on these BF, which is covered by above BF ratios. We also need to change the ratio

$$f_{D^*\pi/D\pi} \equiv \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)} \tag{455}$$

In nominal fit this ratio is set to 0.3.

- We change the ratio $f_{D^*\pi/D\pi}$ to 0.1 and 1.0

and take larger devation as systematic uncertainty. In practice, we use

- $\mathcal{B}(\text{non-resonant } B^+ \to D^-\pi^+\ell^+\nu) = 0.0010$
- $\mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^0\pi^0\ell^+\nu) = 0.0005$
- $\mathcal{B}(\text{non-resonant } B^+ \to D^{*-}\pi^+\ell^+\nu) = 0.0001$
- $\mathcal{B}(\text{non-resonant } B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) = 0.00005$
- $f_{D_0^* D\pi T/D_1 D_2^* T} = 0.87$
- $f_{D_1' D^*\pi T/D_1 D_2^* T} = 0.64$
- $f_{D\pi/D_0^*} = 0.21$
- $f_{D^*\pi/D_1'} = 0.02$

and

- $\mathcal{B}(\text{non-resonant } B^+ \to D^-\pi^+\ell^+\nu) = 0.0006$

- $\mathcal{B}$(non-resonant $B^+ \to \bar{D}^0\pi^0\ell^+\nu$) $= 0.0003$
- $\mathcal{B}$(non-resonant $B^+ \to D^{*-}\pi^+\ell^+\nu$) $= 0.0006$
- $\mathcal{B}$(non-resonant $B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu$) $= 0.0003$
- $f_{D_0^* D\pi T/D_1 D_2^* T} = 0.78$
- $f_{D_1' D^*\pi T/D_1 D_2^* T} = 0.74$
- $f_{D\pi/D_0^*} = 0.13$
- $f_{D^*\pi/D_1'} = 0.13$

24.3 Effect of $B \to D^{(*)}\pi\pi\ell\nu$ decays

24.3.1 $B \to D^{(*)}\pi\pi\ell\nu$ decay BF

As described in section 22, we assign 100 % uncertainty on this BF

- $\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell\nu) = 0.011 \pm 0.011$

We change this BF to $\pm$ one standard deviation and take larger deviation as systematic uncertainty.

24.3.2 Composition

In nominal fit we mix X, X*, Y and Y* with same ratio. To evaluate systematic uncertainty, we cahnge these ratios

- Vector scalar ratio
  We chanege X*/X = Y*/Y = 2 and 0.5
- X Y ratio
  We cahnge X/Y = X*/Y* = 2 and 0.5

and take larger deviations as systematic uncertainty.
$B \to D^{(*)}\pi\pi\ell\nu$ decays may include $B \to D_1\ell\nu$ (a mix of $D_1 \to D\pi^+\pi^0$ and $D_1 \to D\eta$, $\eta \to \pi^+\pi^-$) decays. Thus,

- We mix 50 % of $D_1 \to D\pi\pi$ into nominal $D^{(*)}\pi\ell\nu$

24.4 Input parameters

24.4.1 D_2^* decay branching fractions

As described in section 9.2, we have one parameter,

- $f_{D_2^*} = 1.7 \pm 0.4$

to calculate D_2^* decay branching fractions. We change $f_{D_2^*}$ to $\pm$ one standard deviation and take larger deviation as systematic uncertainties.

24.4.2 $D^{(*)}$ decay branching fractions

The $D^{(*)}$ decay branching fractions

- $\mathcal{B}(D^{*+} \to D^0\pi^+) = 0.677 \pm 0.005$
- $\mathcal{B}(D^0 \to K^-\pi^+) = 0.0389 \pm 0.0005$
- $\mathcal{B}(D^+ \to K^-\pi^+\pi^+) = 0.0922 \pm 0.0021$

are also input parameters of our fitting. We change these BFs to ± one standard deviation and take larger deviation as systematic uncertainties.

24.4.3 t_{+0}

Another input parameter of our fitting is lifetime ratio

- $t_{+0} \equiv \frac{\tau_{B^+}}{\tau_{B^0}} = 1.071 \pm 0.009$

We change t_{+0} to ± one standard deviation and take larger deviation as systematic uncertainties.

24.4.4 f_{+0}

Υ(4S) decay BF ratio

- $f_{+0} \equiv \frac{f_{+-}}{f_{00}} = 1.065 \pm 0.026$

is also our input. We change f_{+0} to ± one standard deviation and take larger deviation as systematic uncertainties.

24.4.5 Luminosity normalization

For $B\bar{B}$ MC

- There is 1.1 % uncertainty (statistical + systematic) on B counting.

For OffPeak data

- There is 0.25 % uncertainty.

Thus we change luminosity normalization factors by one standard deviation and take larger deviation as systematic uncertainties. .

24.5 Various corrections

24.5.1 Beam energy re-weighting

Measured OnPeak beam energy (mean) has uncerainty of 0.2 MeV. Thus we change the mean value in Table 22 by ± 0.2 MeV and take larger differences as systematic uncertainty.

	+0.2 MeV	Normalization	-0.2 MeV	Normalization
Run1	10.577818	0.985596	10.577418	0.990933
Run2	10.578868	1.08792	10.578468	0.989203
Run3	10.578593	1.05449	10.578193	1.01362
Run4	10.577899	0.968835	10.577499	0.987612

24.5.2 Lepton PID correction

We evaluate electron PID systematic uncertainty as follows :

- PID Efficiency
  The following weight, wf, approximates the correction to the PID weights due to the impact of other tracks and photons in the event [50]
  - $wf = 0.966435$ for $p_\ell < 1.0$ GeV
  - $wf = 0.92123 + 0.056185 p_\ell - 0.010980 p_\ell^2$ for $1.0 < p_\ell < 2.5$ GeV
  - $wf = 0.9930675$ for $p_\ell > 2.5$ GeV

  where p_ℓ is the electron momentum.
- Fake rate
  We increase and decrease fake rate correction factor by 15 % and we take larger deviation as systematic uncertainty.

We evaluate muon PID systematic uncertainty as follows :

- Effect on the shape of histograms
  We can evaluate this by fitting without PID correction but keeping the same average efficiency. Average efficiencies are listed below

Run1	0.9349
Run2	0.8876
Run3	0.9211
Run4	0.9075

- Overall scale
  We can take appropriate numbers from previous analysis. Then we multiply PID correction factors by the number and perform fitting.
  We take
  - 4 % for muon [reference]?

  We take larger deviation as systematic uncertainty.
- Fake rate
  We increase and decrease fake rate correction factor by 15 % and we take larger deviation as systematic uncertainty.

24.5.3 Kaon PID correction

We compare the fit results with and without kaon PID correction and take differences as systematic uncertainty.

24.5.4 Tracking efficiency correction

We compare the fit results with and without tracking efficiency correction.

24.5.5 Radiative corrections : PHOTOS

This is explained in section 21.

24.5.6 Bremssrtrahlung

Due to emission of bremsstrahlung photons, the electron loses energy when traveling through the detector material. Since the amount of detector material has unertainty, the effect due to Bremssrtrahlung also has uncertainty. This was studied in detail in [51]. We apply this method in our fitting. First, we create 3D histograms corresponding to more and less detector materials. Then, We perform fit with the two sets of histograms. We take larger deviations as systeatic uncertainties.

24.5.7 Vertexing

The vertexing cut is described in the event selection section 6. Nonimal cuts are

$$\begin{aligned} P^D(\chi^2, dof) &> 0.001 \\ P^B(\chi^2, dof) &> 0.01 \end{aligned} \tag{456}$$

Since this is tight enough, we change them to looser cuts

$$\begin{aligned} P^D(\chi^2, dof) &> 0.0001 \\ P^B(\chi^2, dof) &> 0.001 \end{aligned} \tag{457}$$

to evaluate systematic uncertainty.

	Electron						Muon					
	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	$\mathcal{G}(1)\|V_{cb}\|$	$\mathcal{F}(1)\|V_{cb}\|$	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	$\mathcal{G}(1)\|V_{cb}\|$	$\mathcal{F}(1)\|V_{cb}\|$
(R_1,R_2) set A	0.45	2.78	0.69	-0.38	0.60	0.71	0.47	2.70	0.73	-0.40	0.61	0.70
(R_1,R_2) set B	-0.38	1.04	-0.18	0.30	-0.31	0.49	-0.43	0.98	-0.19	0.30	-0.32	0.48
D^{**} slope	-1.11	-2.65	-0.09	-0.10	-0.68	-0.91	-1.04	-2.73	-0.12	-0.12	-0.60	-0.97
D^{**} B_2	-0.84	0.58	-0.11	0.20	-0.53	0.29	-1.00	0.61	-0.12	0.22	-0.59	0.31
FF total	**1.51**	**4.02**	**0.73**	**0.53**	**1.10**	**1.29**	**1.57**	**4.01**	**0.77**	**0.56**	**1.09**	**1.32**
$\mathcal{B}(D^{(*)}\pi\ell\nu)$	0.42	-0.37	-0.11	-0.88	0.19	-0.56	0.77	-0.43	-0.01	-0.95	0.40	-0.62
$f_{D_2^*/D_1}$	-0.32	0.14	-0.33	0.14	-0.35	0.12	-0.41	0.15	-0.36	0.16	-0.40	0.13
$f_{D_0^* D\pi/D_1 D_2^*}$	-2.16	1.16	-1.52	0.97	-2.02	0.86	-2.91	1.27	-1.53	1.03	-2.34	0.94
$f_{D_1^* D^*\pi/D_1 D_2^*}$	1.10	-0.70	0.82	-0.45	1.03	-0.45	1.46	-0.72	0.85	-0.50	1.19	-0.49
$f_{D\pi/D_0^*}$	-0.75	-1.14	0.31	0.15	-0.27	-0.30	-0.70	-1.10	0.27	0.16	-0.24	-0.29
$f_{D^*\pi/D_1^*}$	-0.18	-0.05	-0.12	0.19	-0.16	0.08	-0.26	-0.04	-0.14	0.21	-0.21	0.09
NR D^*/D ratio	0.70	-0.14	0.27	-0.16	0.53	-0.10	0.83	-0.13	0.27	-0.17	0.58	-0.12
$D^{(*)}\pi\ell\nu$ **total**	**2.69**	**1.82**	**1.81**	**1.42**	**2.38**	**1.17**	**3.55**	**1.89**	**1.83**	**1.53**	**2.76**	**1.27**
$\mathcal{B}(D^{(*)}\pi\pi\ell\nu)$	1.14	-2.03	0.25	-1.29	0.77	-1.30	1.87	-1.77	0.40	-1.21	1.18	-1.19
X*/X & Y*/Y	0.59	-1.18	0.09	-0.28	0.38	-0.52	0.70	-1.04	0.08	-0.24	0.41	-0.47
X/Y & X*/Y*	0.73	-0.85	0.20	-0.66	0.51	-0.61	1.03	-0.79	0.24	-0.63	0.66	-0.58
$D_1 \to D\pi\pi$	2.12	-1.59	0.74	-1.08	1.60	-1.06	2.60	-1.53	0.75	-1.07	1.77	-1.04
$D^{(*)}\pi\pi\ell\nu$ **total**	**2.58**	**2.96**	**0.81**	**1.83**	**1.89**	**1.86**	**3.44**	**2.68**	**0.89**	**1.75**	**2.27**	**1.75**

Table 59: [**Systematics (R_1 and R_2 fixed fit)**] Systematic uncertainties. Numbers are given in %.

	Electron						Muon					
	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	$\mathcal{G}(1)\|V_{cb}\|$	$\mathcal{F}(1)\|V_{cb}\|$	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	$\mathcal{G}(1)\|V_{cb}\|$	$\mathcal{F}(1)\|V_{cb}\|$
$f_{D_2^*}$	-0.11	-0.01	-0.09	0.06	-0.11	0.03	-0.13	-0.01	-0.09	0.06	-0.11	0.03
D^{*+} BF	0.70	-0.01	0.43	-0.34	0.61	-0.17	0.76	-0.01	0.41	-0.34	0.60	-0.17
D^+ BF	-1.37	-0.42	-2.16	0.31	-1.85	0.02	-1.33	-0.43	-2.08	0.28	-1.73	-0.00
D^0 BF	0.67	0.01	-0.20	-1.64	0.29	-0.82	0.89	0.10	-0.26	-1.69	0.35	-0.81
t_{+0}	0.24	0.17	0.62	0.27	0.45	0.19	0.20	0.17	0.58	0.28	0.39	0.20
f_{+0}	0.84	0.44	0.66	-0.54	0.81	-0.13	0.86	0.49	0.57	-0.53	0.74	-0.10
MC Lumi	0.00	-0.00	-1.11	-1.11	-0.56	-0.55	0.00	-0.00	-1.11	-1.11	-0.56	-0.55
OffPeak Lumi	0.05	0.01	-0.02	-0.00	0.02	0.00	0.06	0.00	-0.02	-0.00	0.02	-0.00
Input total	**1.90**	**0.63**	**2.64**	**2.13**	**2.25**	**1.03**	**1.98**	**0.68**	**2.55**	**2.16**	**2.12**	**1.02**
Beam eng	-0.90	0.64	1.28	-0.55	-1.11	0.48	1.22	-0.10	1.25	-0.66	1.28	-0.36
L PID shape	0.00	0.00	0.00	0.00	0.00	0.00	-3.19	0.05	3.00	4.12	-0.20	2.06
L PID eff	0.48	0.17	1.20	0.83	0.88	0.47	0.09	-0.07	-4.10	-4.17	-1.98	-2.09
L PID Fake	0.03	0.01	-0.01	-0.01	0.02	-0.00	2.52	0.71	-0.58	-0.50	1.03	-0.01
Kaon PID	0.06	0.83	0.27	0.24	0.17	0.39	0.97	0.73	0.35	0.29	0.69	0.39
Tracking eff	-0.93	-0.46	-3.31	-2.02	-2.19	-1.16	-0.54	-0.30	-3.33	-2.11	-1.96	-1.16
PHOTOS	-2.98	-1.07	-2.87	-0.72	-2.98	-0.71	-0.71	-0.62	-0.82	-0.24	-0.78	-0.33
Brems	0.07	-0.00	-0.13	-0.29	-0.03	-0.14	0.00	0.00	0.00	0.00	0.00	0.00
Vertexing	0.81	-0.67	0.63	0.60	0.78	0.08	1.69	-0.77	0.96	0.54	1.38	0.01
Correcs total	**3.38**	**1.71**	**4.77**	**2.46**	**4.04**	**1.57**	**4.75**	**1.46**	**6.37**	**6.32**	**3.67**	**3.21**
Backg total	**1.39**	**1.12**	**0.64**	**0.34**	**1.07**	**0.51**	**1.58**	**1.09**	**0.67**	**0.38**	**1.16**	**0.49**
Total	**5.76**	**5.72**	**5.89**	**4.04**	**5.74**	**3.21**	**7.48**	**5.53**	**7.23**	**7.10**	**5.77**	**4.25**

Table 60: [**Systematics (R_1 and R_2 fixed fit)**] Systematic uncertainties. Numbers are given in %.

24.6 Background

As describe in section 9, we re-weight branching fractions of background components. We change those BF to $\pm$ one standard deviation and take larger deviation as systematic uncertainties. Results are given in Table 61. We use different method for inclusive $B \to D$ decays as explained in the following subsection.

24.6.1 D momentum dependent inclusive $B \to D(D_s)$ BF re-weighting

As described in section 9.1.3, we fit for D (D_s) momentum dependent weight by 4-th (3-rd) order polynomial, and we use the fitted function to re-weight MC. To estimate the systematic uncertainty due to the uncertainty on weights, we do the following procedure :

1. We deviate each weight using Gaussian random numbers.
2. We fit the distribution of weight vs. D momentum using 3rd order polynomial.
3. We re-weight MC by the fitted functions to create 3D histograms.
4. Using the histograms, we perform fitting.
5. We iterate the above 1 to 4 400 times.
6. We make distributions of the deviations from the 400 fit results.
7. We take the standard deviation of each distribution as a systematic uncertainty.

To estimate the systematic uncertainty due to fit function, we do the following

- Use 7-th (4-th) order polynomial for D (D_s) momentum dependent weight fitting.
- Use binned correction without fitting.

and add uncertainties by quadrature.

24.6.2 Correlated cascade $B \to D^{(*)}\tau\nu$ background

In case of UncrorrCL and CascL backgrounds, we correct $B \to D^{(*)}\tau\nu$ decay branching fractions. Especially, approximately a half of CascL events are the case where the lepton comes from τ. For the decays $B^+ \to \bar{D}^0\tau^+\nu$, $B^+ \to \bar{D}^{*0}\tau^+\nu$ and $B^0 \to D^-\tau^+\nu$ we use Babar results [52]. For the decay $B^0 \to D^{*-}\tau^+\nu$ we take an average of Babar [52]

$$B^0 \to D^{*-}\tau^+\nu = 0.0162 \pm 0.0031 \pm 0.0010 \pm 0.005 \tag{458}$$

and Belle [53] results.

$$B^0 \to D^{*-}\tau^+\nu = 0.0202 \pm 0.004 \pm 0.0037 \pm 0.005 \tag{459}$$

Averaged BF, SP5/SP6 values and their ratios are

Decay mode	SP5/SP6	Measured [52][53]	Ratio
$B^+ \to \bar{D}^0\tau^+\nu$	0.0070	$0.0092 \pm 0.0026 \pm 0.0011 \pm 0.006$	1.3142 ± 0.4123
$B^+ \to \bar{D}^{*0}\tau^+\nu$	0.0160	$0.0174 \pm 0.0033 \pm 0.0010 \pm 0.006$	1.0875 ± 0.2188
$B^0 \to D^-\tau^+\nu$	0.0070	$0.0086 \pm 0.0024 \pm 0.0011 \pm 0.006$	1.2286 ± 0.3868
$B^0 \to D^{*-}\tau^+\nu$	0.0160	0.0173 ± 0.0028	1.0795 ± 0.1762

Thus we change total amount of correlated cascade $B \to D^{(*)}\tau\nu$ background by ± 30 % and take larger deviation as systematic uncertainty.

24.7 Covariance Matrices

We construct full covariance matrices to keep track of correlation. For statistical uncertainty, covariance matrices are given by the fit.
When we estimate systematic uncertainty, we vary the value of a quantity (k-th source of systematic uncertainty) and take differences, Δ^{sys}, between fitted values and nominal values of parameters. For i-th parameter

$$\Delta^{sys}_{ik} = \text{(fitted value - nominal value) of } i\text{-th parameter} \tag{460}$$

Then, the ij element of a systematic error matrix is given by

$$\text{cov}^{sys}_{ij} = \sum_k \Delta^{sys}_{ik} \cdot \Delta^{sys}_{jk} \tag{461}$$

In cases where we do two fits (one with a + variation and the other with a - variation), we take larger deviation. Total covariance is given by the sum of statistical and systematic covariances.

Statistical and systematic correlations between fitted variables are given in Table 57 and 62 respectively.

	Electron						Muon					
	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	$\mathcal{G}(1)\|V_{cb}\|$	$\mathcal{F}(1)\|V_{cb}\|$	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	$\mathcal{G}(1)\|V_{cb}\|$	$\mathcal{F}(1)\|V_{cb}\|$
$B \to D$ wght	1.32	0.93	0.46	0.27	0.97	0.42	1.43	0.92	0.44	0.27	0.99	0.42
$B \to D$ diff fit	-0.15	-0.07	-0.09	-0.05	-0.13	-0.05	-0.40	-0.07	-0.20	-0.00	-0.31	-0.03
$B \to D$ no fit	-0.37	-0.60	-0.42	-0.14	-0.42	-0.26	-0.36	-0.52	-0.42	-0.11	-0.40	-0.23
$B \to D$ **total**	**1.38**	**1.11**	**0.63**	**0.31**	**1.06**	**0.50**	**1.53**	**1.06**	**0.64**	**0.29**	**1.11**	**0.48**
$D^0 \to K^{*-}\ell\nu$	-0.02	-0.01	-0.01	-0.00	-0.01	-0.00	-0.07	-0.04	-0.04	-0.04	-0.06	-0.01
$D^0 \to K^-\ell\nu$	-0.05	-0.02	-0.02	-0.00	-0.04	-0.01	-0.12	-0.05	-0.06	-0.05	-0.09	-0.02
$D^0 \to \pi^-\ell\nu$	-0.02	-0.01	-0.01	-0.00	-0.01	-0.00	-0.06	-0.04	-0.04	-0.04	-0.05	-0.01
$D^0 \to \rho^-\ell\nu$	-0.01	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.04	-0.03	-0.04	-0.04	-0.01
$D^+ \to K^{*0}\ell\nu$	-0.01	-0.01	-0.01	-0.00	-0.01	-0.00	-0.06	-0.04	-0.04	-0.04	-0.05	-0.01
$D^+ \to K^0\ell\nu$	-0.11	-0.03	-0.05	-0.00	-0.09	-0.01	-0.25	-0.08	-0.11	-0.05	-0.19	-0.02
$D^+ \to \pi^0\ell\nu$	-0.03	-0.01	-0.01	-0.00	-0.02	-0.00	-0.06	-0.04	-0.04	-0.04	-0.05	-0.01
$D^+ \to \rho^0\ell\nu$	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.04	-0.03	0.04	-0.04	-0.01
$D^+ \to \omega\ell\nu$	-0.00	-0.00	-0.00	-0.00	-0.00	-0.00	-0.04	-0.04	-0.03	-0.04	-0.04	-0.01
$D_s^+ \to \phi\ell\nu$	-0.00	0.00	-0.00	-0.00	-0.00	0.00	-0.19	0.12	-0.07	-0.09	-0.13	0.01
$D_s^+ \to \eta\ell\nu$	-0.03	0.06	-0.00	-0.01	-0.02	0.01	-0.06	0.10	0.03	-0.06	-0.04	0.02
$D_s^+ \to \eta'\ell\nu$	-0.00	0.00	-0.00	-0.00	-0.00	0.00	-0.04	0.04	0.03	-0.04	-0.03	0.01
$B^+ \to \bar{X}^0\ell^+\nu$	-0.05	-0.10	-0.04	-0.01	-0.05	-0.04	-0.09	-0.12	-0.05	-0.06	-0.07	-0.04
$B^0 \to X^-\ell^+\nu$	-0.09	-0.09	-0.05	-0.02	-0.08	-0.04	-0.14	-0.12	-0.08	-0.06	-0.11	-0.05
CascL Tau	-0.04	-0.02	0.01	-0.15	-0.02	-0.08	0.05	-0.00	0.05	-0.16	0.05	-0.08
Other bkg	**0.17**	**0.15**	**0.09**	**0.15**	**0.14**	**0.10**	**0.41**	**0.27**	**0.21**	**0.25**	**0.32**	**0.11**
Backg total	**1.39**	**1.12**	**0.64**	**0.34**	**1.07**	**0.51**	**1.58**	**1.09**	**0.67**	**0.38**	**1.16**	**0.49**

Table 61: [**Systematics (R_1 and R_2 fixed fit)**] Systematic uncertainties. Numbers are given in %.

	Electron				Muon			
	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$	ρ_D^2	ρ^2	$\mathcal{B}(D^0)$	$\mathcal{B}(D^{*0})$
ρ_D^2	1							
ρ^2	-0.02	1						
$\mathcal{B}(D\ell\nu)$	0.74	0.12	1					
$\mathcal{B}(D^*\ell\nu)$	-0.21	0.36	0.33	1				
ρ_D^2	0.73	-0.18	0.44	-0.35	1			
ρ^2	-0.06	0.98	0.05	0.31	-0.15	1		
$\mathcal{B}(D\ell\nu)$	0.44	0.02	0.63	0.18	0.13	-0.00	1	
$\mathcal{B}(D^*\ell\nu)$	-0.17	0.19	0.13	0.54	-0.49	0.17	0.69	1

Table 62: [**Systematic correlation coefficients (R_1 and R_2 fixed fit)**] Systematic correlation coefficients.

25 Electron and Muon Combined Results

25.1 Method to combine electron and muon results

We combine electron and muon results by minimizing a χ^2 cnstruted by electron and muon results. The electron and muon fit results are given in Table 64. We put these values to x_i ($i = 0-5$ for electron and $i = 6-11$ for muon). We sum statistical and systematic covariance matrices to get a full 12×12 covariance (error) matrix, $M(\mathrm{err})_{ij}$. Setting $y_0 = y_6 = \rho_D^2$, $y_1 = y_7 = \rho^2$, $y_2 = y_8 = R_1$, $y_3 = y_9 = R_2$, $y_4 = y_{10} = \mathcal{B}(D\ell\nu)$ and $y_5 = y_{11} = \mathcal{B}(D^*\ell\nu)$, χ^2 is given by

$$\chi^2 = \sum_{i,j}(y_i - x_i)M(\mathrm{err})^{-1}_{ij}(y_j - x_j) \tag{462}$$

where $M(\mathrm{err})^{-1}_{ij}$ is the inverse error matrix. By minimizing the χ^2 we can get y_i.

25.2 Separation of statistical and systematic uncertainties

To estimate the size of sytematic uncertainties, we fit to all lepton (= electron and muon summed) sample. Fit results are

$$\begin{aligned}&\rho_D^2 = 1.234 \pm 0.037\,(3.0\%), \quad \rho^2 = 1.236 \pm 0.019\,(1.5\%)\\ &\mathcal{B}(\bar{D}^0\ell^+\nu) = 0.02359 \pm 0.00025\,(1.1\%),\ \ \mathcal{B}(\bar{D}^{*0}\ell^+\nu) = 0.05271 \pm 0.00025\,(0.5\%)\end{aligned} \tag{463}$$

25.3 Combined results

Combined fit results are given in Table 56. Correlations of combined fit are given in Table 63. $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$ are calculated to be

$$\begin{aligned}\mathcal{G}(1)|V_{cb}| &= (43.8 \pm 0.8(stat.) \pm 2.3(syst.)) \times 10^{-3}\\ \mathcal{F}(1)|V_{cb}| &= (35.7 \pm 0.2(stat.) \pm 1.2(syst.)) \times 10^{-3}\end{aligned} \tag{464}$$

Corrlations are

- $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$: -0.05
- $\mathcal{G}(1)|V_{cb}|$ and ρ_D^2 : +0.63
- $\mathcal{F}(1)|V_{cb}|$ and ρ^2 : +0.56

Then we extract $|V_{cb}|$

$$\begin{aligned}D\ell\nu &\Rightarrow |V_{cb}| = (40.5 \pm 0.8 \pm 2.1 \pm 0.9) \times 10^{-3}\\ D^*\ell\nu &\Rightarrow |V_{cb}| = (38.5 \pm 0.2 \pm 1.3 \pm 1.0) \times 10^{-3}\end{aligned} \tag{465}$$

The errors are statistical, sytematic and theory errors respectively. Theory errors come from the uncertainties on $\mathcal{G}(1)$ and $\mathcal{F}(1)$. Results are consistent with each other

	ρ_D^2	ρ^2	$\mathcal{B}(D^0\ell\nu)$	$\mathcal{B}(D^{*0}\ell\nu)$
ρ_D^2	1			
ρ^2	-0.131	1		
$\mathcal{B}(D^0\ell\nu)$	+0.598	+0.055	1	
$\mathcal{B}(D^{*0}\ell\nu)$	-0.283	+0.310	+0.302	1

Table 63: [**Combined fit correlation coefficients (**R_1 **and** R_2 **fixed fit**] Correlations between parameters.

26 Discussion

26.1 $B \to D^* \ell \nu$ form factors

26.1.1 Resluts by floating R_1 and R_2

Here, we give results by floating R_1 and R_2. Electron fit results, muon fit results and combined fit results are listed in Table 64. Correlations of combined fit are given in Table 65. $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$ are calculated to be

$$\begin{aligned} \mathcal{G}(1)|V_{cb}| &= (43.2 \pm 0.9 \pm 2.3) \times 10^{-3} \\ \mathcal{F}(1)|V_{cb}| &= (35.4 \pm 0.3 \pm 1.0) \times 10^{-3} \end{aligned} \tag{466}$$

Corrlations are

- $\mathcal{G}(1)|V_{cb}|$ and $\mathcal{F}(1)|V_{cb}|$: -0.016
- $\mathcal{G}(1)|V_{cb}|$ and ρ_D^2 : +0.923
- $\mathcal{F}(1)|V_{cb}|$ and ρ^2 : +0.408

26.1.2 Comparison with previous BaBar results

We compare our results with previous Babar measurements [40][57]. The [40] use the decay mode $B^0 \to D^{*-}\ell^+\nu$, and results are

$$\begin{aligned} \rho^2 &= 1.191 \pm 0.048 \pm 0.028 \\ R_1 &= 1.429 \pm 0.061 \pm 0.044 \\ R_2 &= 0.827 \pm 0.038 \pm 0.022 \end{aligned} \tag{467}$$

The analysis in Ref.[57] use the decay mode $B^- \to D^{*0}\ell^-\bar{\nu}$. This analysis use R_1 and R_2 from the above measurement as input. The result is

$$\rho^2 = 1.15 \pm 0.06 \pm 0.08 \tag{468}$$

Our result by fixing R_1 and R_2 is

$$\rho^2 = 1.211 \pm 0.071 \tag{469}$$

This agreeds with previous mesurements. Our results by floating R_1 and R_2 are

$$\begin{aligned} \rho^2 &= 1.325 \pm 0.099 \\ R_1 &= 1.555 \pm 0.160 \\ R_2 &= 0.659 \pm 0.105 \end{aligned} \tag{470}$$

	Electron	Muon	Combined
ρ_D^2	$1.24 \pm 0.05 \pm 0.09$	$1.12 \pm 0.07 \pm 0.09$	$1.18 \pm 0.04 \pm 0.07$
ρ^2	$1.32 \pm 0.05 \pm 0.09$	$1.31 \pm 0.06 \pm 0.09$	$1.31 \pm 0.04 \pm 0.09$
R_1	$1.56 \pm 0.09 \pm 0.14$	$1.50 \pm 0.10 \pm 0.16$	$1.53 \pm 0.07 \pm 0.14$
R_2	$0.68 \pm 0.07 \pm 0.09$	$0.70 \pm 0.08 \pm 0.10$	$0.67 \pm 0.05 \pm 0.09$
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$ (%)	$2.40 \pm 0.04 \pm 0.15$	$2.26 \pm 0.04 \pm 0.17$	$2.33 \pm 0.03 \pm 0.13$
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$ (%)	$5.48 \pm 0.05 \pm 0.23$	$5.32 \pm 0.06 \pm 0.37$	$5.46 \pm 0.04 \pm 0.22$
χ^2/ndof (P-value)	418/468 (0.95)	490/464 (0.20)	2.0/6 (0.92)

Table 64: [**Fit results by floating R_1 and R_2**] Fit results.

Combined Fit	ρ_D^2	ρ^2	R_1	R_2	$\mathcal{B}(D\ell\nu)$	$\mathcal{B}(D^*\ell\nu)$
ρ_D^2	1					
ρ^2	-0.426	1				
R_1	-0.212	+0.736	1			
R_2	+0.506	-0.782	-0.719	1		
$\mathcal{B}(D\ell\nu)$	+0.598	-0.031	+0.140	+0.097	1	
$\mathcal{B}(D^*\ell\nu)$	-0.319	+0.418	+0.144	-0.315	+0.217	1

Table 65: [**Correlation coefficients (floating R_1 and R_2, Electron and Muon Combined Fit)**] Correlations between parameters.

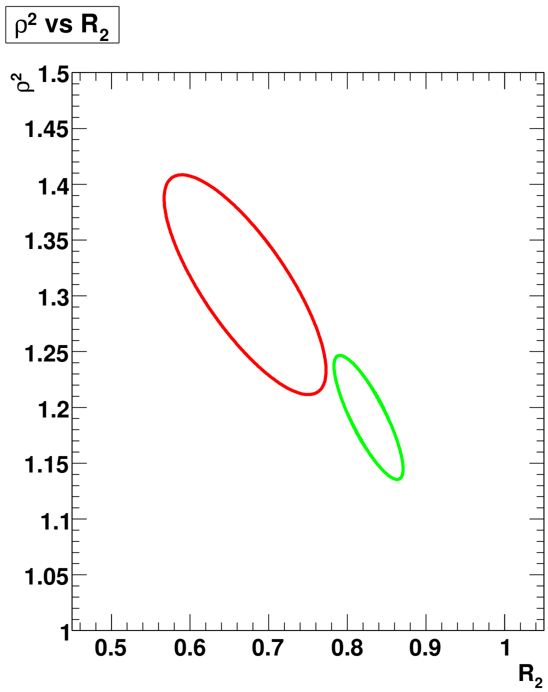


Figure 91: [ρ^2 **and** R_2 **comparison**] Comparison of R_1 and R_2. Red is our results and green is [40]. The contours are $\Delta\chi^2 = 1$ which corresponds to 39 % C.L. in 2 dimensions.

We combine our results with [40]. The fit gives χ^2/ndf (P-value) = 2.2/3 (0.53); thus, agreement is good. Combined results are

$$\begin{aligned} \rho^2 &= 1.21 \pm 0.05 \\ R_1 &= 1.43 \pm 0.08 \\ R_2 &= 0.80 \pm 0.04 \end{aligned} \tag{471}$$

We also compare ρ^2 and R_2, which is given in Figure 91. Red is our results and green is [40].

In [57] derivatives

$$\frac{\partial \rho^2}{\partial R_1} = 0.0303, \quad \frac{\partial \rho^2}{\partial R_2} = -1.22 \tag{472}$$

are given. Thus, if [57] apply our R_1 and R_2 result, they could have got the slope which agrees with our slope.

26.2 Branching fractions

Please see our papaer draft BAD1781 for latest numbers.

26.3 Isospin symmetry

We try to fit without isospin symmetry constrints on $B \to D^{(*)}\ell\nu$ decays. Results are **Electron :** Results including systematic uncertainties are

$$\begin{aligned}
&\rho_D^2 = 1.23 \pm 0.05(\text{stat.}, 4.1\%) \pm 0.12(\text{syst.}, 9.9\%)\\
&\rho^2 = 1.34 \pm 0.05(\text{stat.}, 4.0\%) \pm 0.09(\text{syst.}, 6.9\%)\\
&R_1 = 1.59 \pm 0.09(\text{stat.}, 5.7\%) \pm 0.14(\text{syst.}, 9.1\%)\\
&R_2 = 0.66 \pm 0.07(\text{stat.}, 10.6\%) \pm 0.09(\text{syst.}, 13.7\%)\\
&\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) = 0.0245 \pm 0.0006(\text{stat.}, 2.4\%) \pm 0.0018(\text{syst.}, 7.3\%)\\
&\mathcal{B}(B^0 \to D^-\ell^+\nu) = 0.0222 \pm 0.0005(\text{stat.}, 2.1\%) \pm 0.0015(\text{syst.}, 6.7\%)\\
&\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) = 0.0528 \pm 0.0012(\text{stat.}, 2.3\%) \pm 0.0031(\text{syst.}, 5.9\%)\\
&\mathcal{B}(B^0 \to D^{*-}\ell^+\nu) = 0.0528 \pm 0.0017(\text{stat.}, 3.3\%) \pm 0.0058(\text{syst.}, 11.0\%)
\end{aligned} \tag{473}$$

To check isospin symmetry, we take ratios

$$\frac{\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)}{t_{+0}\mathcal{B}(B^0 \to D^-\ell^+\nu)} = 1.03 \pm 0.11 \tag{474}$$

and

$$\frac{\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)}{t_{+0}\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)} = 0.93 \pm 0.12 \tag{475}$$

where t_{+0} is the lifetime ratio of B^+ and B^0. Thus, isospin symmetry holds.
Muon : Results including systematic uncertainties are

$$\begin{aligned}
&\rho_D^2 = 1.12 \pm 0.07(\text{stat.}, 6.3\%) \pm 0.10(\text{syst.}, 9.2\%)\\
&\rho^2 = 1.32 \pm 0.07(\text{stat.}, 5.0\%) \pm 0.09(\text{syst.}, 6.9\%)\\
&R_1 = 1.51 \pm 0.11(\text{stat.}, 7.0\%) \pm 0.15(\text{syst.}, 9.6\%)\\
&R_2 = 0.69 \pm 0.08(\text{stat.}, 12.2\%) \pm 0.10(\text{syst.}, 14.6\%)\\
&\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) = 0.0225 \pm 0.0006(\text{stat.}, 2.9\%) \pm 0.0016(\text{syst.}, 7.1\%)\\
&\mathcal{B}(B^0 \to D^-\ell^+\nu) = 0.0213 \pm 0.0005(\text{stat.}, 2.6\%) \pm 0.0016(\text{syst.}, 7.6\%)\\
&\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) = 0.0522 \pm 0.0014(\text{stat.}, 2.8\%) \pm 0.0041(\text{syst.}, 7.8\%)\\
&\mathcal{B}(B^0 \to D^{*-}\ell^+\nu) = 0.0505 \pm 0.0021(\text{stat.}, 4.1\%) \pm 0.0063(\text{syst.}, 12.6\%)
\end{aligned} \tag{476}$$

To check isospin symmetry, we take ratios

$$\frac{\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)}{t_{+0}\mathcal{B}(B^0 \to D^-\ell^+\nu)} = 0.99 \pm 0.11 \tag{477}$$

and

$$\frac{\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)}{t_{+0}\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)} = 0.97 \pm 0.15 \tag{478}$$

Thus, isospin symmetry holds.

27 Averaging with other BaBar results

The results obtained here can be combined with other BaBar measurements. This has been done using a simple Maple worksheet as well as using the COMBOS program used by HFAG. **We recommend that the averages from COMBOS be used in any official BaBar publication.** They are more comprehensive, adjusting both values and errors for changes in input parameters. The other results can be considered a cross-check on COMBOS.

The results we combine are

1. The current measurements from the global fit,. The central values correspond to the combined $e+\mu$ samples. The errors are taken from the electron sample except in those cases where there are real differences between the e and μ samples, namely radiative corrections, bremsstrahlung and PID. In these cases the error is given by the smaller of the two errors (this is what happens when solving the χ^2 minimization used in averaging the electron and muon results).

2. Measurement of the $\overline{B}^0 \to D^{*+}\ell\overline{\nu}$ branching fraction, form factor parameters and $|V_{cb}|$ [40]. The paper quotes

$$\begin{aligned}
\mathcal{F}(1)|V_{cb}| &= (34.4 \pm 0.3 \pm 1.1) \times 10^{-3} && (479)\\
\rho^2_{D^*} &= 1.191 \pm 0.048 \pm 0.028 && (480)\\
R_1 &= 1.429 \pm 0.061 \pm 0.044 && (481)\\
R_2 &= 0.827 \pm 0.038 \pm 0.022 && (482)\\
\mathcal{B}(\overline{B}^0 \to D^{*+}\ell\overline{\nu}) &= (4.69 \pm 0.04 \pm 0.34)\% && (483)
\end{aligned}$$

   We rescale the branching fraction and $\mathcal{F}(1)|V_{cb}|$ values to correspond to $f_{+0} = 1.065 \pm 0.026$ to find $\mathcal{B}(\overline{B}^0 \to D^{*+}\ell\overline{\nu}) = 4.75\%$ and $\mathcal{F}(1)|V_{cb}| = 34.6 \times 10^{-3}$.

3. Measurement of the $B^- \to D^{*0}\ell\overline{\nu}$ branching fraction, $\rho^2_{D^*}$, and $|V_{cb}|$ [57]. The paper quotes

$$\begin{aligned}
\mathcal{F}(1)|V_{cb}| &= (35.9 \pm 0.6 \pm 1.4) \times 10^{-3} && (484)\\
\rho^2_{D^*} &= 1.16 \pm 0.06 \pm 0.08 && (485)\\
\mathcal{B}(B^- \to D^{*0}\ell\overline{\nu}) &= (5.56 \pm 0.08 \pm 0.41)\% && (486)
\end{aligned}$$

   We rescale the branching fraction and $\mathcal{F}(1)|V_{cb}|$ values to correspond to $f_{+0} = 1.065 \pm 0.026$ and $\mathcal{B}(D^0 \to K^-\pi^+) = (3.912 \pm 0.048)\%$ to find $\mathcal{B}(B^- \to D^{*0}\ell\overline{\nu}) = 5.30$ and $\mathcal{F}(1)|V_{cb}| = 35.0 \times 10^{-3}$. The fractional errors are kept the same.

4. Study of the $\overline{B} \to D^*\ell\overline{\nu}$ and $\overline{B} \to D\ell\overline{\nu}$ branching fractions using B-tagged events [4]. This includes measurements of B^- and $\overline{B}^0$ decays which need to be averaged amongst themselves (not done in the published paper). The paper quotes

$$\mathcal{B}(B^- \to D^0\ell\overline{\nu}) = (2.33 \pm 0.09 \pm 0.09)\% \qquad (487)$$

$$\mathcal{B}(B^- \to D^{*0}\ell\overline{\nu}) = (5.83 \pm 0.15 \pm 0.30)\% \tag{488}$$
$$\mathcal{B}(\overline{B}^0 \to D^{+}\ell\overline{\nu}) = (2.21 \pm 0.11 \pm 0.12)\% \tag{489}$$
$$\mathcal{B}(\overline{B}^0 \to D^{*+}\ell\overline{\nu}) = (5.49 \pm 0.16 \pm 0.25)\% \tag{490}$$

Given the use of many individual decay modes for the D^0 and D^+ reconstruction we do not rescale the result (which uses $\mathcal{B}(D^0 \to K^-\pi^+) = 3.80\%$).

We proceed as follows. The statistical correlations amongst these analyses are assumed to be negligeable because

- Measurements 2 and 3 use disjoint samples
- The two measurements in 4 have essentially no overlap with each other, and have tiny overlap with the untagged measurements.
- The global fit measurements are a large superset of the other measurements and in any case have very tiny statistical errors.

The systematic errors, however, need to be considered more carefully. Some systematic variations can be clearly identified in each analysis, e.g. the variation of D decay branching fractions, or the error from B counting. In such cases the corresponding systematic uncertainties on the measured quantities are taken as fully correlated. However, not all systematic errors are assessed using the same variations in the different analyses. To determine the correlation amongst these other errors, we group the systematics into the following categories. In some cases the systematic error in a particular category for a particular analysis is zero (e.g. the tagged analysis has no uncertainty from B counting).

- Detector effects, including reconstruction efficiencies for charged tracks and photons, vertexing, particle identification and bremsstrahlung.
- Specific issues associated with soft pion reconstruction
- Backgrounds, including both higher mass $\overline{B} \to X_c\ell\overline{\nu}$ decays and non-semileptonic background.
- R_1 and R_2
- $\mathcal{B}(D^{*0} \to D^0\pi^0)$
- $\mathcal{B}(D^{*+} \to D^0\pi^+)$
- $\mathcal{B}(D^+ \to K^-\pi^+\pi^+)$
- $\mathcal{B}(D^0 \to K^-\pi^+)$
- B lifetime ratio
- $\Upsilon(4S) \to B^+B^-/\Upsilon(4S) \to B^0\overline{B}^0$

- B counting
- B momentum distribution (from the beam energy uncertainty)
- Radiative corrections
- B tagging issues (m_{ES} fits, normalization to s.l. BF, ...)

Within each category the systematic errors in a given analysis are combined in quadrature. These quadrature sums are then treated as fully correlated across the analyses being averaged. Uncertainties that don't fall into any of these categories (e.g. the D^{*0} production rate in $e^+e^- \to c\overline{c}$ events, relevant to measurement 3) are taken as uncorrelated.

This approach almost certainly overestimates the systematic correlations, which in turn leads to a larger uncertainty on the average (which is conservative) and a lower χ^2 probability for the average (i.e. is pessimistic about how consistent the measurements are). Splitting the error sources up more finely would reduce the correlations. However, it may not be realistic to focus too narrowly on individual variations, especially in complex areas like detector modeling.

Having constructed the full covariance matrix, the averaging procedure is as described for averaging the electron and muon samples in the global fit analysis, namely a simple χ^2 minimization. When averaging branching fractions for $\overline{B}^0$ and B^- we account for the lifetime ratio when constructing the χ^2.

27.1 $\mathcal{B}(B^- \to D^0\ell\overline{\nu})$ average

The measured values are

$$\begin{aligned}
\text{Global fit } \mathcal{B}(B^- \to D^0\ell\overline{\nu}) &= (2.36 \pm 0.03 \pm 0.12)\% && (491)\\
\text{tagged } \overline{B}^0\ \mathcal{B}(\overline{B}^0 \to D^+\ell\overline{\nu}) &= (2.21 \pm 0.11 \pm 0.12)\% && (492)\\
\text{tagged } B^-\ \mathcal{B}(B^- \to D^0\ell\overline{\nu}) &= (2.33 \pm 0.09 \pm 0.09)\% && (493)
\end{aligned}$$

The inputs used in building the covariance are shown in Table 66. Those rows surrounded by horizontal lines are the ones used in creating the covariance; the rows above in each case are summed in quadrature. The result using the COMBOS program of HFAG is

$$\mathcal{B}(B^- \to D^0\ell\overline{\nu}) = 2.33 \pm 0.09 \qquad (494)$$

with $P(\chi^2) = 0.85$.

27.2 $\mathcal{B}(B^- \to D^{*0}\ell\overline{\nu})$ average

The input values are

$$\begin{aligned}
\text{Global fit } \mathcal{B}(B^- \to D^{*0}\ell\overline{\nu}) &= (5.37 \pm 0.02 \pm 0.21)\% && (495)\\
\overline{B}^0 \to D^{*+}\ell\overline{\nu}\ \mathcal{B}(\overline{B}^0 \to D^{*+}\ell\overline{\nu}) &= (4.75 \pm 0.04 \pm 0.34)\% && (496)\\
\overline{B}^0 \to D^{*+}\ell\overline{\nu}\ \mathcal{B}(B^- \to D^{*0}\ell\overline{\nu}) &= (5.30 \pm 0.08 \pm 0.39)\% && (497)
\end{aligned}$$

$$\begin{aligned}
\text{tagged } \overline{B}^0 \;\; \mathcal{B}(\overline{B}^0 \to D^{*+}\ell\overline{\nu}) &= (5.49 \pm 0.16 \pm 0.12)\% && (498)\\
\text{tagged } B^- \;\; \mathcal{B}(B^- \to D^{*0}\ell\overline{\nu}) &= (5.83 \pm 0.15 \pm 0.09)\% && (499)
\end{aligned}$$

The inputs used in building the covariance are shown in Table 66. A negative value indicates that the fitted quantity decreases when the parameter value increases. This is important for external parameters, like branching fractions, to allow post-facto adjustments. Those rows surrounded by horizontal lines are the ones used in creating the covariance; the rows above in each case are summed in quadrature.

The corresponding average from the COMBOS program of HFAG is

$$\mathcal{B}(B^- \to D^{*0}\ell\overline{\nu}) = 5.47 \pm 0.19 \tag{500}$$

with $P(\chi^2) = 0.38$.

27.3 $\rho^2_{D^*}$ average

The measured values are

$$\begin{aligned}
\text{Global fit} \quad \rho^2_{D^*} &= 1.21 \pm 0.02 \pm 0.07 && (501)\\
\overline{B}^0 \to D^{*+}\ell\overline{\nu} \;\; \rho^2_{D^*} &= 1.191 \pm 0.048 \pm 0.028 && (502)\\
B^- \to D^{*0}\ell\overline{\nu} \;\; \rho^2_{D^*} &= 1.16 \pm 0.06 \pm 0.08 && (503)
\end{aligned}$$

The inputs used in building the covariance are shown in Table 67. Those rows surrounded by horizontal lines are the ones used in creating the covariance; the rows above in each case are summed in quadrature.

The corresponding average from the COMBOS program of HFAG is

$$\rho^2_{D^*} = 1.19 \pm 0.044 \tag{504}$$

with $P(\chi^2) = 0.90$.

27.4 $\mathcal{F}(1)|V_{cb}|$ average

The measured values are

$$\begin{aligned}
\text{Global fit} \quad \mathcal{F}(1)|V_{cb}| &= 35.7 \pm 0.2 \pm 1.2 && (505)\\
\overline{B}^0 \to D^{*+}\ell\overline{\nu} \;\; \mathcal{F}(1)|V_{cb}| &= 34.6 \pm 0.4 \pm 1.0 && (506)\\
B^- \to D^{*0}\ell\overline{\nu} \;\; \mathcal{F}(1)|V_{cb}| &= 35.0 \pm 0.6 \pm 1.4 && (507)
\end{aligned}$$

The inputs used in building the covariance are shown in Table 67. A negative value indicates that the fitted quantity decreases when the parameter value increases. This is important for external parameters, like branching fractions, to allow post-facto adjustments. Those rows surrounded by horizontal lines are the ones used in creating the covariance; the rows above in each case are summed in quadrature.

The corresponding average from the COMBOS program of HFAG is

$$\mathcal{F}(1)|V_{cb}| = 34.8 \pm 0.8 \tag{508}$$

with $P(\chi^2) = 0.33$.

item	global fit		$D^{*+}\ell\overline{\nu}$	$D^{*0}\ell\overline{\nu}$	tagged $\overline{B}^0$		tagged B^-	
	$D\ell\overline{\nu}$	$D^*\ell\overline{\nu}$	$D^*\ell\overline{\nu}$	$D^*\ell\overline{\nu}$	$D\ell\overline{\nu}$	$D^*\ell\overline{\nu}$	$D\ell\overline{\nu}$	$D^*\ell\overline{\nu}$
Lepton PID scale	1.20	0.83						
Lepton PID Fake	-0.01	-0.01						
Kaon PID	0.27	0.24						
Tracking eff	-3.29	-2.02						
Bremsstrahlung	-0.00	-0.00						
Vertexing	0.63	0.59						
Detector	3.57	2.27	3.89	2.89	1.57	1.96	1.64	2.28
Soft π^0				3.62				1.3
Soft π^+			3.80			1.2		
D^{**} slope	-0.09	-0.10						
D^{**} B_2	-0.10	0.19						
$\mathcal{B}(B^+ \to D^{(*)}\pi\ell\nu)$	-0.12	0.87						
$f_{D_2^*/D_1}$	-0.37	0.16						
$f_{D_0^* D\pi/D_1 D_2^*}$	-1.58	1.01						
$f_{D_1^* D^*\pi/D_1 D_2^*}$	0.93	-0.52						
$f_{D\pi/D_0^*}$	0.29	0.14						
$f_{D^*\pi/D_1^*}$	-0.10	0.17						
NR D^*/D ratio	0.27	-0.14						
$\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell\nu)$	0.26	-1.29						
X*/X(Y*/Y) ratio	0.09	-0.28						
X/Y(X*/Y*) ratio	0.21	-0.66						
$D_1 \to D\pi\pi$	0.74	-1.09						
$f_{D_2^*}$	-0.11	0.08						
Other background	0.63	0.34						
Background	2.18	2.38	1.70	0.36	1.0	1.0	0.6	0.6
(R_1,R_2) set A	0.69	-0.38						
(R_1,R_2) set B	-0.18	0.30						
(R_1,R_2)	0.71	0.48		0.5		0.8		0.8
ρ_D					0.4		0.4	
D^{*0} BF				-4.7				-2.8
D^{*+} BF	0.42	-0.34	-0.85			-0.8		
D^+ BF	-2.03	0.30			-4.1	-1.8		
D^0 BF	-0.21	-1.57	-2.5	-1.8		-1.8	-2.3	-2.1
t_{+0}	0.61	0.27	0.64					
f_{+0}	0.63	-0.54	+1.5	-1.6				
Number of $B\overline{B}$ events	-1.11	-1.11	-1.11	-1.11				
B momentum distribution	1.28	-0.55	3.0	0.6				
PHOTOS	-0.82	-0.24	1.5	1.4				
B tagging related					3.02	2.52	2.25	2.64
Uncorrelated				0.2				
Total	5.16	3.96	7.39	7.30	5.43	4.50	3.68	5.20

Table 66: Systematic uncertainties on $\mathcal{B}(\overline{B} \to D\ell\overline{\nu})$ and $\mathcal{B}(\overline{B} \to D^*\ell\overline{\nu})$, given in %.

item	global fit		$D^{*+}\ell\overline{\nu}$		$D^{*0}\ell\overline{\nu}$	
	$\rho^2_{D^*}$	$\mathcal{F}(1)\vert V_{cb}\vert$	$\rho^2_{D^*}$	$\mathcal{F}(1)\vert V_{cb}\vert$	$\rho^2_{D^*}$	$\mathcal{F}(1)\vert V_{cb}\vert$
Lepton PID scale	0.17	0.47				
Lepton PID Fake	0.01	-0.00				
Kaon PID	0.82	0.39				
Tracking eff	-0.45	-1.16				
Bremsstrahlung	-0.00	-0.00				
Vertexing	-0.67	0.08				
Detector	1.16	1.31	1.24	2.01	2.06	1.53
Soft π^0					3.5	2.06
Soft π^+			1.1	1.32		
D^{**} slope	-2.62	-0.90				
D^{**} B_2	0.56	0.28				
$\mathcal{B}(B^+ \to D^{(*)}\pi\ell\nu)$	-0.35	-0.55				
$f_{D_2^*/D_1}$	0.15	0.13				
$f_{D_0^* D\pi/D_1 D_2^*}$	1.24	0.92				
$f_{D_1^* D^*\pi/D_1 D_2^*}$	-0.78	-0.51				
$f_{D\pi/D_0^*}$	-1.09	-0.29				
$f_{D^*\pi/D_1^*}$	-0.04	0.07				
NR D^*/D ratio	-0.13	-0.10				
$\mathcal{B}(B^+ \to D^{(*)}\pi\pi\ell\nu)$	-2.03	-1.31				
X*/X(Y*/Y) ratio	-1.18	-0.53				
X/Y(X*/Y*) ratio	-0.85	-0.61				
$D_1 \to D\pi\pi$	-1.59	-1.06				
$f_{D_2^*}$	-0.01	0.04				
Other background	1.12	0.52				
Background	4.55	2.49	1.08	0.31	0.71	0.32
(R_1,R_2) set A	2.76	0.71				
(R_1,R_2) set B	1.03	0.49				
(R_1,R_2)	2.95	0.86			4.7	0.4
D^{*0} BF					0.0	-2.3
D^{*+} BF	-0.01	-0.17	0	-0.37		
D^+ BF	-0.40	0.02				
D^0 BF	0.01	-0.78	0	-0.8	0	-0.9
t_{+0}	0.17	0.19	0	-0.29	0	-0.3
f_{+0}	0.43	-0.13	+0.3	+0.72	0	-0.8
Number of $B\overline{B}$ events	-0.00	-0.55	0	-0.55	0	-0.55
OffPeak Luminosity	0.01	0.00			0.4	0.1
B momentum distribution	0.63	0.47	1.1		2.5	1.0
PHOTOS	-0.62	-0.33	0.3	0.55	0.4	0.5
Uncorrelated					1.06	0.36
Total	5.65	3.16	2.3	2.9	6.8	3.9

Table 67: Systematic uncertainties on $\rho^2_{D^*}$ and $\mathcal{F}(1)|V_{cb}|$, given in %.

28 Conclusion

Please see our papaer draft BAD1781 for latest numbers.

A Decay modes to consider

A.1 Semileptonic $B \to D^{(*,**)}\ell\nu$ decays

A.1.1 Semileptonic B^+ decays

There are three types of decay modes (Figure 92).

- $B^+ \to \bar{D}^{(*,**)0}\ell^+\nu_\ell$
- $B^+ \to \bar{D}^{(*)0}\pi^0\ell^+\nu_\ell$
- $B^+ \to D^{(*)-}\pi^+\ell^+\nu_\ell$

A.1.2 Semileptonic B^0 decays

There are three types of decay modes (Figure 93).

- $B^0 \to D^{(*,**)-}\ell^+\nu_\ell$
- $B^0 \to D^{(*)-}\pi^0\ell^+\nu_\ell$
- $B^0 \to \bar{D}^{(*)0}\pi^-\ell^+\nu_\ell$

A.2 D^* and D^{**} decays

A.2.1 $D^{*,**+}$ decays

There are three types of decay modes (Figure 94).

- $D^{*,**+} \to D^{(*)0}\pi^+$
- $D^{*,**+} \to D^{(*)+}\pi^0$
- $D^{*,**+} \to D^{(*)+}\gamma$

Note that $D^{**+} \to D^{*+}\gamma$ is an electromagnetic process. Thus, it is negligible compared to strong processes. In case of $D^{*+} \to D^{*+}\gamma$, because strong processes are suppressed by phase space, the EM process amplitude has comparable size to strong processes.

A.2.2 $D^{*,**0}$ decays

There are three types of decay modes (Figure 95).

- $D^{*,**0} \to D^{(*)+}\pi^-$
- $D^{*,**0} \to D^{(*)0}\pi^0$
- $D^{*,**0} \to D^{(*)0}\gamma$

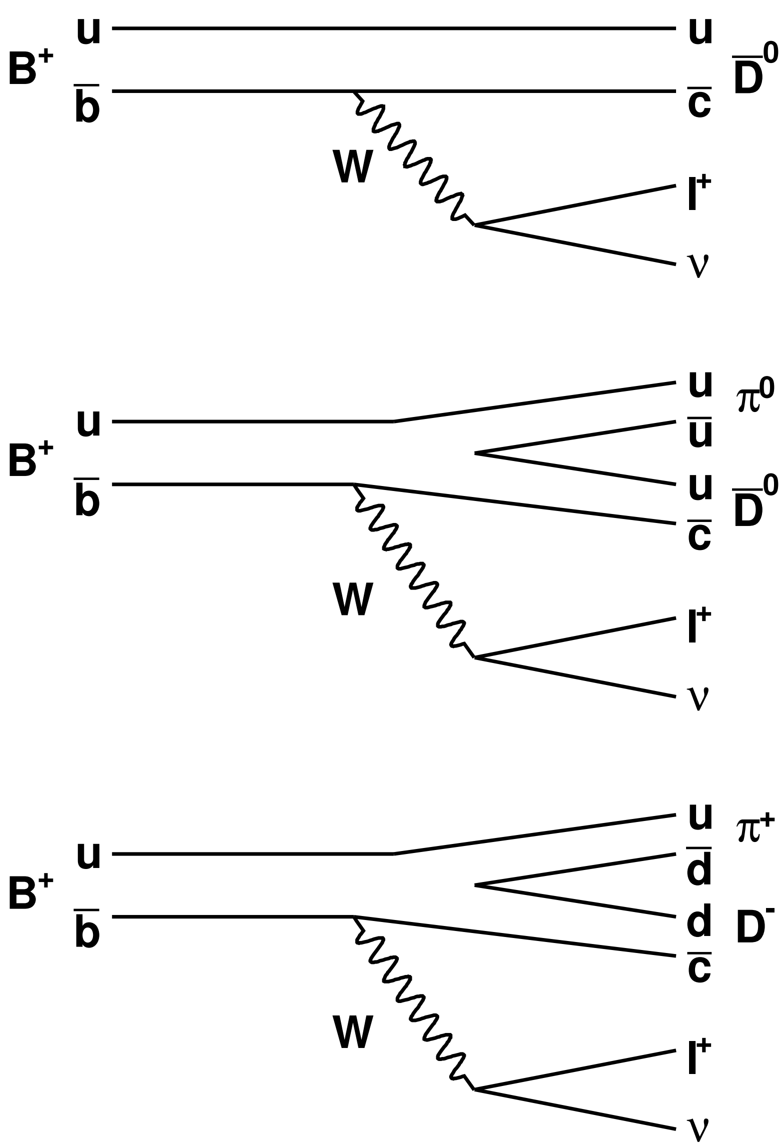


Figure 92: Feynman diagrams of semileptonic $B^+ \to D^{(*,**)}\ell\nu$ decays

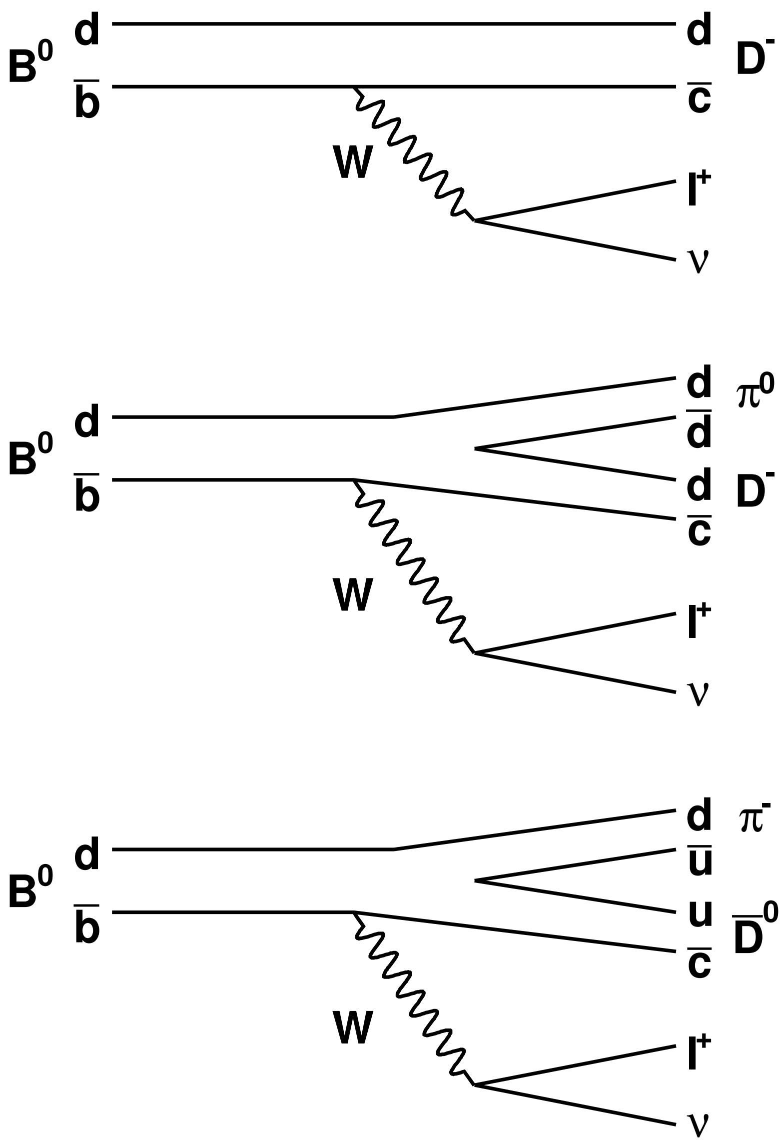


Figure 93: Feynman diagrams of semileptonic $B^0 \rightarrow D^{(*,**)}\ell\nu$ decays

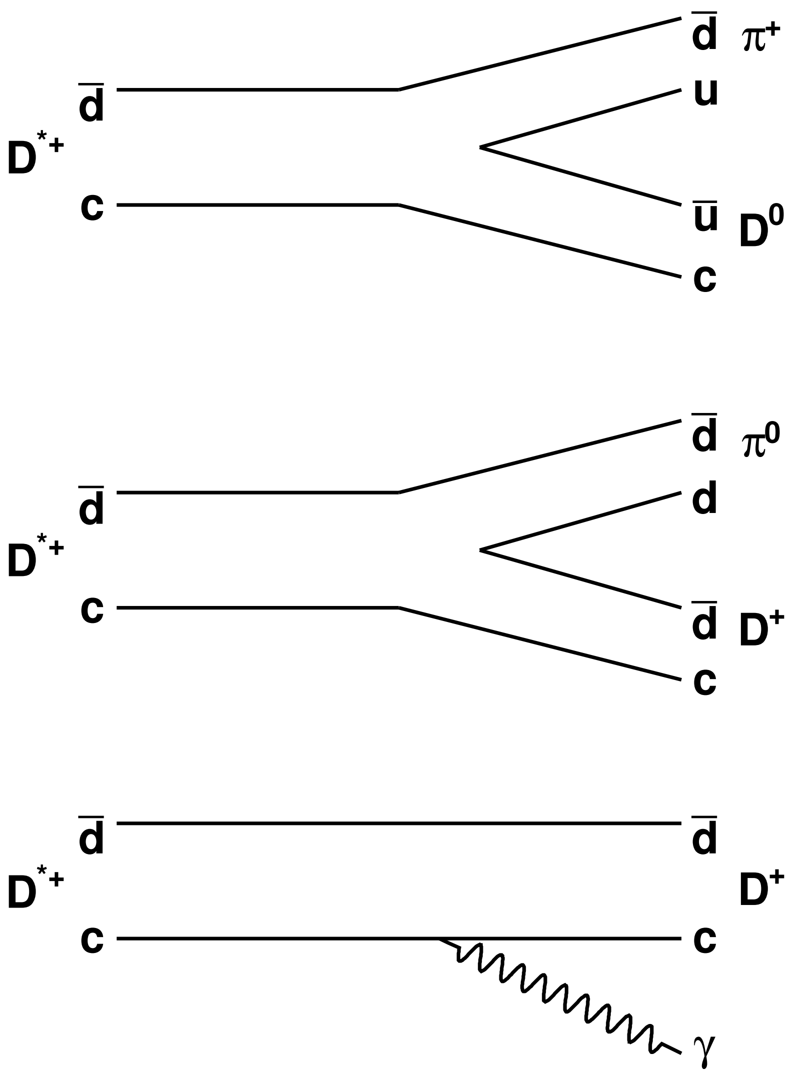


Figure 94: Feynman diagrams of $D^{*,**+}$ decays

Note that

- $D^{*0} \to D^+\pi^-$ is not allowed
  because $m_{D^{*0}} < m_{D^+} + m_{\pi^-}$
- $D^{**0} \to D^{*0}\gamma$ is negligible

A.3 D^0 and D^+ decays

A.3.1 D^0 decays

There are four decay modes used in the reconstruction of D^0 (Figure 96).

- $D^0 \to K^-\pi^+$
- $D^0 \to K^-\pi^+\pi^+\pi^-$
- $D^0 \to K^-\pi^+\pi^0$
- $D^0 \to K_s^0\pi^+\pi^-$

A.3.2 D^+ decays

There are two decay modes used in the reconstruction of D^+ (Figure 97).

- $D^+ \to K^-\pi^+\pi^+$
- $D^+ \to \bar{K}_s^0\pi^+$

A.3.3 π^0 and K_s^0

π^0 is reconstructed from 2γ.
K_s^0 is reconstructed from $\pi^0\pi^0$ or $\pi^+\pi^-$.

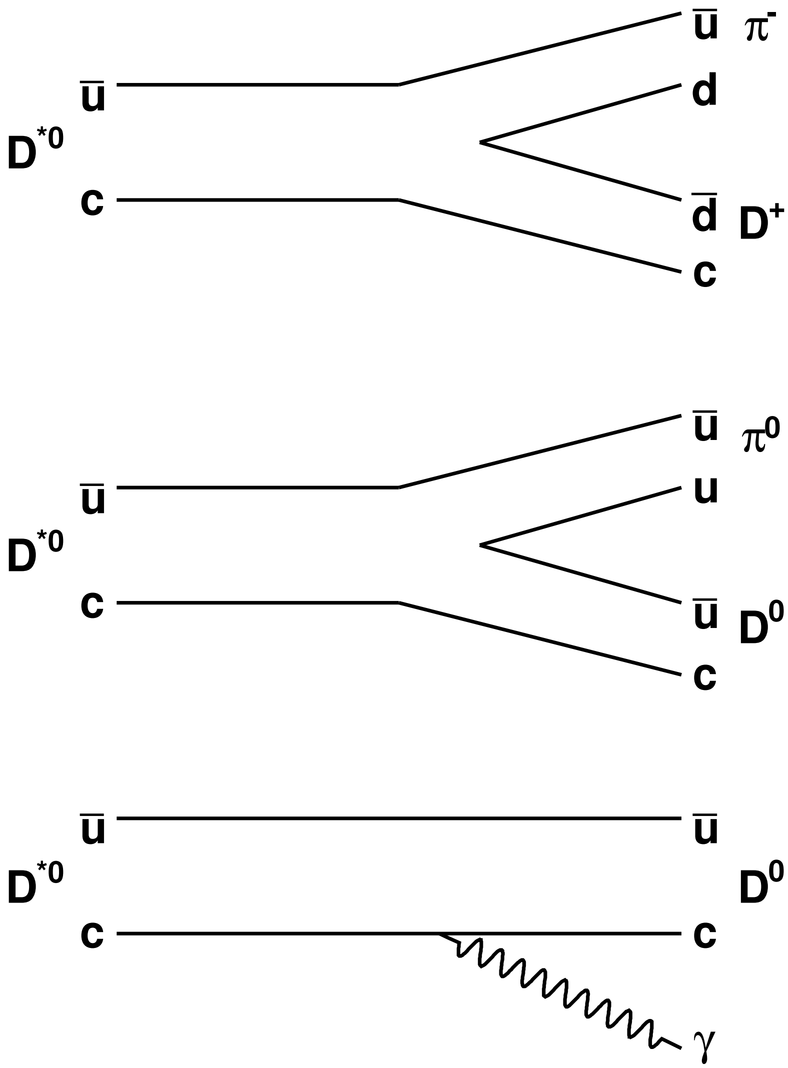


Figure 95: Feynman diagrams of $D^{*,**0}$ decays

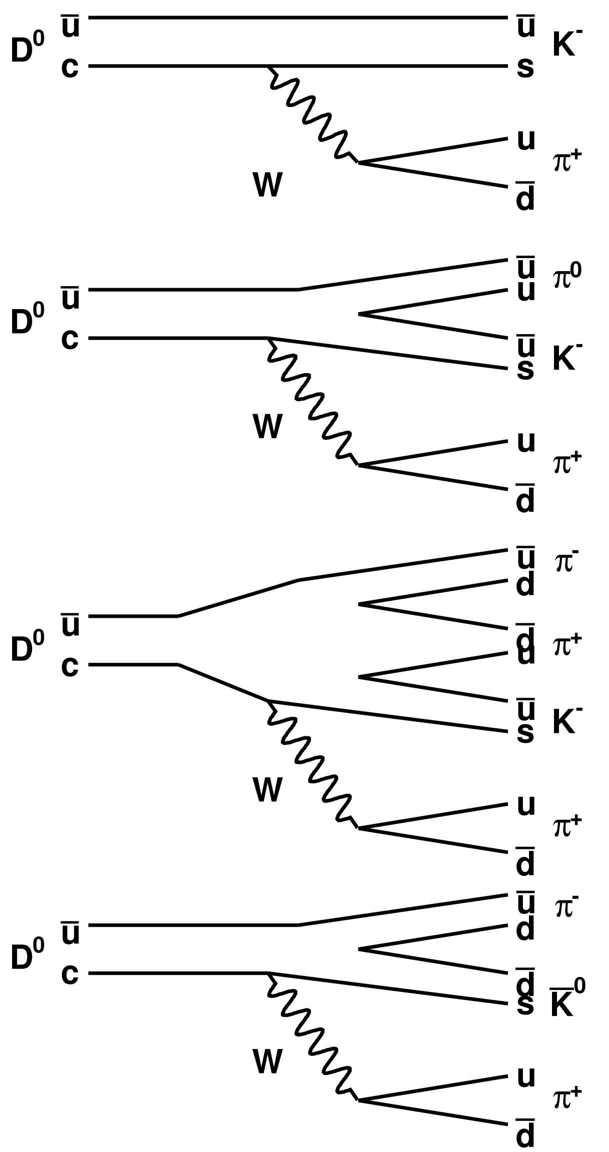


Figure 96: Feynman diagrams of D^0 decays

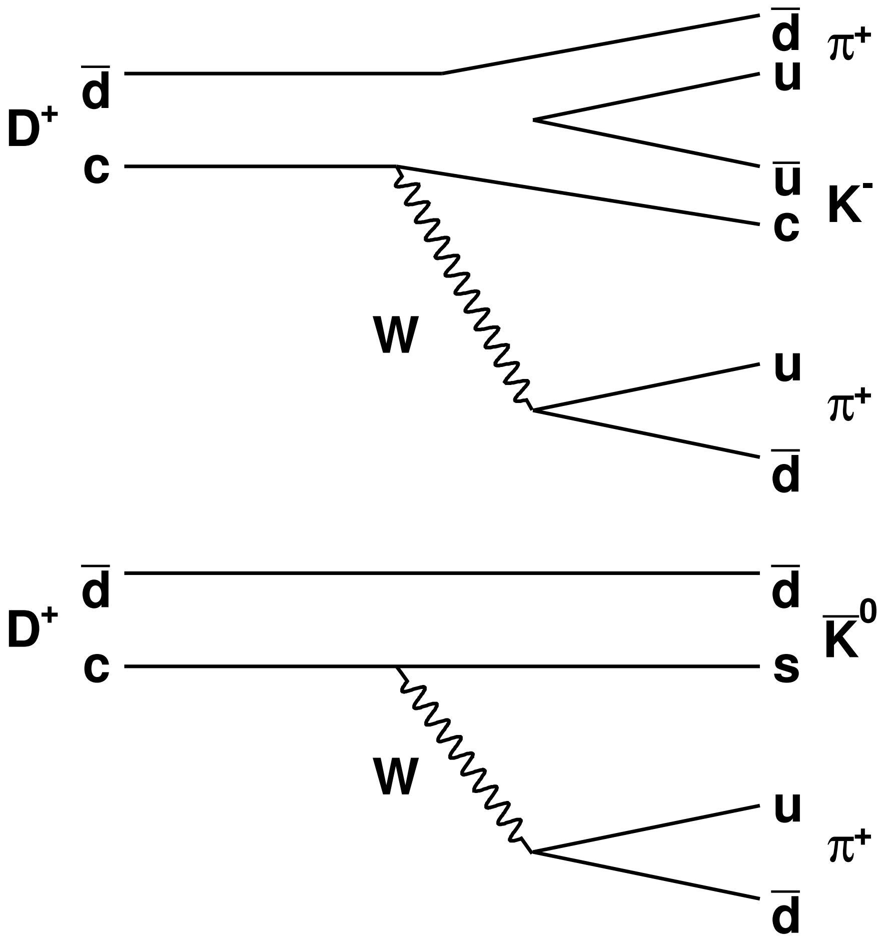


Figure 97: Feynman diagrams of D^+ decays

A.4 Decay chain

A.4.1 B^+ decay chain

1. $B^+ \to \bar{D}^0\ell^+\nu_\ell$ (D^0)
2. $B^+ \to \bar{D}^{*0}\ell^+\nu_\ell$
   - $\to D^-\pi^+$ (not possible)
   - $\to \bar{D}^0\pi^0$ (D^0)
   - $\to \bar{D}^0\gamma$ (D^0)
3. $B^+ \to \bar{D}^{**0}\ell^+\nu_\ell$
   - $\to D^-\pi^+$ (D^+)
   - $\to \bar{D}^0\pi^0$ (D^0)
   - $\to \bar{D}^0\gamma$ (D^0)
   - $\to D^{*-}\pi^+$
     - $\to \bar{D}^0\pi^-$ (D^0)
     - $\to D^-\pi^0$ (D^+)
     - $\to D^-\gamma$ (negligible)
   - $\to \bar{D}^{*0}\pi^0$
     - $\to D^-\pi^+$ (D^+)
     - $\to \bar{D}^0\pi^0$ (D^0)
     - $\to \bar{D}^0\gamma$ (D^0)
   - $\to \bar{D}^{*0}\gamma$ (negligible)
4. $B^+ \to \bar{D}^0\pi^0\ell^+\nu_\ell$ (D^0)
5. $B^+ \to D^-\pi^+\ell^+\nu_\ell$ (D^+)

6. $B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu_\ell$
   - $\to D^-\pi^+$ (not possible)
   - $\to \bar{D}^0\pi^0$ (D^0)
   - $\to \bar{D}^0\gamma$ (D^0)
7. $B^+ \to D^{*-}\pi^+\ell^+\nu_\ell$
   - $\to \bar{D}^0\pi^-$ (D^0)
   - $\to D^-\pi^0$ (D^+)
   - $\to D^-\gamma$ (D^+)

A.4.2 B^0 decay chain

1. $B^0 \to D^-\ell^+\nu_\ell$ (D^0)
2. $B^0 \to D^{*-}\ell^+\nu_\ell$
   - $\to \bar{D}^0\pi^-$ (D^0)
   - $\to D^-\pi^0$ (D^+)
   - $\to D^-\gamma$ (D^+)

3. $B^0 \to D^{**-}\ell^+\nu_\ell$
   - $\to \bar{D}^0\pi^-$ (D^0)
   - $\to D^-\pi^0$ (D^+)
   - $\to D^-\gamma$ (D^+)
   - $\to D^{*-}\pi^0$
     - $\to \bar{D}^0\pi^-$ (D^0)
     - $\to D^-\pi^0$ (D^+)
     - $\to D^-\gamma$ (negligible)
   - $\to \bar{D}^{*0}\pi^-$
     - $\to D^-\pi^+$ (not possible)
     - $\to \bar{D}^0\pi^0$ (D^0)
     - $\to \bar{D}^0\gamma$ (D^0)
   - $\to \bar{D}^{*-}\gamma$ (negligible)
4. $B^0 \to D^-\pi^0\ell^+\nu_\ell$ (D^+)
5. $B^0 \to \bar{D}^0\pi^-\ell^+\nu_\ell$ (D^0)
6. $B^0 \to D^{*-}\pi^0\ell^+\nu_\ell$
   - $\to \bar{D}^0\pi^-$ (D^0)
   - $\to D^-\pi^0$ (D^+)
   - $\to D^-\gamma$ (D^+)
7. $B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu_\ell$
   - $\to D^-\pi^+$ (not possible)
   - $\to \bar{D}^0\pi^0$ (D^0)
   - $\to \bar{D}^0\gamma$ (D^0)

B Classification of D^{**}

B.1 Angular momentum of a meson

In atomic spectroscopy, there are two useful schemes to add angular momentum: LS-coupling and jj-coupling.

- LS-coupling

$$\begin{aligned}\vec{S} &= \vec{s}_1 + \vec{s}_2 \\ \vec{J} &= \vec{L} + \vec{S}\end{aligned} \tag{509}$$

- jj-coupling

$$\begin{aligned}\vec{j}_1 &= \vec{L} + \vec{s}_1 \\ \vec{J} &= \vec{j}_1 + \vec{s}_2\end{aligned} \tag{510}$$

where s, l and j are the spin, orbital angular momentum and total angular momentum of individual particles, and S, L and J are the spin, orbital angular momentum and total angular momentum of the system.

B.2 Parity of a meson ($Q\bar{q}$-system) : $P = (-1)^{L+1}$

The parity of

- Quark [3] : $P_Q = +1$
- Anti-quark : $P_{\bar{q}} = -1$
- Orbital angular momentum L : $(-1)^L$

Thus, the parity of a $Q\bar{q}$-system is given by

$$P = (P_Q)(P_{\bar{q}})(-1)^L = (+1)(-1)(-1)^L = (-1)^{L+1} \tag{511}$$

Hence, $P = (-)$ for $L = 0$ and $P = (+)$ for $L = 1$.

B.3 Standard classification of mesons

For equal-mass meson system, it is traditional to couple the orbital angular momentum, $\vec{L}$, with the total spin of the quark and anti-quark, $\vec{S}$ (LS-coupling scheme).

$$\begin{aligned}\vec{S} &= \vec{s}_1 + \vec{s}_2 \\ \vec{J} &= \vec{L} + \vec{S}\end{aligned} \tag{512}$$

This is the standard practice for light mesons and heavy quarkonium ($c\bar{c}$ and $b\bar{b}$). The good quantum numbers are then S, L and J.
There are spin-singlet and spin-triplet levels because S is always $S = 0$ or $S = 1$.

[3] One could define the parity of quark to be (-1). Then its of anti-quark is (+1). But, as far as P(quark) = - P(anti-quark), the final result won't change.

	Particle	J^P
$^1S_0$	D	0^-
$^3S_1$	D^*	1^-
$^1P_1$	D_1	1^+
$^3P_0$	D_0^*	0^+
$^3P_1$	D_1^*	1^+
$^3P_2$	D_2^*	2^+

Table 68: Traditional spectroscopic notation of D, D^* and D^{**} mesons.

- For $L = 0$ and $S = 0$, $J = 0$: $^1S_0$
- For $L = 0$ and $S = 1$, $J = 1$: $^3S_1$
- For $L = 1$ and $S = 0$, $J = 1$: $^1P_1$
- For $L = 1$ and $S = 1$, $J = 0, 1, 2$: $^3P_0$, $^3P_1$ or $^3P_2$

In general, for $S = 0$ and $S = 1$

$$^1L_L \quad \text{and} \quad ^3L_{L-1,L,L+1} \tag{513}$$

We may apply this classification to D, D^* and D^{**} mesons. It is summarized in Table 68. In this classification D^{**} consists of one singlet (D_1) and a triplet (D_0^*, D_1^* and D_2^*).

B.4 Heavy Quark Effective Theory (HQET) and D, D^* and D^{**} mesons

In HQET, the spin symmetry suggests that the spin of heavy quark is conserved. Thus, for the mesons consist of heavy and light quarks, we can apply jj coupling scheme. First, we can add the angular momentum (L) and the spin of the light quark (s_l) to get the angular momentum of light constituent (j_l). Then we can add the spin of the heavy quark (s_h) to j_l.

$$\begin{aligned} \vec{j}_l &= \vec{L} + \vec{s}_l \\ \vec{J} &= \vec{j}_l + \vec{s}_h \end{aligned} \tag{514}$$

For $L = 0$ and $s_l = 1/2$, $j_l = 1/2$ only, and for $L = 1$ and $s_l = 1/2$, $j_l = 1/2$ or $j_l = 3/2$. Hence,

- $L = 0$, $j_l = 1/2 \rightarrow J = 0$ (D) or 1 (D^*)
- $L = 1$, $j_l = 1/2 \rightarrow J = 0$ (D_0^*) or 1 (D_1')

j_l^P	Particles	J^P	mass (GeV)
$\frac{1}{2}^+$	D, D^*	0^-, 1^-	1.971
$\frac{1}{2}^-$	D_0^*, D_1'	0^+, 1^+	2.40
$\frac{3}{2}^-$	D_1, D_2^*	1^+, 2^+	2.445

Table 69: Spin parity of charmed meson doublets

Initial state	Final state	ΔL	description
D_0^*	$D\pi$	0	broad S-wave transition
	$D^*\pi$	1	parity violation
D_1'	$D\pi$	1	parity violation
	$D^*\pi$	0	broad S-wave transition
		2	heavy quark spin symmetry violation
D_1	$D\pi$	1	parity violation
	$D^*\pi$	0	heavy quark spin symmetry violation
		2	narrow D-wave transition
D_2^*	$D\pi$	2	narrow D-wave transition
	$D^*\pi$	2	narrow D-wave transition

Table 70: D^{**} decay modes.

- $L = 1$, $j_l = 3/2 \to J = 1$ (D_1) or 2 (D_2^*)

These are summarized in Table 69. In this classification, D^{**} consists of two doublets : (D_0^*, D_1') and (D_1, D_2^*).

B.5 Decay modes of D^{**}

The HQET based classification implies that the $j = 3/2$ states will decay only through the d-wave ($D_1 \to D^*$ and $D_2^* \to D$) and so will be narrow. The $j = 1/2$ states can decay through the s-wave ($D_0^* \to D$ and $D_1' \to D^*$) and, in general, will be broad. This is shown in Figure 98 and summarized in Table 70. We consider this in detail.

Spin-parity of $D\pi$ system is

- Since $D : S = 0, P = (-)$ and $\pi : S = 0, P = (-)$
- if $L = 0$, $J = 0$ and $P = (-)(-)(-)^0 = (+)$, then $J^P = 0^+$
- if $L = 1$, $J = 1$ and $P = (-)(-)(-)^1 = (-)$, then $J^P = 1^-$

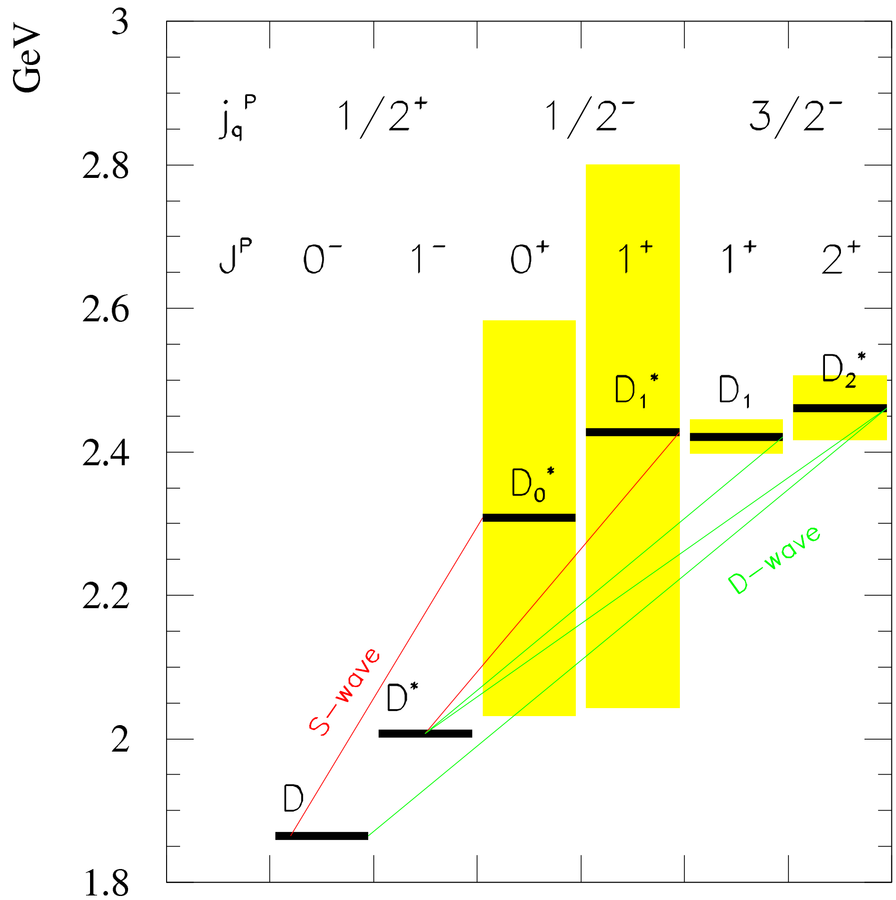


Figure 98: Charmed meson levels and transitions. $j_q = j_l$ in the text. The yellow bands show the width of each D^{**} state [15]

.

- if $L = 2,\ J = 2$ and $P = (-)(-)(-)^2 = (+)$, then $J^P = 2^+$

Note that $P = (-)^L$ for a orbital angular momentum L system. Spin-parity of $D^*\pi$ system is

- Since $D^* : S = 1, P = (-)$ and $\pi : S = 0, P = (-)$
- if $L = 0,\ J = 1$ and $P = (-)(-)(-)^0 = (+)$, then $J^P = 1^+$
- if $L = 1,\ J = 2, 1, 0$ and $P = (-)(-)(-)^1 = (-)$, then $J^P = 0^-, 1^-, 2^-$
- if $L = 2,\ J = 3, 2, 1$ and $P = (-)(-)(-)^2 = (+)$, then $J^P = 1^+, 2^+, 3^+$

Thus,

- $J^P = 0^+$ state (D_0^*) can decay only to $D\pi$ and $\Delta L = 0$
- $J^P = 1^+$ state (D_1 and D_1') can decay only to $D^*\pi$ and $\Delta L = 0, 2$
- $J^P = 2^+$ state (D_2^*) can decay both to $D\pi$ and to $D^*\pi$ and $\Delta L = 2$

Next, we consider angular momentum of D_1 and D_1' decays.

- $D_1(L = +1, s_l = +1/2, j_l = +3/2; s_h = -1/2, J = +1)$
  $\to D^*(s_l = +1/2, s_h = +1/2, j_D = +1) + \pi(j_\pi = 0) + (L = 0); J = +1$
  This violates heavy quark spin symmetry ($s_h = -1/2 \to s_h = +1/2$)
- $D_1(L = +1, s_l = +1/2, j_l = +3/2; s_h = -1/2, J = +1)$
  $\to D^*(s_l = -1/2, s_h = -1/2, j_D = -1) + \pi(j_\pi = 0) + (L = +2); J = +1$
  This conserves heavy quark spin symmetry ($s_h = -1/2 \to s_h = -1/2$)
- $D_1'(L = +1, s_l = -1/2, j_l = +1/2; s_h = +1/2, J = +1)$
  $\to D^*(s_l = +1/2, s_h = +1/2, j_D = +1) + \pi(j_\pi = 0) + (L = 0); J = +1$
  This conserves heavy quark spin symmetry ($s_h = +1/2 \to s_h = +1/2$)
- $D_1'(L = +1, s_l = -1/2, j_l = +1/2; s_h = +1/2, J = +1)$
  $\to D^*(s_l = -1/2, s_h = -1/2, j_D = -1) + \pi(j_\pi = 0) + (L = +2); J = +1$
  This violates heavy quark spin symmetry ($s_h = +1/2 \to s_h = -1/2$)

C Isospin Symmetry

C.1 Semileptonic B decays

From isospin symmetry, these B decay rates are identical

$$\begin{aligned}
\Gamma(B^+ \to \bar{D}^0\ell^+\nu) &= \Gamma(B^0 \to D^-\ell^+\nu) \\
\Gamma(B^+ \to \bar{D}^{*0}\ell^+\nu) &= \Gamma(B^0 \to D^{*-}\ell^+\nu) \\
\Gamma(B^+ \to \bar{D}^{**0}\ell^+\nu) &= \Gamma(B^0 \to D^{**-}\ell^+\nu)
\end{aligned} \tag{515}$$

Isospin symmetry may be applied to non-resonant $B \to D^{(*)}\pi\ell\nu$ decays.

$$\begin{aligned}
\Gamma(B^+ \to D^-\pi^+\ell^+\nu) &= \Gamma(B^0 \to \bar{D}^0\pi^-\ell^+\nu) \\
\Gamma(B^+ \to D^{*-}\pi^+\ell^+\nu) &= \Gamma(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)
\end{aligned} \tag{516}$$

$$\begin{aligned}
\Gamma(B^+ \to D^-\pi^+\ell^+\nu) &= 2\Gamma(B^+ \to \bar{D}^0\pi^0\ell^+\nu) \\
\Gamma(B^0 \to \bar{D}^0\pi^-\ell^+\nu) &= 2\Gamma(B^0 \to D^-\pi^0\ell^+\nu) \\
\Gamma(B^+ \to D^{*-}\pi^+\ell^+\nu) &= 2\Gamma(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu) \\
\Gamma(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) &= 2\Gamma(B^0 \to D^{*-}\pi^0\ell^+\nu)
\end{aligned} \tag{517}$$

The origin of the factor 2 is, for example,

$$\begin{aligned}
&B^+(u\bar{b}) : |1/2, 1/2> \\
&D^-(d\bar{c}) : |1/2, -1/2> \\
&\pi^+(u\bar{d}) : |1, 1> \\
&D^-\pi^+ : |1, 1> \times |1/2, -1/2> = \sqrt{\tfrac{1}{3}}|3/2, 1/2> + \sqrt{\tfrac{2}{3}}|1/2, 1/2> \\
&\bar{D}^0(u\bar{c}) : |1/2, 1/2> \\
&\pi^0(u\bar{u} \text{ or } d\bar{d}) : |1, 0> \\
&\bar{D}^0\pi^0 : |1, 0> \times |1/2, 1/2> = \sqrt{\tfrac{2}{3}}|3/2, 1/2> - \sqrt{\tfrac{1}{3}}|1/2, 1/2>
\end{aligned} \tag{518}$$

hence

$$\Gamma(B^+ \to D^-\pi^+\ell^+\nu) : \Gamma(B^+ \to \bar{D}^0\pi^0\ell^+\nu) = \frac{2}{3} : \frac{1}{3} = 2 : 1 \tag{519}$$

Branching fractions are not same because of the difference in total rate (Γ) or life time (τ) of B^+ and B^0. We define

$$t_{+0} \equiv \frac{\tau_{B^+}}{\tau_{B^0}} = \frac{\Gamma_{B^0}}{\Gamma_{B^+}} \tag{520}$$

and

$$t_{0+} \equiv \frac{\tau_{B^0}}{\tau_{B^+}} = \frac{\Gamma_{B^+}}{\Gamma_{B^0}} \tag{521}$$

So branching fractions are, for example,

$$\frac{\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)}{\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)} = \frac{\Gamma(B^+ \to D^*\ell\nu)}{\Gamma(B^0 \to D^*\ell\nu)}\frac{\Gamma_{B^0}}{\Gamma_{B^+}} = t_{+0} \tag{522}$$

Thus, due to isospin symmetry, we have the following relations between branching fractions of semileptonic B decays.

$$\begin{aligned}
&\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^-\ell^+\nu)\\
&\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)\\
&\mathcal{B}(B^+ \to \bar{D}^{**0}\ell^+\nu) = t_{+0}\mathcal{B}(B^0 \to D^{**-}\ell^+\nu)\\
&\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)\\
&= t_{+0}\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)\\
&\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) = 2\mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)\\
&= t_{+0}\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) = 2t_{+0}\mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)
\end{aligned} \tag{523}$$

C.2 D^{**} decays

Isospin symmetry may be applied to D^{**} decays as well.

$$\begin{aligned}
\Gamma(D^{**0} \to D^+\pi^-) &= \Gamma(D^{**+} \to D^0\pi^+)\\
\Gamma(D^{**0} \to D^0\pi^0) &= \Gamma(D^{**+} \to D^+\pi^0)\\
\Gamma(D^{**0} \to D^{*+}\pi^-) &= \Gamma(D^{**+} \to D^{*0}\pi^+)\\
\Gamma(D^{**0} \to D^{*0}\pi^0) &= \Gamma(D^{**+} \to D^{*+}\pi^0)
\end{aligned} \tag{524}$$

And also

$$\begin{aligned}
\Gamma(D^{**0} \to D^+\pi^-) &= 2\Gamma(D^{**0} \to D^0\pi^0)\\
\Gamma(D^{**+} \to D^0\pi^+) &= 2\Gamma(D^{**+} \to D^+\pi^0)\\
\Gamma(D^{**0} \to D^{*+}\pi^-) &= 2\Gamma(D^{**0} \to D^{*0}\pi^0)\\
\Gamma(D^{**+} \to D^{*0}\pi^+) &= 2\Gamma(D^{**+} \to D^{*+}\pi^0)
\end{aligned} \tag{525}$$

This factor 2 has the same origin as B decay case. For example,

$$\begin{aligned}
&D^{**+} : |1/2, 1/2>\\
&D^0 : |1/2, -1/2>\\
&\pi^+ : |1, 1>\\
&D^0\pi^+ : |1, 1> \times |1/2, -1/2> = \sqrt{\tfrac{1}{3}}|3/2, 1/2> + \sqrt{\tfrac{2}{3}}|1/2, 1/2>\\
&D^+ : |1/2, 1/2>\\
&\pi^0 : |1, 0>\\
&D^+\pi^0 : |1, 0> \times |1/2, 1/2> = \sqrt{\tfrac{2}{3}}|3/2, 1/2> - \sqrt{\tfrac{1}{3}}|1/2, 1/2>
\end{aligned} \tag{526}$$

hence

$$\Gamma(D^{**+} \to D^0\pi^+) : \Gamma(D^{**+} \to D^+\pi^0) = \frac{2}{3} : \frac{1}{3} = 2 : 1 \tag{527}$$

Supposing total rates or lifetimes of D^{**0} and D^{**+} are same, branching fractions for D^{**} decays are

$$\begin{aligned}
&\mathcal{B}(D^{**0} \to D^+\pi^-) = 2\mathcal{B}(D^{**0} \to D^0\pi^0) = \mathcal{B}(D^{**+} \to D^0\pi^+) = 2\mathcal{B}(D^{**+} \to D^+\pi^0) \\
&\mathcal{B}(D^{**0} \to D^{*+}\pi^-) = 2\mathcal{B}(D^{**0} \to D^{*0}\pi^0) = \mathcal{B}(D^{**+} \to D^{*0}\pi^+) = 2\mathcal{B}(D^{**+} \to D^{*+}\pi^0)
\end{aligned} \tag{528}$$

D Values used in SP5/SP6 MC and related to our fit

The values used in SP5/SP6 MC are given in Tables 71-78. Calculated and used values are given in relevant sections in the text.

D.1 Definition of total or summed branching fractions, and branching fraction ratios

In section 5, we define the following total or summed branching fractions.

1. $\mathcal{BT}(B^+ \to \bar{D}^{**0}\ell^+\nu)$
   $\equiv \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)$

2. $\mathcal{BT}(B^0 \to D^{**-}\ell^+\nu)$
   $\equiv \mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) + \mathcal{B}(B^0 \to D_1^-\ell^+\nu) + \mathcal{B}(B^0 \to D_1'^-\ell^+\nu) + \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)$

3. $\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu) \equiv \mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)$

4. $\mathcal{BT}(B^0 \to D_1^-/D_2^{*-}\ell^+\nu) \equiv \mathcal{B}(B^0 \to D_1^-\ell^+\nu) + \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)$

5. $\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu)$
   $\equiv \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(\bar{D}_2^{*0} \to D^-\pi^+) + \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)\mathcal{B}(\bar{D}_0^{*0} \to D^-\pi^+)$
   $+\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)$

6. $\mathcal{BT}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)$
   $\equiv \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*-} \to \bar{D}^0\pi^-) + \mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)\mathcal{B}(D_0^{*-} \to \bar{D}^0\pi^-)$
   $+\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)$

7. $\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)$
   $\equiv \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) + \mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)$

8. $\mathcal{BT}(B^0 \to D_0^{*-}/(D\pi)^-\ell^+\nu)$
   $\equiv \mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) + \mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) + \mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)$

9. $\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu)$
   $\equiv \mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\mathcal{B}(\bar{D}_1^0 \to D^{*-}\pi^+) + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(\bar{D}_2^{*0} \to D^{*-}\pi^+)$
   $+\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu)\mathcal{B}(\bar{D}_1'^0 \to D^{*-}\pi^+) + \mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)$

10. $\mathcal{BT}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)$
    $\equiv \mathcal{B}(B^0 \to D_1^-\ell^+\nu)\mathcal{B}(D_1^- \to \bar{D}^{*0}\pi^-) + \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*-} \to \bar{D}^{*0}\pi^-)$
    $+\mathcal{B}(B^0 \to D_1'^-\ell^+\nu)\mathcal{B}(D_1'^- \to \bar{D}^{*0}\pi^-) + \mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)$

11. $\mathcal{BT}(B^+ \to \bar{D}_1'^0/(D^*\pi)^0\ell^+\nu)$
    $\equiv \mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) + \mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)$

12. $\mathcal{BT}(B^0 \to D_1'^-/(D^*\pi)^-\ell^+\nu)$
    $\equiv \mathcal{B}(B^0 \to D_1'^-\ell^+\nu) + \mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) + \mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)$

13. $\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu)$
$\equiv \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)$
$+\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}^0\pi^0\ell^+\nu)$
$+\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) + \mathcal{B}(B^+ \to \bar{D}^{*0}\pi^0\ell^+\nu)$

14. $\mathcal{BT}(B^0 \to D^{**-}/(D\pi)^-\ell^+\nu)$
$\equiv \mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu) + \mathcal{B}(B^0 \to D_1^-\ell^+\nu) + \mathcal{B}(B^0 \to D_1'^-\ell^+\nu) + \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)$
$+\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) + \mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)$
$+\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) + \mathcal{B}(B^0 \to D^{*-}\pi^0\ell^+\nu)$

We also define the following branching fraction ratios :

1. $f_{D_2^*} \equiv \frac{\mathcal{B}(D_2^{*+} \to D^0\pi^+)}{\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+)}$

2. $f_{D_2^*/D_1} \equiv \frac{\mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)}$

3. $f_{D\pi/D_0^*} \equiv \frac{\mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)}$

4. $f_{D^*\pi/D_1'} \equiv \frac{\mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu)}{\mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu)}$

5. $f_{D_0^*D\pi T/D_1D_2^*T} \equiv \frac{\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)}$

6. $f_{D_1'D^*\pi T/D_1D_2^*T} \equiv \frac{\mathcal{BT}(B^+ \to \bar{D}_1'^0/(D^*\pi)^0\ell^+\nu)}{\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0}\ell^+\nu)}$

D.2 Measured $B \to D^{(*)}\pi^+\ell^+\nu$ branching fractions

$B \to D^{(*)}\pi^+\ell^+\nu$ branching fractions given in Table 71 and 72 includes not only non-resonant contribution but also $B \to D^{(*)}\pi^+\ell^+\nu$ contributions. Thus,

$$\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) = \mathcal{B}(B^+ \to \bar{D}_0^{*0}\ell^+\nu)\mathcal{B}(D_0^{*0} \to D^+\pi^-) + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^+\pi^-) + \mathcal{B}(B^+ \to D^-\pi^+\ell^+\nu) \tag{529}$$

$$\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu) = \mathcal{B}(B^+ \to \bar{D}_1^0\ell^+\nu)\mathcal{B}(D_1^0 \to D^{*+}\pi^-) + \mathcal{B}(B^+ \to \bar{D}_1'^0\ell^+\nu)\mathcal{B}(D_1'^0 \to D^{*+}\pi^-) + \mathcal{B}(B^+ \to \bar{D}_2^{*0}\ell^+\nu)\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-) + \mathcal{B}(B^+ \to D^{*-}\pi^+\ell^+\nu) \tag{530}$$

$$\mathcal{BT}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) = \mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)\mathcal{B}(D_0^{*+} \to D^0\pi^+) + \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*+} \to D^0\pi^+) + \mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) \tag{531}$$

$$\mathcal{BT}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) = \mathcal{B}(B^0 \to D_1^-\ell^+\nu)\mathcal{B}(D_1^+ \to D^{*0}\pi^+) + \mathcal{B}(B^0 \to D_1'^-\ell^+\nu)\mathcal{B}(D_1'^+ \to D^{*0}\pi^+) + \mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+) + \mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) \tag{532}$$

Thus, using SP5/SP6 MC values

$$\begin{aligned}\mathcal{BT}(B^+ \to D^-\pi^+\ell^+\nu) &= 0.0020 \times 0.6667 + 0.0037 \times 0.459 + 0.0060 \\ &= 0.00903\end{aligned} \tag{533}$$

$$\begin{aligned}\mathcal{BT}(B^+ \to D^{*-}\pi^+\ell^+\nu) &= 0.0056 \times 0.6667 + 0.0037 \times 0.6667 \\ &\quad +0.0037 \times 0.209 + 0.0020 \\ &= 0.00897\end{aligned} \tag{534}$$

$$\begin{aligned}\mathcal{BT}(B^0 \to \bar{D}^0\pi^-\ell^+\nu) &= 0.0020 \times 0.6667 + 0.0037 \times 0.459 + 0.0060 \\ &= 0.00903\end{aligned} \tag{535}$$

$$\begin{aligned}\mathcal{BT}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu) &= 0.0056 \times 0.6667 + 0.0037 \times 0.6667 \\ &\quad +0.0037 \times 0.209 + 0.0020 \\ &= 0.00897\end{aligned} \tag{536}$$

D.3 Inclusive $\mathcal{B}(B \to X_c\ell\nu)$

Inclusive branching fractions in SP5/SP6 MC are given by the sum of exclusive branching fractions :

$$\mathcal{B}(B^+ \to \bar{D}^0\ell^+\nu)+\mathcal{B}(B^+ \to \bar{D}^{*0}\ell^+\nu)+\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0\ell^+\nu) = \mathcal{B}(B^+ \to X_c\ell^+\nu) \tag{537}$$

and

$$\mathcal{B}(B^0 \to D^-\ell^+\nu)+\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)+\mathcal{BT}(B^0 \to D^{**-}/(D\pi)^-\ell^+\nu) = \mathcal{B}(B^0 \to X_c\ell^+\nu) \tag{538}$$

In SP5/SP6 the following value is used :

$$\mathcal{B}(B^+ \to X_c\ell\nu) = \mathcal{B}(B^0 \to X_c\ell\nu) = 0.104 \tag{539}$$

Measured values form PDG [12] are

$$\begin{aligned}\mathcal{B}(B \to X\ell\nu) &= 0.1081 \pm 0.0014 \\ \mathcal{B}(B \to X_u\ell\nu) &= 0.00233 \pm 0.00022\end{aligned} \tag{540}$$

Thus,

$$\begin{aligned}\mathcal{B}(B \to X_c\ell\nu) &= (0.1081 \pm 0.0014) - (0.00233 \pm 0.00022) \\ &= 0.10577 \pm 0.001417\end{aligned} \tag{541}$$

Combining this with lifetime ratio [12]

$$t_{+0} \equiv \frac{\tau_{B^+}}{\tau_{B^0}} = 1.071 \pm 0.009 \tag{542}$$

Branching fraction	SP5/SP6	Measurement [reference]
$\mathcal{B}(B^+ \to \bar{D}^0 \ell^+ \nu)$	0.0210	0.0215 ± 0.0022 [12] $0.0233 \pm 0.0009 \pm 0.0008$ [4]
$\mathcal{B}(B^+ \to \bar{D}^{*0} \ell^+ \nu)$	0.0560	0.065 ± 0.005 [12] $0.0583 \pm 0.0015 \pm 0.0028$ [4]
$\mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+ \nu)$	0.0020	
$\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu)$	0.0056	0.0056 ± 0.0016 [12] $0.00448 \pm 0.00026 \pm 0.00035$ [15]
$\mathcal{B}(B^+ \to \bar{D}_1'^0 \ell^+ \nu)$	0.0037	
$\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu)$	0.0037	$0.00354 \pm 0.00032 \pm 0.00054$ [15]
$\mathcal{B}(B^+ \to D^- \pi^+ \ell^+ \nu)$	0.0060	
$\mathcal{B}(B^+ \to \bar{D}^0 \pi^0 \ell^+ \nu)$	0.0030	
$\mathcal{B}(B^+ \to D^{*-} \pi^+ \ell^+ \nu)$	0.0020	
$\mathcal{B}(B^+ \to \bar{D}^{*0} \pi^0 \ell^+ \nu)$	0.0010	
$\mathcal{BT}(B^+ \to D^- \pi^+ \ell^+ \nu)$	0.00903	0.0052 ± 0.0010 [12] $0.0042 \pm 0.0006 \pm 0.0003$ [4]
$\mathcal{BT}(B^+ \to D^{*-} \pi^+ \ell^+ \nu)$	0.00897	0.0063 ± 0.0015 [12] $0.0059 \pm 0.0005 \pm 0.0004$ [4]

Table 71: B^+ decay branching fractions.

we can calculate B^+ and B^0 separately. By assuming B^+ and B^0 contributions are equal,

$$\mathcal{B}(B^+ \to \bar{X}_c^0 \ell^+ \nu) + \mathcal{B}(B^0 \to X_c^- \ell^+ \nu) = 2\mathcal{B}(B \to X_c \ell \nu) \tag{543}$$

From isospin symmetry

$$\mathcal{B}(B^+ \to \bar{X}_c^0 \ell^+ \nu) = t_{+0} \mathcal{B}(B^0 \to X_c^- \ell^+ \nu) \tag{544}$$

Thus

$$\mathcal{B}(B^+ \to \bar{X}_c^0 \ell^+ \nu) = \frac{2t_{+0}}{1+t_{+0}} \mathcal{B}(B \to X_c \ell \nu) = 0.1094 \pm 0.0014 \tag{545}$$

and

$$\mathcal{B}(B^+ \to \bar{X}_c^0 \ell^+ \nu) = \frac{2}{1+t_{+0}} \mathcal{B}(B \to X_c \ell \nu) = 0.1021 \pm 0.0014 \tag{546}$$

Branching fraction	SP5/SP6	Measurement [reference]
$\mathcal{B}(B^0 \to D^-\ell^+\nu)$	0.0210	0.0208 ± 0.0018 [12]
		$0.0221 \pm 0.0011 \pm 0.0009$ [4]
$\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)$	0.0560	0.0529 ± 0.0019 [12]
		$0.0549 \pm 0.0016 \pm 0.0023$ [4]
$\mathcal{B}(B^0 \to D_0^{*-}\ell^+\nu)$	0.0020	$0.0042 \pm 0.0033 \pm 0.022$ [16]
$\mathcal{B}(B^0 \to D_1^{-}\ell^+\nu)$	0.0056	$0.00364 \pm 0.00032 \pm 0.00049$ [15]
$\mathcal{B}(B^0 \to D_1'^{-}\ell^+\nu)$	0.0037	$0.0124 \pm 0.0025 \pm 0.027$ [16]
$\mathcal{B}(B^0 \to D_2^{*-}\ell^+\nu)$	0.0037	$0.00270 \pm 0.00035 \pm 0.00043$ [15]
$\mathcal{BT}(B^0 \to D^{**-}\ell^+\nu)$	0.0150	$0.027 \pm 0.007 \pm 0.002$ [16]
$\mathcal{B}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)$	0.0060	
$\mathcal{B}(B^0 \to D^-\pi^0\ell^+\nu)$	0.0030	
$\mathcal{B}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)$	0.0020	
$\mathcal{B}(B^0 \to \bar{D}^{*-}\pi^0\ell^+\nu)$	0.0010	
$\mathcal{BT}(B^0 \to \bar{D}^0\pi^-\ell^+\nu)$	0.00903	0.0032 ± 0.0010 [12]
		$0.0043 \pm 0.0008 \pm 0.0003$ [4]
$\mathcal{BT}(B^0 \to \bar{D}^{*0}\pi^-\ell^+\nu)$	0.00897	0.0065 ± 0.0015 [12]
		$0.0048 \pm 0.0008 \pm 0.0004$ [4]

Table 72: B^0 decay branching fractions.

Branching fraction	SP5/SP6	AWG average
$\mathcal{B}(B^+ \to \bar{D}_1^0 \ell^+ \nu)$	0.0056	0.0054 ± 0.0006
$\mathcal{B}(B^+ \to \bar{D}_2^{*0} \ell^+ \nu)$	0.0037	0.0042 ± 0.0008
$\mathcal{BT}(B^+ \to \bar{D}_1^0/\bar{D}_2^{*0} \ell^+ \nu)$	0.0093	
$\mathcal{B}(B^+ \to \bar{D}_0^{*0} \ell^+ \nu)$	0.0020	0.0085 ± 0.0020
$\mathcal{B}(B^+ \to D^- \pi^+ \ell^+ \nu)$	0.0060	0.0019 ± 0.0012
$\mathcal{B}(B^+ \to \bar{D}^0 \pi^0 \ell^+ \nu)$	0.0030	0.0010 ± 0.0006
$\mathcal{BT}(B^+ \to \bar{D}_0^{*0}/(D\pi)^0 \ell^+ \nu)$	0.0110	
$\mathcal{B}(B^+ \to \bar{D}_1^{\prime 0} \ell^+ \nu)$	0.0037	0.0045 ± 0.0009
$\mathcal{B}(B^+ \to D^{*-} \pi^+ \ell^+ \nu)$	0.0020	0.0006 ± 0.0004
$\mathcal{B}(B^+ \to \bar{D}^{*0} \pi^0 \ell^+ \nu)$	0.0010	0.0003 ± 0.0002
$\mathcal{BT}(B^+ \to \bar{D}_1^{*0}/(D^*\pi)^0 \ell^+ \nu)$	0.0067	
$\mathcal{BT}(B^+ \to \bar{D}^{**0}/(D\pi)^0 \ell^+ \nu)$	0.0270	
$f_{D_2^*/D_1}$	$37/56 = 0.66$	
$f_{D\pi/D_0^*}$	$60/20 = 3.0$	
$f_{D^*\pi/D_1^*}$	$20/37 = 0.54$	
$f_{D_0^* D\pi T/D_1 D_2^* T}$	$110/93 = 1.18$	
$f_{D_1^* D^*\pi T/D_1 D_2^* T}$	$67/93 = 0.72$	

Table 73: $B^+ \to D^{**}\ell\nu$ and $B^+ \to D^{(*)}\pi\ell\nu$ decay branching fractions and ratios defined in section 5.

Branching fraction	SP5/SP6	Measurement [reference]
$\mathcal{B}(B^+ \to X \ell^+ \nu)$		0.1099 ± 0.0028 [12]
$\mathcal{B}(B^+ \to \bar{X}_c^0 \ell^+ \nu)$	0.1040	0.108 ± 0.004 [12]
$\mathcal{B}(B^0 \to X \ell^+ \nu)$		0.1033 ± 0.0028 [12]
$\mathcal{B}(B^0 \to X_c^- \ell^+ \nu)$	0.1040	0.101 ± 0.004 [12]
$\mathcal{B}(B \to X \ell^+ \nu)$		0.1024 ± 0.0015 [12]
$\mathcal{B}(B \to X_c \ell \nu)$	0.1040	0.1057 ± 0.0015 [12]
$\mathcal{B}(B \to X_u \ell \nu)$		0.00233 ± 0.00022 [12]
$\mathcal{B}(B \to X_c e \nu(\gamma), E_e > 1.5\mathrm{GeV})$		$0.0454 \pm 0.0003 \pm 0.0008$ [59]

Table 74: Inclusive B decay branching fractions.

Branching fraction	SP5/SP6	Measurement [reference]
$\mathcal{B}(D_0^{*+} \to D^0\pi^+)$	0.6667	
$\mathcal{B}(D_0^{*+} \to D^+\pi^0)$	0.3333	
$\mathcal{B}(D_0^{*0} \to D^+\pi^-)$	0.6667	
$\mathcal{B}(D_0^{*0} \to D^0\pi^0)$	0.3333	
$\mathcal{B}(D_1^{+} \to D^{*0}\pi^+)$	0.6667	
$\mathcal{B}(D_1^{+} \to D^{*+}\pi^0)$	0.3333	
$\mathcal{B}(D_1^{0} \to D^{*+}\pi^-)$	0.6667	
$\mathcal{B}(D_1^{0} \to D^{*0}\pi^0)$	0.3333	
$\mathcal{B}(D_1'^{+} \to D^{*0}\pi^+)$	0.6667	
$\mathcal{B}(D_1'^{+} \to D^{*+}\pi^0)$	0.3333	
$\mathcal{B}(D_1'^{0} \to D^{*+}\pi^-)$	0.6667	
$\mathcal{B}(D_1'^{0} \to D^{*0}\pi^0)$	0.3333	
$f_{D_2^*}$	$459/209 = 2.2$	$\frac{\mathcal{B}(D_2^{*+}\to D^0\pi^+)}{\mathcal{B}(D_2^{*+}\to D^{*0}\pi^+)} = 1.9 \pm 1.1 \pm 0.3$ [12] $\frac{\mathcal{B}(D_2^{*0}\to D^+\pi^-)}{\mathcal{B}(D_2^{*0}\to D^{*+}\pi^-)} = 2.3 \pm 0.6$ [12]
$\mathcal{B}(D_2^{*+} \to D^0\pi^+)$	0.4590	
$\mathcal{B}(D_2^{*+} \to D^+\pi^0)$	0.2290	
$\mathcal{B}(D_2^{*+} \to D^{*0}\pi^+)$	0.2090	
$\mathcal{B}(D_2^{*+} \to D^{*+}\pi^0)$	0.1030	
$\mathcal{B}(D_2^{*0} \to D^+\pi^-)$	0.4590	
$\mathcal{B}(D_2^{*0} \to D^0\pi^0)$	0.2290	
$\mathcal{B}(D_2^{*0} \to D^{*+}\pi^-)$	0.2090	
$\mathcal{B}(D_2^{*0} \to D^{*0}\pi^0)$	0.1030	

Table 75: D^{**} decay branching fractions.

Branching fraction	SP5/SP6	Measurement [reference]
$\mathcal{B}(D^{*+} \to D^0\pi^+)$	0.6830	0.677 ± 0.005 [12]
$\mathcal{B}(D^{*+} \to D^+\pi^0)$	0.3060	0.307 ± 0.005 [12]
$\mathcal{B}(D^{*+} \to D^+\gamma)$	0.0110	0.016 ± 0.004 [12]
$\mathcal{B}(D^{*0} \to D^0\pi^0)$	0.6190	0.619 ± 0.029 [12]
$\mathcal{B}(D^{*0} \to D^0\gamma)$	0.3810	0.381 ± 0.029 [12]
$\mathcal{B}(D^0 \to K^-\pi^+)$	0.0383	0.0382 ± 0.0007 [12] $0.04007 \pm 0.00037 \pm 0.00070$ [28] $0.03891 \pm 0.00035 \pm 0.00059 \pm 0.00035$ [29]
$\mathcal{B}(D^+ \to K^-\pi^+\pi^+)$	0.0900	0.0951 ± 0.0034 [12] $0.0915 \pm 0.0010 \pm 0.0016 \pm 0.0007$ [29]

Table 76: $D^{(*)}$ decay branching fractions.

Constant	SP5/SP6	Measurement [reference]
t_{+0}	1	1.071 ± 0.009 [12]
f_{+0}	1	1.037 ± 0.029 [12]

Table 77: Constants.

Parameter	SP5/SP6	Measurement [reference]
ρ^2	0.92	$1.145 \pm 0.059 \pm 0.030 \pm 0.035$ [39]
R_1	1.18	$1.396 \pm 0.060 \pm 0.035 \pm 0.027$ [39]
R_2	0.72	$0.885 \pm 0.040 \pm 0.022 \pm 0.013$ [39]
ρ^2	0.92	$1.191 \pm 0.048 \pm 0.028$ [40]
R_1	1.18	$1.429 \pm 0.061 \pm 0.044$ [40]
R_2	0.72	$0.827 \pm 0.038 \pm 0.022$ [40]
$\mathcal{F}(1)\lvert V_{cb}\rvert$		$(34.4 \pm 0.3 \pm 1.1) \times 10^{-3}$ [40]
$\mathcal{B}(B^0 \to D^{*-}\ell^+\nu)$	0.0560	$0.0469 \pm 0.0004 \pm 0.0034$ [40]
ρ^2	0.92	$1.16 \pm 0.06 \pm 0.08$ [57]
$\mathcal{F}(1)\lvert V_{cb}\rvert$		$(35.9 \pm 0.6 \pm 1.4) \times 10^{-3}$ [57]
$\mathcal{B}(B^- \to D^{*0}e^-\bar{\nu}_e)$	0.0560	$0.0556 \pm 0.0008 \pm 0.0041$ [57]

Table 78: $B \to D^*\ell\nu$ FF parameters. "Measurement" is the most recent measured value from Babar. Definition of these parameters are given in Appendix I. Note that different parametrizations of ρ^2 was used in SP5/SP6 MC and [39]. See details in section 10

Decay mode	SP5/SP6	Measurement [4]
$\mathcal{B}(B^- \to D^0\ell^-\bar{\nu})$	0.0210	$0.0233 \pm 0.0009 \pm 0.0009$
$\mathcal{B}(B^- \to D^{*0}\ell^-\bar{\nu})$	0.0560	$0.0583 \pm 0.0015 \pm 0.0030$
$\mathcal{B}(B^- \to D^+\pi^-\ell^-\bar{\nu})$	0.00903	$0.0042 \pm 0.0006 \pm 0.0003$
$\mathcal{B}(B^- \to D^{*+}\pi^-\ell^-\bar{\nu})$	0.00897	$0.0059 \pm 0.0005 \pm 0.0004$
$\mathcal{B}(B^- \to D^{(*)}\pi\ell^-\bar{\nu})$	0.0270	$0.0152 \pm 0.0012 \pm 0.0010$
$\mathcal{B}(\bar{B}^0 \to D^+\ell^-\bar{\nu})$	0.0210	$0.0221 \pm 0.0011 \pm 0.0012$
$\mathcal{B}(\bar{B}^0 \to D^{*+}\ell^-\bar{\nu})$	0.0560	$0.0549 \pm 0.0016 \pm 0.0025$
$\mathcal{B}(\bar{B}^0 \to D^0\pi^+\ell^-\bar{\nu})$	0.00903	$0.0043 \pm 0.0008 \pm 0.0003$
$\mathcal{B}(\bar{B}^0 \to D^{*0}\pi^+\ell^-\bar{\nu})$	0.00897	$0.0048 \pm 0.0008 \pm 0.0004$
$\mathcal{B}(\bar{B}^0 \to D^{(*)}\pi\ell^-\bar{\nu})$	0.0270	$0.0137 \pm 0.0017 \pm 0.0010$

Table 79: $B \to D^{(*)}(\pi)\ell\nu$ branching fractions.

E BToDlnu skim rates and luminosities

The event selection rates and luminosities of R16b BToDlnu skims of data and MC is summarized in Table 81 and 82.
The luminosities of BpBm and B0B0bar MC are calculated by the generated number of events and B counting number in the OnPeak table 81. For example, in case of BpBm MC Run2, the generated number of events is 146536000. The B counting number for Run2 OnPeak is 65697538.1 ± 723956.4, which corresponds to the luminosity of 59.402357 fb^{-1}. We assume the half of the $B\bar{B}$ is B^+B^-. Thus, the luminosity is

$$\frac{146536000}{(65697538.1 \pm 723956.4)/2} \times 59.402357 = 264.99 \pm 2.92 \ \mathrm{fb}^{-1}$$

The luminosities of ccbar, uds and tautau MC are calculated by the generated number of events and their cross section. Cross sections are taken from the Babar Physics Book [60] and given in Table 80. The luminosities are calculated based on the cross sections and the number of events before skimming. For example, in case of ccbar Run4 MC, as the number of events before skimming is 188718626, its luminosity is

$$\frac{188718626}{1.30 \times 10^3} = 1.452 \times 10^5 \mathrm{pb}^{-1} = 145.2 \ \mathrm{fb}^{-1}$$

E.1 B counting

The measurement of the number of $\Upsilon(4S)$ mesons, N^0_Υ, is done as follows.

- Apply hadronic event selection to both OnPeak (N^{OnPeak}_{MH}) and OffPeak ($N^{OffPeak}_{\mathrm{MH}}$) data.
- Then, the number of $\Upsilon(4S)$, N_Υ, satisfying the selection criteria is given by

$$N_\Upsilon = N^{OnPeak}_{\mathrm{MH}} - N^{ScaledOffPeak}_{\mathrm{MH}} \tag{547}$$

This $N^{ScaledOffPeak}_{\mathrm{MH}}$ is not the measuerd number but a scaled number based on $\mu^+\mu^-$ pair production.

$$N^{ScaledOffPeak}_{\mathrm{MH}} = \kappa N^{OffPeak}_{\mathrm{MH}} \frac{N^{OnPeak}_{\mu^+\mu^-}}{N^{OffPeak}_{\mu^+\mu^-}} \tag{548}$$

where κ is the factor, which is close to unity, to correct for possible variations in cross sections and efficiencies with center of mass energy. Then, N^0_Υ is given by

$$N^0_\Upsilon = \frac{N_\Upsilon}{\epsilon_{B\bar{B}}} \tag{549}$$

where $\epsilon_{B\bar{B}}$ is the efficiency for $B\bar{B}$ to satisfy the selection criteria.

Type of MC	Cross section (nb)
$B\bar{B}$	1.05
$c\bar{c}$	1.30
uds	2.09
$\tau^+\tau^-$	0.94

Table 80: Cross sections used to calculate MC luminosities.

Data set	Selection rate	Luminosity (fb^{-1})	Number of BBbar
OnPeak Run1	8792976/245552690	17.807136	19411539.8 ± 214782.8
Run2	31374579/852649140	59.402357	65697538.1 ± 723956.4
Run3	17527801/438622047	30.630558	33754342.8 ± 372583.9
Run4	57952809/1448776065	99.405417	$109449337.9 \pm 1205214.0$
Total		207.245468	
OffPeak Run1	515760/28946433	2.330076	
Run2	1674989/90665305	6.833280	
Run3	659867/31451100	2.394734	
Run4	2878531/132941301	9.929434	
Total		21.487524	

Table 81: BToDlnu Data skims event selection rates and luminosities.

Data set	Selection rate	Luminosity (fb^{-1})
BpBm Run1	6920906/27176000	49.86 ± 0.55
Run2	39029539/146536000	264.99 ± 2.92
Run3	23854196/86492000	156.98 ± 1.73
Run4	89759580/323822000	588.21 ± 6.48
Total		1060.04 ± 7.34
B0B0bar Run1	7421839/28762000	52.77 ± 0.58
Run2	30164526/111924000	202.40 ± 2.23
Run3	17081965/61158000	111.00 ± 1.23
Run4	90093404/321150000	583.36 ± 6.42
Total		949.53 ± 6.93
ccbar Run1	1897239/21346000	16.420
Run2	9838664/99972000	76.708
Run3	6127861/57700000	44.385
Run4	25956105/246602000	189.69
Total		327.20
uds Run1	948025/30168000	14.434
Run2	6799899/176242000	84.326
Run3	5639210/124040000	59.349
Run4	15259514/346551000	165.81
Total		323.919
tautau Run1	135022/22098000	23.509
Run2	695973/99806000	106.18
Run3	544465/69852000	74.311
Run4	1472037/154900000	164.79
Total		368.79

Table 82: BToDlnu MC skims event selection rates and luminosities.

F $\cos\theta_{BY}$

F.1 $B \to D\ell\nu$ decays

In the decay of $B \to Dl + \nu$, 4-momentum conservation gives

$$p_B = p_{Dl} + p_\nu \tag{550}$$

As the mass of neutrino is equal to zero

$$0 = p_\nu^2 = (p_B - p_{Dl})^2 = p_B^2 + p_{Dl}^2 - 2p_B \cdot p_{Dl} = m_B^2 + m_{Dl}^2 - 2p_B \cdot p_{Dl} \tag{551}$$

Here,

$$p_B \cdot p_{Dl} = E_B E_{Dl} - \vec{p}_B \cdot \vec{p}_{Dl} = E_B E_{Dl} - |\vec{p}_B||\vec{p}_{Dl}| \cos\Theta_{B-Dl} \tag{552}$$

Hence

$$\begin{aligned} 0 &= m_B^2 + m_{Dl}^2 - 2(E_B E_{Dl} - |\vec{p}_B||\vec{p}_{Dl}| \cos\Theta_{B-Dl}) \\ \Rightarrow m_B^2 + m_{Dl}^2 &= 2E_B E_{Dl} - 2|\vec{p}_B||\vec{p}_{Dl}| \cos\Theta_{B-Dl} \end{aligned} \tag{553}$$

Therefore

$$\cos\Theta_{B-Dl} = \frac{2E_B E_{Dl} - m_B^2 - m_{Dl}^2}{2|\vec{p}_B||\vec{p}_{Dl}|} \tag{554}$$

In practice, E_B is given by the half of the beam energy in the CM frame. $|\vec{p}_B|$ is given by

$$|\vec{p}_B| = \sqrt{E_B^2 - m_B^2} \tag{555}$$

F.2 With an additional pion

If there are additional particles, $\cos\Theta_{B-Dl} < -1$ is possible. Let us consider an additional pion, for example, from D^*. Conservation of 4-momentum is

$$p_B = p_{Dl} + p_\pi + p_\nu \tag{556}$$

Then,

$$\begin{aligned} &(p_\nu + p_\pi)^2 = (p_B - p_{Dl})^2 \\ &\Rightarrow p_\nu^2 + p_\pi^2 + 2p_\nu \cdot p_\pi = p_B^2 + p_{Dl}^2 - 2p_B \cdot p_{Dl} \\ &\Rightarrow 0 + m_\pi^2 + 2p_\nu \cdot p_\pi = m_B^2 + m_{Dl}^2 - 2p_B \cdot p_{Dl} \end{aligned} \tag{557}$$

Since $m_\pi \ll m_B$,

$$0 + 0 + 2p_\nu \cdot p_\pi = m_B^2 + m_{Dl}^2 - 2p_B \cdot p_{Dl} \tag{558}$$

and also

$$\begin{aligned} p_\nu \cdot p_\pi &= E_\nu E_\pi - \vec{p}_\nu \cdot \vec{p}_\pi = E_\nu E_\pi - |\vec{p}_\nu||\vec{p}_\pi| \cos\Theta_{\nu-\pi} \\ &= E_\nu E_\pi - E_\nu E_\pi \cos\Theta_{\nu-\pi} = E_\nu E_\pi (1 - \cos\Theta_{\nu-\pi}) \geq 0 \end{aligned} \tag{559}$$

Let the angle between B and Dl be Θ'_{B-Dl}. Then,

$$p_B \cdot p_{Dl} = E_B E_{Dl} - \vec{p}_B \cdot \vec{p}_{Dl} = E_B E_{Dl} - |\vec{p}_B||\vec{p}_{Dl}| \cos\Theta'_{B-Dl} \tag{560}$$

Hence,

$$\begin{aligned}\cos\Theta'_{B-Dl} &= \frac{2E_BE_{Dl}-m_B^2-m_{Dl}^2}{2|\vec{p}_B||\vec{p}_{Dl}|} + \frac{2p_\nu\cdot p_\pi}{2|\vec{p}_B||\vec{p}_{Dl}|} \\ &= \cos\Theta_{B-Dl} + \alpha\end{aligned} \tag{561}$$

where

$$\alpha = \frac{2p_\nu\cdot p_\pi}{2|\vec{p}_B||\vec{p}_{Dl}|} \geq 0 \tag{562}$$

Considering

$$-1 \leq \cos\Theta'_{B-Dl} \leq 1 \tag{563}$$

$\cos\Theta_{B-Dl} = \cos\Theta'_{B-Dl} - \alpha$ is

$$-1-\alpha \leq \cos\Theta_{B-Dl} \leq 1-\alpha \leq 1 \tag{564}$$

Since $\alpha \geq 0$, the lower end of $\cos\Theta_{B-Dl}$ can be smaller than -1, whereas the upper end remains to 1.

G If D and l come from different B

In the case that D and l came from different B, $\cos\Theta_{B-Dl} > 1$ is possible.

$$\begin{aligned}&E_{Dl} = E_D + E_l = \sqrt{m_D^2+|\vec{p}_D|^2} + \sqrt{m_l^2+|\vec{p}_l|^2} = \sqrt{m_D^2+|\vec{p}_D|^2} + |\vec{p}_l| \\ &|\vec{p}_{Dl}|^2 = |\vec{p}_D+\vec{p}_l|^2 = |\vec{p}_D|^2 + |\vec{p}_l|^2 + 2|\vec{p}_D||\vec{p}_l|\cos\theta\end{aligned} \tag{565}$$

and

$$\begin{aligned}m_{Dl}^2 &= E_{Dl}^2 - |\vec{p}_{Dl}|^2 \\ &= m_D^2 + |\vec{p}_D|^2 + |\vec{p}_l|^2 + 2|\vec{p}_l|\sqrt{m_D^2+|\vec{p}_D|^2} - |\vec{p}_D|^2 - |\vec{p}_l|^2 - 2|\vec{p}_D||\vec{p}_l|\cos\theta \\ &= m_D^2 + 2|\vec{p}_l|\sqrt{m_D^2+|\vec{p}_D|^2} - 2|\vec{p}_D||\vec{p}_l|\cos\theta \\ &= m_D^2 + 2|\vec{p}_l|(\sqrt{m_D^2+|\vec{p}_D|^2} - |\vec{p}_D|\cos\theta)\end{aligned} \tag{566}$$

If D and l come from different B, p_D and p_l are not restricted by energy momentum conservation. The $|\vec{p}_D|$ and $|\vec{p}_l|$ can be any value. For example, if $|\vec{p}_l| \simeq |\vec{p}_D|$ and $\cos\theta \simeq -1$, $|\vec{p}_{Dl}|$ can be very small. However,

$$\begin{aligned}&2E_BE_{Dl} - m_B^2 - m_{Dl}^2 \\ &\simeq 2E_B(\sqrt{m_D^2+|\vec{p}_D|^2} + |\vec{p}_D|) - m_B^2 - m_D^2 - 2|\vec{p}_D|(\sqrt{m_D^2+|\vec{p}_D|^2} + |\vec{p}_D|) \\ &= 2(\sqrt{m_D^2+|\vec{p}_D|^2} + |\vec{p}_D|)(E_B - |\vec{p}_D|) - m_B^2 - m_D^2\end{aligned} \tag{567}$$

is not necessarily small. For example, if $|\vec{p}_D| \ll m_D$

$$2E_BE_{Dl} - m_B^2 - m_{Dl}^2 \simeq 2m_DE_B - m_B^2 - m_D^2 \simeq -(m_B-m_D)^2 \tag{568}$$

Hence

$$\cos\Theta_{B-Dl} = \frac{2E_BE_{Dl}-m_B^2-m_{Dl}^2}{2|\vec{p}_B||\vec{p}_{Dl}|} \tag{569}$$

can be large negative number. And, if $|\vec{p}_D| \simeq 1.2\, m_D$

$$2E_B E_{Dl} - m_B^2 - m_{Dl}^2 \simeq 5.4\, m_D(m_B - m_D) - m_B^2 - m_D^2 > 0 \tag{570}$$

Hence

$$\cos\Theta_{B-Dl} = \frac{2E_B E_{Dl} - m_B^2 - m_{Dl}^2}{2|\vec{p}_B||\vec{p}_{Dl}|} \tag{571}$$

can be larger than 1.

H Velocity transfer w

H.1 w and q^2

In the framework of Heavy Quark Effective Theory (HQET) it is convenient to use the velocity transfer w instead of momentum transfer q^2. In semileptonic $B \to D^{(*)}\ell\nu$ decays, w is defined as the product of 4-velocities of B and $D^{(*)}$

$$w \equiv v_B \cdot v_{D^{(*)}} = \frac{p_B \cdot p_{D^{(*)}}}{m_B m_{D^{(*)}}} \tag{572}$$

The momentum transfer q from B to $D^{(*)}$ (or the invariant mass of the virtual W) is given by

$$q = p_\ell + p_\nu = p_B - p_{D^{(*)}} \tag{573}$$

Thus

$$\begin{aligned} &q^2 = (p_B - p_{D^{(*)}})^2 = p_B^2 + p_{D^{(*)}}^2 - 2p_B \cdot p_{D^{(*)}} = m_B^2 + m_{D^{(*)}}^2 - 2p_B \cdot p_{D^{(*)}} \\ &\Rightarrow p_B \cdot p_{D^{(*)}} = \tfrac{m_B^2 + m_{D^{(*)}}^2 - q^2}{2} \end{aligned} \tag{574}$$

Hence

$$w = \frac{m_B^2 + m_{D^{(*)}}^2 - q^2}{2m_B m_{D^{(*)}}} \tag{575}$$

w is a Lorentz invariant and linearly related to q^2.
w corresponds to the relativistic boost γ of the $D^{(*)}$ in the B rest frame.

$$\gamma_{D^{(*)}} = \frac{E_{D^{(*)}}}{m_{D^{(*)}}} \tag{576}$$

H.2 $1 \leq w \lesssim 1.5$

The range of w and q^2 are restricted by kinematics of the decay. When $q^2 = 0$, w is maximum

$$w_{\rm max} = \frac{m_B^2 + m_{D^{(*)}}^2}{2m_B m_{D^{(*)}}} \tag{577}$$

Using $m_B = 5279$, $m_D = 1867$ and $m_{D^*} = 2008$ MeV,

$$\frac{m_B^2 + m_D^2}{2m_B m_D} = 1.590, \qquad \frac{m_B^2 + m_{D^*}^2}{2m_B m_{D^*}} = 1.504 \tag{578}$$

Using $m_{D_0^*} = 2400$, $m_{D_1'} = 2430$, $m_{D_1} = 2420$ and $m_{D_2^*} = 2460$ MeV,

$$\begin{aligned} \frac{m_B^2 + m_{D_0^*}^2}{2m_B m_{D_0^*}} = 1.327, &\qquad \frac{m_B^2 + m_{D_1'}^2}{2m_B m_{D_1'}} = 1.316, \\ \frac{m_B^2 + m_{D_1}^2}{2m_B m_{D_1}} = 1.320, &\qquad \frac{m_B^2 + m_{D_2^*}^2}{2m_B m_{D_2^*}} = 1.306 \end{aligned} \tag{579}$$

The minimum of the w is

$$w_{\text{min}} = 1 \tag{580}$$

which corresponds to

$$q^2_{\text{max}} = (m_B - m_{D^{(*)}})^2 \tag{581}$$

where

$$(m_B - m_D)^2 = 11.64, \quad (m_B - m_{D^*})^2 = 10.69 \ (\text{GeV}/c^2)^2 \tag{582}$$

and

$$(m_B - m_{D_0^*})^2 = 8.29, \quad (m_B - m_{D_1'})^2 = 8.12 \ (\text{GeV}/c^2)^2, \tag{583}$$

$$(m_B - m_{D_1})^2 = 8.17, \quad (m_B - m_{D_2^*})^2 = 7.95 \ (\text{GeV}/c^2)^2 \tag{584}$$

H.3 Practical calculation of w

We can calculate w by the following formula

$$w = \frac{p_B \cdot p_{D^{(*)}}}{m_B m_{D^{(*)}}} = \frac{E_B E_{D^{(*)}} - |\vec{P}_B||\vec{P}_{D^{(*)}}| \cos\theta_{B-D}}{m_B m_{D^{(*)}}} \tag{585}$$

In practice, E_B is the half of the beam energy, and $|\vec{P}_B|$ is given by

$$|\vec{P}_B| = \sqrt{E_B^2 - m_B^2} \tag{586}$$

$\cos\theta_{B-D}$ can be calculated from $\cos\theta_{B-Dl}$ (see Appendix F and Figure 29).

I Form Factors (FF) of $B \to D^{(*,**)}X\ell\nu$ decays

I.1 Matrix elements, form factors and decay rates

The amplitude of $B \to D^{(*,**)}\ell\nu$ decay is given by

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}V_{cb}L^{\mu}H_{\mu} \tag{587}$$

where the leptonic current can be written in terms of Dirac spinors u_l and v_ν

$$L^{\mu} = \bar{u}_l\gamma^{\mu}(1-\gamma_5)v_{\nu} \tag{588}$$

The hadronic current H_μ can be written in terms of quark spinors b and c

$$H^{\mu} =< D|\bar{c}\gamma^{\mu}(1-\gamma_5)b|B >=< D|V^{\mu}|B > - < D|A^{\mu}|B > \tag{589}$$

where

$$\begin{aligned} V^{\mu} &= \bar{c}\gamma^{\mu}b \quad \text{(vector current)} \\ A^{\mu} &= \bar{c}\gamma^{\mu}\gamma_5 b \quad \text{(axial-vector current)} \end{aligned} \tag{590}$$

The matrix elements $< D|V^{\mu}|B >$ and $< D|A^{\mu}|B >$ can be expressed by form factors. In constituent quark models such as ISGW2 model [31][32], form factors are functions of momentum transfer q^2, whereas in HQET (Heavy Quark Effective Theory) models, it is more natural to express form factors as functions of the velocity transfer w

$$w = v \cdot v' = v_B \cdot v_D = \frac{p_B \cdot p_D}{m_B m_D} \tag{591}$$

We also use the following mass ratios

$$r = \frac{m_D}{m_B}, \quad R = \frac{2\sqrt{m_B m_D}}{m_B + m_D} \tag{592}$$

where D in the v_D, p_D and m_D may be D^*, D_0^*, D_1', D_1 or D_2^* depending on the final state.

I.1.1 $B \to D\ell\nu$

In the ISGW2 model, the matrix elements are given by

$$< D(p')|V^{\mu}|B(p) >= f_+(q^2)(p+p')^{\mu} + f_-(q^2)(p-p')^{\mu} \tag{593}$$

$$< D(p')|A^{\mu}|B(p) >= 0 \tag{594}$$

In HQET models,

$$\frac{< D(v')|V^{\mu}|B(v) >}{\sqrt{m_D m_B}} = h_+(v+v')^{\mu} + h_-(v-v')^{\mu} \tag{595}$$

$$\frac{< D(v')|A^\mu|B(v) >}{\sqrt{m_D m_B}} = 0 \tag{596}$$

We can relate ISGW2 form factors with HQET form factors using $p = m_B v$ and $p' = m_D v'$. By comparing the coefficients of v^μ and v'^μ,

$$f_+ m_B + f_- m_B = \sqrt{m_D m_B}(h_+ + h_-) \tag{597}$$
$$f_+ m_D - f_- m_D = \sqrt{m_D m_B}(h_+ - h_-) \tag{598}$$

Thus

$$h_+ = \frac{m_B + m_D}{2\sqrt{m_D m_B}}\left(f_+ + \frac{m_B - m_D}{m_B + m_D} f_-\right) \tag{599}$$
$$h_- = \frac{m_B + m_D}{2\sqrt{m_D m_B}}\left(\frac{m_B - m_D}{m_B + m_D} f_+ + f_-\right) \tag{600}$$

Hence,

$$h_+ = \frac{1}{R}\left(f_+ + \frac{1-r}{1+r} f_-\right) \tag{601}$$
$$h_- = \frac{1}{R}\left(\frac{1-r}{1+r} f_+ + f_-\right) \tag{602}$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D\ell\nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{3/2} \mathcal{J}_D(w) \tag{603}$$

where

$$\mathcal{J}_D(w) = [(1+r)h_+ - (1-r)h_-]^2 \tag{604}$$

I.1.2 $B \to D^*\ell\nu$

In the ISGW2 model, the matrix elements are given by

$$< D^*(p', \varepsilon)|V^\mu|B(p) >= -ig(q^2)\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu (p+p')_\alpha (p-p')_\beta \tag{605}$$

$$< D^*(p', \varepsilon)|A^\mu|B(p) >= f(q^2)\varepsilon^{*\mu} + a_+(q^2)(\varepsilon^* \cdot p)(p+p')^\mu + a_-(q^2)(\varepsilon^* \cdot p)(p-p')^\mu \tag{606}$$

In HQET models,

$$\frac{< D^*(v', \varepsilon)|V^\mu|B(v) >}{\sqrt{m_{D^*} m_B}} = ih_V \epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu v'_\alpha v_\beta \tag{607}$$

$$\frac{< D^*(v', \varepsilon)|A^\mu|B(v) >}{\sqrt{m_{D^*} m_B}} = h_{A_1}(w+1)\varepsilon^{*\mu} - (h_{A_2} v^\mu + h_{A_3} v'^\mu)(\varepsilon^* \cdot v) \tag{608}$$

We can relate ISGW2 form factors with HQET form factors using $p = m_B v$ and $p' = m_D v'$. First, we use $< D^*(p', \varepsilon)|V^\mu|B(p) >$. We can simplify

$$\begin{aligned}&\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu (p+p')_\alpha (p-p')_\beta = \epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu (m_B v + m_{D^*} v')_\alpha (m_B v - m_{D^*} v')_\beta \\ &= -2 m_B m_{D^*} \epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu v_\alpha v'_\beta\end{aligned} \tag{609}$$

where we used the fact that $\epsilon^{\mu\nu\alpha\beta}$ is antisymmetric but $v_\alpha v_\beta$ and $v'_\alpha v'_\beta$ are symmetric between α and β. Thus,

$$2 m_B m_{D^*} g = \sqrt{m_{D^*} m_B}\, h_V \tag{610}$$

Hence

$$h_V = 2\sqrt{m_{D^*} m_B}\, g = m_B R(1+r) g \tag{611}$$

Next, we use $< D^*(p', \varepsilon)|A^\mu|B(p) >$. By comparing the coefficient of $\varepsilon^{*\mu}$,

$$f = \sqrt{m_{D^*} m_B} h_{A_1} (w+1) = \frac{m_B R(1+r)}{2} (w+1) h_{A_1} \tag{612}$$

Thus

$$h_{A_1} = \frac{2}{m_B R(1+r)(w+1)} f \tag{613}$$

By comparing the coefficients of v^μ and v'^μ,

$$m_B a_+ + m_B a_- = \sqrt{m_{D^*} m_B} h_{A_2} \tag{614}$$

$$m_{D^*} a_+ - m_{D^*} a_- = \sqrt{m_{D^*} m_B} h_{A_3} \tag{615}$$

Thus

$$h_{A_3} + r h_{A_2} = \frac{m_{D^*} m_B}{\sqrt{m_{D^*} m_B}} \cdot 2 a_+ = m_B R(1+r) a_+ \tag{616}$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D^* \ell\nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2} \mathcal{J}_{D^*}(w) \tag{617}$$

where

$$\begin{aligned}\mathcal{J}_{D^*}(w) = {} & 2(1 - 2wr + r^2)\left[(w+1)^2 h_{A_1}^2 + (w^2-1) h_V^2\right] \\ & + \left[(w+1)(w-r) h_{A_1} - (w^2-1)(h_{A_3} + r h_{A_2})\right]^2\end{aligned} \tag{618}$$

I.1.3 $B \to D_0^* \ell\nu$

In the ISGW2 model, the matrix elements are given by

$$< D_0^*(p')|V^\mu|B(p) >= 0 \tag{619}$$

$$< D_0^*(p')|A^\mu|B(p) >= u_+(q^2)(p+p')^\mu + u_-(q^2)(p-p')^\mu \tag{620}$$

In HQET models,

$$\frac{< D_0^*(v')|V^\mu|B(v) >}{\sqrt{m_{D_0^*} m_B}} = 0 \tag{621}$$

$$\frac{< D_0^*(v')|A^\mu|B(v) >}{\sqrt{m_{D_0^*} m_B}} = g_+(v+v')^\mu + g_-(v-v')^\mu \tag{622}$$

We can relate the form factors as we did in the $B \to D\ell\nu$ case

$$g_+ = \frac{1}{R}\left(u_+ + \frac{1-r}{1+r}u_-\right) \tag{623}$$

$$g_- = \frac{1}{R}\left(\frac{1-r}{1+r}u_+ + u_-\right) \tag{624}$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D_0^*\ell\nu)}{dw} = \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{3/2}\, \mathcal{J}_{D_0^*}(w) \tag{625}$$

where

$$\mathcal{J}_{D_0^*}(w) = [(1+r)g_+ - (1-r)g_-]^2 \tag{626}$$

I.1.4 $B \to D_1'\ell\nu$

In the ISGW2 model, the matrix elements are given by

$$< D_1'(p',\varepsilon)|V^\mu|B(p) >= l(q^2)\varepsilon^{*\mu} + c_+(q^2)(\varepsilon^*\cdot p)(p+p')^\mu + c_-(q^2)(\varepsilon^*\cdot p)(p-p')^\mu \tag{627}$$

$$< D_1'(p',\varepsilon)|A^\mu|B(p) >= -iq(q^2)\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu (p+p')_\alpha (p-p')_\beta \tag{628}$$

In HQET models,

$$\frac{< D_1'(v',\varepsilon)|V^\mu|B(v) >}{\sqrt{m_{D_1'} m_B}} = g_{V_1}\varepsilon^{*\mu} + (g_{V_2}v^\mu + g_{V_3}v'^\mu)(\varepsilon^*\cdot v) \tag{629}$$

$$\frac{< D_1(v',\varepsilon)|A^\mu|B(v) >}{\sqrt{m_{D_1} m_B}} = ig_A\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu v_\alpha v'_\beta \tag{630}$$

We can relate the form factors as we did in the $B \to D^*\ell\nu$ case

$$g_A = m_B R(1+r)q \tag{631}$$

and

$$g_{V_1} = \frac{2}{m_B R(1+r)} l \tag{632}$$

$$g_{V_3} + r g_{V_2} = m_B R(1+r)c_+ \tag{633}$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D_1'\ell\nu)}{dw} = \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2}\, \mathcal{J}_{D_1'}(w) \tag{634}$$

where

$$\begin{aligned}\mathcal{J}_{D_1'}(w) = &\left[(w-r)g_{V_1} + (w^2-1)(g_{V_3}+rg_{V_2})\right]^2 \\ &+2(1-2rw+r^2)[g_{V_1}^2 + (w^2-1)g_A^2]\end{aligned} \tag{635}$$

I.1.5 $B \to D_1 \ell \nu$

In the ISGW2 model, the matrix elements are given by

$$< D_1'(p',\varepsilon)|V^\mu|B(p) >= r(q^2)\varepsilon^{*\mu} + s_+(q^2)(\varepsilon^* \cdot p)(p+p')^\mu + s_-(q^2)(\varepsilon^* \cdot p)(p-p')^\mu \quad (636)$$

$$< D_1'(p',\varepsilon)|A^\mu|B(p) >= -iv(q^2)\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu (p+p')_\alpha (p-p')_\beta \quad (637)$$

In HQET models,

$$\frac{< D_1(v',\varepsilon)|V^\mu|B(v) >}{\sqrt{m_{D_1} m_B}} = f_{V_1}\varepsilon^{*\mu} + (f_{V_2} v^\mu + f_{V_3} v'^\mu)(\varepsilon^* \cdot v) \quad (638)$$

$$\frac{< D_1(v',\varepsilon)|A^\mu|B(v) >}{\sqrt{m_{D_1} m_B}} = i f_A \epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu v_\alpha v'_\beta \quad (639)$$

We can relate the form factors as we did in the $B \to D^* \ell \nu$ case

$$f_A = m_B R(1+r)v \quad (640)$$

and

$$f_{V_1} = \frac{2}{m_B R(1+r)} r \quad (641)$$

$$f_{V_3} + r f_{V_2} = m_B R(1+r) s_+ \quad (642)$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D_1 \ell \nu)}{dw} = \frac{G_F^2 |V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2} \, \mathcal{J}_{D_1}(w) \quad (643)$$

where

$$\begin{aligned}\mathcal{J}_{D_1}(w) = &\left[(w-r)f_{V_1} + (w^2-1)(f_{V_3} + r f_{V_2})\right]^2 \\ &+2(1-2rw+r^2)[f_{V_1}^2 + (w^2-1) f_A^2]\end{aligned} \quad (644)$$

I.1.6 $B \to D_2^* \ell \nu$

In the ISGW2 model, the matrix elements are given by

$$< D_2^*(p',\varepsilon)|V^\mu|B(p) >= -ih(q^2)\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_{\nu\sigma} p^\sigma (p+p')_\alpha (p-p')_\beta \quad (645)$$

$$\begin{aligned}&< D_2^*(p',\varepsilon)|A^\mu|B(p) > \\ &= k(q^2)\varepsilon^{*\mu\alpha}p_\alpha + b_+(q^2)\varepsilon^*_{\alpha\beta}p^\alpha p^\beta (p+p')^\mu + b_-(q^2)\varepsilon^*_{\alpha\beta}p^\alpha p^\beta (p-p')^\mu\end{aligned} \quad (646)$$

In HQET models,

$$\frac{< D_2^*(v',\varepsilon)|V^\mu|B(v) >}{\sqrt{m_{D_2^*} m_B}} = i k_V \epsilon^{\mu\nu\alpha\beta}\varepsilon^*_{\nu\sigma} v^\sigma v_\alpha v'_\beta \quad (647)$$

$$\frac{< D_2^*(v',\varepsilon)|A^\mu|B(v) >}{\sqrt{m_{D_2^*} m_B}} = k_{A_1}\varepsilon^{*\mu\alpha}v_\alpha + (k_{A_2} v^\mu + k_{A_3} v'^\mu)\varepsilon^{*\alpha\beta} v_\alpha v'_\beta \quad (648)$$

We can relate the form factors as we did in the $B \to D^*\ell\nu$ case. Notice that we pick up an extra m_B from $\varepsilon^{*\mu\alpha}p_\alpha$.

$$k_V = m_B^2 R(1+r)v \tag{649}$$

and

$$k_{A_1} = \frac{2}{R(1+r)}r \tag{650}$$

$$k_{A_3} + r k_{A_2} = m_B^2 R(1+r)s_+ \tag{651}$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D_2^*\ell\nu)}{dw} = \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{3/2}\, \mathcal{J}_{D_2^*}(w) \tag{652}$$

where

$$\begin{aligned}\mathcal{J}_{D_2^*}(w) = \frac{2}{3}\left[(w-r)k_{A_1} + (w^2-1)(k_{A_3}+rk_{A_2})\right]^2 \\ +(1-2rw+r^2)[k_{A_1}^2 + (w^2-1)k_V^2]\end{aligned} \tag{653}$$

I.2 Traditional notation

We summarize a traditional notation of form factors and decay rates as a reference.

I.2.1 $B \to D\ell\nu$

Traditionally, the form factors $F_0(q^2)$ and $F_1(q^2)$ are used.

$$\begin{aligned}&< D(p')|V^\mu|B(p) > \\ &= F_1(q^2)\left[(p+p')^\mu - \tfrac{m_B^2-m_D^2}{q^2}q^\mu\right] + F_0(q^2)\tfrac{m_B^2-m_D^2}{q^2}q^\mu\end{aligned} \tag{654}$$

where

$$q^\mu = p^\mu - p'^\mu \tag{655}$$

We can relate these form factors with $f_+(q^2)$ and $f_-(q^2)$

$$\begin{aligned}F_1(q^2) &= f_+(q^2) \\ F_0(q^2) &= f_+(q^2) + \tfrac{q^2}{m_B^2-m_D^2}f_-(q^2)\end{aligned} \tag{656}$$

The form factors $F_1(q^2)$ and $F_0(q^2)$ can be associated with the exchange of particles with quantum numbers $J^P = 1^-$ and $J^P = 0^+$, respectively. There is no singular behavior at $q^2 = 0$ because $F_1(0) = F_0(0)$. In practice the terms proportional to q^μ are nearly always negligible. Thus

$$< D(p')|V^\mu|B(p) >= F_1(q^2)(p+p')^\mu \tag{657}$$

or

$$< D(p')|V^\mu|B(p) >= f_+(q^2)(p+p')^\mu \tag{658}$$

The differential decay rate is given by

$$\frac{d\Gamma(B \to D\ell\nu)}{dq^2} = \frac{G_F^2|V_{cb}|^2}{24\pi^3}|\vec{P}_D|^3|f_+(q^2)|^2$$
$$= \frac{G_F^2|V_{cb}|^2}{192\pi^3 m_B^3}\left[(q^2 - m_B^2 - m_D^2)^2 - 4m_B^2 m_D^2\right]^{3/2}|f_+(q^2)|^2 \tag{659}$$

where we used the following calculation. In the B rest frame,

$$p = (m_B, \vec{0}), \quad p' = (\sqrt{m_D^2 + |\vec{P}_D|^2}, \vec{p}') \tag{660}$$

Thus

$$\begin{aligned} &q^2 = (p - p')^2 = p^2 + p'^2 - 2p \cdot p' = m_B^2 + m_D^2 - 2m_B\sqrt{m_D^2 + |\vec{P}_D|^2} \\ &\Rightarrow 4m_B^2\left(m_D^2 + |\vec{P}_D|^2\right) = (q^2 - m_B^2 - m_D^2)^2 \\ &\Rightarrow |\vec{P}_D| = \tfrac{1}{2m_B}\left[(q^2 - m_B^2 - m_D^2)^2 - 4m_B^2 m_D^2\right]^{1/2} \end{aligned} \tag{661}$$

We can derive $\frac{d\Gamma(B\to D\ell\nu)}{dq^2}$ from $\frac{d\Gamma(B\to D\ell\nu)}{dw}$ (Equation (603)). Since

$$w = \frac{m_B^2 + m_D^2 - q^2}{2m_B m_D} \quad \Rightarrow \quad dw = -\frac{dq^2}{2m_B m_D} \tag{662}$$

the left-hand side is

$$\frac{d\Gamma(B \to D\ell\nu)}{dw} = 2m_B m_D \frac{d\Gamma(B \to D\ell\nu)}{dq^2} \tag{663}$$

where we removed a negative sign because it is not relevant to the decay rate, which is positive. The right-hand side is

$$\begin{aligned} &\frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2 - 1)^{3/2}|(1 + r)h_+ - (1 - r)h_-|^2 \\ &= \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3}\left(\frac{m_D}{m_B}\right)^3\left[\left(\frac{m_B^2 + m_D^2 - q^2}{2m_B m_D}\right)^2 - 1\right]^{3/2} \\ &\qquad \times\left|\left(1 + \frac{m_D}{m_B}\right)h_+ - \left(1 - \frac{m_D}{m_B}\right)h_-\right|^2 \\ &= \frac{G_F^2|V_{cb}|^2}{48\pi^3}\frac{m_B^5 m_D^3 (m_B + m_D)^2}{m_B^3 (8m_B^3 m_D^3) m_B^2}[(m_B^2 + m_D^2 - q^2)^2 - 4m_B^2 m_D^2]^{3/2} \\ &\qquad \times\left|h_+ - \frac{m_B - m_D}{m_B + m_D}h_-\right|^2 \\ &= \frac{G_F^2|V_{cb}|^2}{192\pi^3 m_B^3}\frac{(m_B + m_D)^2}{2}[(m_B^2 + m_D^2 - q^2)^2 - 4m_B^2 m_D^2]^{3/2} \\ &\qquad \times\left|\frac{2\sqrt{m_B m_D}}{m_B + m_D}f_+\right|^2 \\ &= 2m_B m_D \frac{G_F^2|V_{cb}|^2}{192\pi^3 m_B^3}[(m_B^2 + m_D^2 - q^2)^2 - 4m_B^2 m_D^2]^{3/2}|f_+|^2 \end{aligned} \tag{664}$$

Thus, we recover

$$\frac{d\Gamma(B \to D\ell\nu)}{dq^2} = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_B^3} [(m_B^2 + m_D^2 - q^2)^2 - 4m_B^2 m_D^2]^{3/2} |f_+|^2 \tag{665}$$

I.2.2 $B \to D^* \ell\nu$

In this case

$$< D^*(p', \varepsilon)|V^\mu|B(p) >= \frac{2i\epsilon^{\mu\nu\alpha\beta}}{m_B + m_{D^*}} \varepsilon_\nu^* p'_\alpha p_\beta V(q^2) \tag{666}$$

and

$$\begin{aligned} < D^*(p', \varepsilon)|A^\mu|B(p) >= &-(m_B + m_{D^*})\varepsilon^{*\mu} A_1(q^2) \\ &+\frac{\varepsilon^* \cdot q}{m_B + m_{D^*}} (p + p')^\mu A_2(q^2) \\ &+2m_{D^*} \frac{\varepsilon^* \cdot q}{q^2} q^\mu A_3(q^2) \\ &-2m_{D^*} \frac{\varepsilon^* \cdot q}{q^2} q^\mu A_0(q^2) \end{aligned} \tag{667}$$

where

$$A_3(q^2) = \frac{m_B + m_{D^*}}{2m_{D^*}} A_1(q^2) - \frac{m_B - m_{D^*}}{2m_{D^*}} A_2(q^2) \tag{668}$$

with $A_0(0) = A_3(0)$. The form factors $A_1(q^2)$ and $A_2(q^2)$ can be associated with the exchange of particles with quantum numbers $J^P = 1^+$, whereas $V(q^2)$ is associated with $J^P = 1^-$.
We can relate these form factors with $g(q^2)$, $f(q^2)$, $a_+(q^2)$ and $a_-(q^2)$

$$\begin{aligned} V(q^2) &= (m_B + m_{D^*})g(q^2) \\ A_1(q^2) &= \tfrac{1}{m_B + m_{D^*}} f(q^2) \\ A_2(q^2) &= (m_B + m_{D^*})a_+(q^2) \\ A_3(q^2) - A_0(q^2) &= \tfrac{q^2}{2m_{D^*}} a_-(q^2) \end{aligned} \tag{669}$$

Here, we used

$$\begin{aligned} \epsilon^{\mu\nu\alpha\beta} p_\alpha p_\beta &= \epsilon^{\mu\nu\alpha\beta} p'_\alpha p'_\beta = 0 \\ \epsilon^{\mu\nu\alpha\beta} p_\alpha p'_\beta &= -\epsilon^{\mu\nu\alpha\beta} p'_\alpha p_\beta \end{aligned} \tag{670}$$

Again, terms proportional to q^μ only play an important role only for the case $l = \tau$. Thus those are negligible in our case

$$\begin{aligned} < D^*(p', \varepsilon)|A^\mu|B(p) >= &-(m_B + m_{D^*})\varepsilon^{*\mu} A_1(q^2) \\ &+\frac{\varepsilon^* \cdot q}{m_B + m_{D^*}} (p + p')^\mu A_2(q^2) \end{aligned} \tag{671}$$

or

$$< D^*(p', \varepsilon)|A^\mu|B(p) >= \varepsilon^{*\mu} f(q^2) + \varepsilon^* \cdot q(p + p')^\mu a_+(q^2) \tag{672}$$

I.3 ISGW2 model

ISGW2 model [31][32] is a quark model before the Heavy Quark Effective Theory (HQET) Form Factors (FF) are developed. It provides a method to calculate $B \to D^{(*,**)}\ell\nu$ decay FF. In this model, the quark-model B and D masses are defined as

$$\begin{aligned} \tilde{m}_B &= m_b + m_u = m_b + m_d \\ \tilde{m}_D &= m_c + m_u = m_c + m_d \end{aligned} \tag{673}$$

where m_u, m_d, m_c and m_b are u, d, c and b quark masses, respectively. In ISGW2 model,

$$\begin{aligned} m_d &= m_u = 0.33\text{GeV} \\ m_c &= 1.82\text{GeV} \\ m_b &= 5.20\text{GeV} \end{aligned} \tag{674}$$

are used. Using, these quark masses, ISGW2 define

$$\mu_\pm = \left(\frac{1}{m_c} \pm \frac{1}{m_b}\right)^{-1} \tag{675}$$

The hyperfine-averaged physical mass of a HQS multiplet with light degrees of freedom having angular momentum j_l is

$$\bar{m}_{j_l} = \frac{j_l+1}{2j_l+1} m_{J=j_l+\frac{1}{2}} + \frac{j_l}{2j_l+1} m_{J=j_l-\frac{1}{2}} \tag{676}$$

For example

$$\begin{aligned} \bar{m}_B &= \tfrac{3}{4} m_B + \tfrac{1}{4} m_{B^*} = 5.31 \\ \bar{m}_D &= \bar{m}_{D^*} = \tfrac{3}{4} m_D + \tfrac{1}{4} m_{D^*} \\ \bar{m}_{D_0^*} &= \bar{m}_{D_1'} = \tfrac{3}{4} m_{D_1'} + \tfrac{1}{4} m_{D_0^*} \\ \bar{m}_{D_1} &= \bar{m}_{D_2^*} = \tfrac{5}{8} m_{D_2^*} + \tfrac{3}{8} m_{D_1} \end{aligned} \tag{677}$$

where

$$\begin{aligned} m_B &= 5.27\text{GeV}, \quad m_{B^*} = 5.33\text{GeV} \\ m_D &= 1.87\text{GeV}, \quad m_{D^*} = 2.01\text{GeV} \\ m_{D_0^*} &= 2.40\text{GeV}, \quad m_{D_1'} = 2.49\text{GeV} \\ m_{D_1} &= 2.42\text{GeV}, \quad m_{D_2^*} = 2.46\text{GeV} \end{aligned} \tag{678}$$

There are also parameters β in the wave functions, and those values in the ISGW2 model are

$$\begin{aligned} \beta_B &= 0.431\text{GeV} \\ \beta_D &= 0.45\text{GeV} \\ \beta_{D^*} &= 0.38\text{GeV} \\ \beta_{D^{**}} &= 0.33\text{GeV} \end{aligned} \tag{679}$$

Also, β_{BX}^2 ($X = D$, D^* or D^{**}) is defined as

$$\beta_{BX}^2 \equiv \frac{1}{2}(\beta_B^2 + \beta_X^2) \tag{680}$$

The momentum transfer is

$$t = (p_B - p_D)^2 = q^2 \tag{681}$$

and the maximum momentum transfer is

$$t_m = (m_B - m_D)^2 \tag{682}$$

where m_D can be m_{D^*} or $m_{D^{**}}$.
In ISGW2 model, the factor $F_n^{(\alpha)}$ is defined as

$$F_n^{(\alpha)} = \left(\frac{\bar{m}_B}{\tilde{m}_B}\right)^{n_B(\alpha)} \left(\frac{\bar{m}_D}{\tilde{m}_D}\right)^{n_X(\alpha)} \left(\frac{\tilde{m}_D}{\tilde{m}_B}\right)^{1/2} \left(\frac{\beta_B\beta_D}{\beta_{BD}^2}\right)^{n/2} \left[1 + \frac{1}{6N}r^2(t_m - t)\right]^{-N} \tag{683}$$

where $N = 2 + n + n'$. n and n' are the harmonic oscillator quantum numbers of the initial and final wave-functions. Therefore, $N = 2$ for S wave to S wave ($B \to D$ or D^*), and $N = 3$ for S wave to P wave ($B \to D^{**}$). $n_B(\alpha)$ and $n_X(\alpha)$ is given in the table

Form factors (α)	$n_B(\alpha)$	$n_X(\alpha)$
$f_+ + f_-$	-1/2	+1/2
$f_+ - f_-$	+1/2	-1/2
g	-1/2	-1/2
f	+1/2	+1/2
$a_+ + a_-$	-3/2	+1/2
$a_+ - a_-$	-1/2	-1/2
$u_+ + u_-$	-1/2	+1/2
$u_+ - u_-$	+1/2	-1/2
$l_{(3/2,1/2)}, r$	+1/2	+1/2
$c_{+(3/2,1/2)} + c_{-(3/2,1/2)}, s_+ + s_-$	-3/2	+1/2
$c_{+(3/2,1/2)} - c_{-(3/2,1/2)}, s_+ - s_-$	-1/2	-1/2
$q_{(3/2,1/2)}, v$	-1/2	-1/2
h	-3/2	-1/2
k	-1/2	+1/2
$b_+ + b_-$	-5/2	+1/2
$b_+ - b_-$	-3/2	-1/2

The form factors in this table are defined later. r^2 is

$$r^2 = \frac{3}{4m_bm_c} + \frac{3m_d^2}{2\bar{m}_B\bar{m}_D\beta_{BD}^2} + \frac{1}{\bar{m}_B\bar{m}_D}\left(\frac{16}{33 - 2N_f}\right)\ln\left[\frac{\alpha_s(\mu_{QM})}{\alpha_s(m_c)}\right] \tag{684}$$

where, if a scale $\mu < 0.6$ GeV,

$$\alpha_s(\mu) = 0.6 \tag{685}$$

and if $\mu > 0.6$ GeV,

$$\alpha_s(\mu) = \frac{12\pi}{33 - 2N_f} \left[\ln \left(\frac{\mu^2}{\Lambda^2_{QCD}} \right) \right]^{-1} \tag{686}$$

where N_f is the number of active flavors below the scale μ. For $\mu = m_c$, $N_f = 3$, and for $\mu = m_b$, $N_f = 4$. Thus, $N_f = 3$ for $b \to c$ decays. The constants in the above equations are

$$\begin{aligned} \mu_{QM} &= 0.1\text{GeV} \\ \Lambda^2_{QCD} &= 0.04\text{GeV}^2 \end{aligned} \tag{687}$$

The factor $R^{(\alpha)}$ is defined as

$$R^{(\alpha)} = C_{cb} \left[f^{(\alpha)} + \tilde{\beta}^{(\alpha)}_{cb}(w) \frac{\alpha_s\left(\sqrt{m_b m_c}\right)}{\pi} \right] \tag{688}$$

where $f^{(\alpha)} = 0$ for $a_+ + a_-$ and $f^{(\alpha)} = 1$ for others.
Here

$$C_{cb} = \left[\frac{\alpha_s(m_b)}{\alpha_s(m_c)} \right]^{a_I} \left[\frac{\alpha_s(m_c)}{\alpha_s(\mu_{QM})} \right]^{a_L(w)} \tag{689}$$

is independent of α and has

$$a_I = -\frac{6}{33 - 2N_f} \tag{690}$$

and

$$a_L(w) = -\frac{8}{33 - 2N'_f} [w r(w) - 1] \tag{691}$$

with

$$r(w) = \frac{1}{\sqrt{w^2 - 1}} \ln \left(w + \sqrt{w^2 - 1} \right) \tag{692}$$

and

$$w = 1 + \frac{t_m - t}{2 m_B m_D} \tag{693}$$

where m_D may be m^*_D or m^{**}_D depending on the decay modes. N_f is the number of active flavors below the scale m_b and N'_f is the number of active flavors below the scale m_c. Thus, $N_f = 4$ and $N'_f = 3$. In ISGW2, $\tilde{w}$ is defined as

$$\tilde{w} = 1 + \frac{t_m - t}{2 \bar{m}_B \bar{m}_D} \tag{694}$$

For w near 1

$$\left[\frac{\alpha_s(m_c)}{\alpha_s(\mu_{QM})} \right]^{a_L(w)} = 1 - \frac{2}{3} \left(\frac{8}{33 - 2N'_f} \right) \ln \left[\frac{\alpha_s(\mu_{QM})}{\alpha_s(m_c)} \right] (w - 1) \tag{695}$$

In SP5/SP6, $\left[\frac{\alpha_s(m_c)}{\alpha_s(\mu_{QM})}\right]^{a_L(w)} = 1$ was used.
At $w = 1$, $\tilde{\beta}_{cb}$ are

$$\begin{array}{l}
\tilde{\beta}_{cb}^{(f_+ + f_-)}(1) = \gamma_{cb} - \frac{2}{3}\chi_{cb} \\
\tilde{\beta}_{cb}^{(f_+ - f_-)}(1) = \gamma_{cb} + \frac{2}{3}\chi_{cb} \\
\tilde{\beta}_{cb}^{(g)}(1) = \gamma_{cb} + \frac{2}{3} \\
\tilde{\beta}_{cb}^{(f)}(1) = \gamma_{cb} - \frac{2}{3} \\
\tilde{\beta}_{cb}^{(a_+ + a_-)}(1) = -1 - \chi_{cb} + \frac{4}{3}\frac{1}{(1-z_{cb})} + \frac{2}{3}\frac{1+z_{cb}}{(1-z_{cb})^2}\gamma_{cb} \\
\tilde{\beta}_{cb}^{(a_+ - a_-)}(1) = -\frac{1}{3} - \chi_{cb} + \frac{4}{3}\frac{1}{(1-z_{cb})} + \left[1 - \frac{2}{3}\frac{1+z_{cb}}{(1-z_{cb})^2}\right]\gamma_{cb}
\end{array} \tag{696}$$

where

$$\gamma_{cb} = \frac{2z_{cb}}{1-z_{cb}} \ln\left(\frac{1}{z_{cb}}\right) - 2 \tag{697}$$

and

$$\chi_{cb} = -1 - \frac{\gamma_{cb}}{1-z_{cb}} \tag{698}$$

with

$$z_{cb} = \frac{m_c}{m_b} \tag{699}$$

w dependence of $\tilde{\beta}$ is weak and negligible.
In ISGW [32], LL-coupling was used, and ISGW2 [31] also follows the convention. But, ISGW2 aware that jj-coupling is a better representation of heavy-light mesons. Thus, ISGW2 also gives jj-coupling type of form factors for D_1 and D_1'.

I.3.1 $B \to D\ell\nu$ (1^2S_0)

Matrix element is

$$< D(p')|V^\mu|B(p) >= f_+(q^2)(p+p')^\mu + f_-(q^2)(p-p')^\mu \tag{700}$$

where f_+ and f_- are the form factors. These are calculated as follows :

$$f_+ + f_- = \left[2 - \frac{\tilde{m}_D}{m_c}\left(1 - \frac{m_d m_c}{2\mu_+ \tilde{m}_D}\frac{\beta_B^2}{\beta_{BD}^2}\right)\right] F_3^{(f_+ + f_-)} R^{(f_+ + f_-)} \tag{701}$$

and

$$f_+ - f_- = \frac{\tilde{m}_B}{m_c}\left(1 - \frac{m_d m_c}{2\mu_+ \tilde{m}_D}\frac{\beta_B^2}{\beta_{BD}^2}\right) F_3^{(f_+ - f_-)} R^{(f_+ - f_-)} \tag{702}$$

Thus, we get f_+ and f_- from these two equations. For example,

$$f_+ = \frac{(f_+ + f_-) + (f_+ - f_-)}{2} \tag{703}$$

I.3.2 $B \to D^*\ell\nu$ (1^3S_0)

Matrix element is

$$< D^*(p',\varepsilon)|V^\mu|B(p) >= i\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu(p+p')_\alpha(p-p')_\beta g(q^2) \tag{704}$$

and

$$< D^*(p',\varepsilon)|A^\mu|B(p) >= \varepsilon^{*\mu} f(q^2) + \varepsilon^* \cdot p(p+p')^\mu a_+(q^2) + \varepsilon^* \cdot p(p-p')^\mu a_-(q^2) \tag{705}$$

Form factors g, f, a_+ and a_- are given by the following equations.

$$f = C_f \tilde{m}_B \left[(1+\tilde{w}) + \frac{m_d(\tilde{w}-1)}{2\mu_+}\right] F_3^{(f)} R^{(f)} \tag{706}$$

where $C_f = 0.989$ for $B \to D^*$.

$$g = \frac{1}{2}\left[\frac{1}{m_c} - \frac{m_d\beta_B^2}{2\mu_-\tilde{m}_D\beta_{BD}^2}\right] F_3^{(g)} R^{(g)} \tag{707}$$

$$\begin{aligned} &a_+ + a_- \\ &= C_{cb}\left[\frac{m_d}{(1+\tilde{w})m_c m_b}\frac{\beta_D^2}{\beta_{BD}^2}\left(1 - \frac{m_d}{2\tilde{m}_B}\frac{\beta_D^2}{\beta_{BD}^2}\right) + \tilde{\beta}^{(a_+ + a_-)}\frac{\alpha_s}{\pi}\tilde{m}_B\right] \\ &\qquad F_3^{(a_+ + a_-)} \end{aligned} \tag{708}$$

$$\begin{aligned} &a_+ - a_- \\ &= -\frac{1}{\tilde{m}_D}\left[\frac{\tilde{m}_B}{m_b} - \frac{m_d}{2\mu_+}\frac{\beta_D^2}{\beta_{BD}^2} + \frac{\tilde{w}m_d\tilde{m}_B}{(1+\tilde{w})m_c m_b}\frac{\beta_D^2}{\beta_{BD}^2}\left(1 - \frac{m_d}{2\tilde{m}_B}\frac{\beta_D^2}{\beta_{BD}^2}\right)\right] \\ &\qquad F_3^{(a_+ - a_-)} R^{(a_+ - a_-)} \end{aligned} \tag{709}$$

I.3.3 $B \to D_0^*\ell\nu$ (1^3P_0)

Matrix element is

$$< D_0^*(p')|V^\mu|B(p) >= 0 \tag{710}$$

and

$$< D_0^*(p')|A^\mu|B(p) >= u_+(q^2)(p+p')^\mu + u_-(q^2)(p-p')^\mu \tag{711}$$

Form factors u_+ and u_- are given by the following two equations.

$$u_+ + u_- = -\sqrt{\frac{2}{3}}\frac{m_d}{\beta_B}F_5^{(u_+ + u_-)} \tag{712}$$

$$u_+ - u_- = \sqrt{\frac{2}{3}}\frac{m_d\tilde{m}_B}{\beta_B\tilde{m}_D}F_5^{(u_+ - u_-)} \tag{713}$$

I.3.4 $B \to D_1' \ell\nu$ (1^3P_1)

Matrix element is

$$< D_1'(p',\varepsilon)|V^\mu|B(p) >= \varepsilon^{*\mu} l(q^2) + \varepsilon^* \cdot p(p+p')^\mu c_+(q^2) + \varepsilon^* \cdot p(p-p')^\mu c_-(q^2) \quad (714)$$

and

$$< D_1'(p',\varepsilon)|A^\mu|B(p) >= i\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu (p+p')_\alpha (p-p')_\beta q(q^2) \quad (715)$$

Form factors l, c_+, c_- and q are given by the following equations.

$$q = -\frac{m_d}{2\tilde{m}_D\beta_B}\left(\frac{5+\tilde{w}}{6}\right) F_5^{(q)} \quad (716)$$

$$l = -\tilde{m}_B\beta_B\left[\frac{1}{\mu_-} + \frac{m_d\tilde{m}_D(\tilde{w}-1)}{\beta_B^2}\left(\frac{5+\tilde{w}}{6m_c} - \frac{1}{2\mu_-}\frac{m_d}{\tilde{m}_D}\frac{\beta_B^2}{\beta_{BD}^2}\right)\right] F_5^{(l)} \quad (717)$$

$$c_+ + c_- = -\frac{m_d\tilde{m}_D}{2m_c\tilde{m}_B\beta_B}\left(1 - \frac{m_d m_c}{2\tilde{m}_D\mu_-}\frac{\beta_B^2}{\beta_{BD}^2}\right) F_5^{(c_+ + c_-)} \quad (718)$$

$$c_+ - c_- = -\frac{m_d\tilde{m}_D}{2m_c\tilde{m}_B\beta_B}\left(\frac{\tilde{w}+2}{3} - \frac{m_d m_c}{2\tilde{m}_D\mu_-}\frac{\beta_B^2}{\beta_{BD}^2}\right) F_5^{(c_+ - c_-)} \quad (719)$$

I.3.5 $B \to D_1 \ell\nu$ (1^1P_1)

Matrix element is

$$< D_1'(p',\varepsilon)|V^\mu|B(p) >= \varepsilon^{*\mu} r(q^2) + \varepsilon^* \cdot p(p+p')^\mu s_+(q^2) + \varepsilon^* \cdot p(p-p')^\mu s_-(q^2) \quad (720)$$

and

$$< D_1'(p',\varepsilon)|A^\mu|B(p) >= i\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu (p+p')_\alpha (p-p')_\beta v(q^2) \quad (721)$$

Form factors r, s_+, s_- and v are given by the following equations.

$$v = \left[\frac{\tilde{m}_B\beta_B}{4\sqrt{2}m_b m_c\tilde{m}_D} + \frac{(\tilde{w}-1)}{6\sqrt{2}}\frac{m_d}{\tilde{m}_D\beta_B}\right] F_5^{(v)} \quad (722)$$

$$r = \frac{\tilde{m}_B\beta_B}{\sqrt{2}}\left[\frac{1}{\mu_+} + \frac{m_d\tilde{m}_D}{3m_c\beta_B^2}(\tilde{w}-1)^2\right] F_5^{(r)} \quad (723)$$

$$s_+ + s_- = \frac{m_d}{\sqrt{2}\tilde{m}_B\beta_B}\left(1 - \frac{m_d}{m_c} + \frac{m_d}{2\mu_+}\frac{\beta_B^2}{\beta_{BD}^2}\right) F_5^{(s_+ + s_-)} \quad (724)$$

$$s_+ - s_- = \frac{m_d}{\sqrt{2}m_c\beta_B}\left(\frac{4-\tilde{w}}{3} - \frac{m_d m_c}{2\tilde{m}_D\mu_+}\frac{\beta_B^2}{\beta_{BD}^2}\right) F_5^{(s_+ - s_-)} \quad (725)$$

I.3.6 $B \to D_2^* \ell \nu$ $(1^3 P_2)$

Matrix element is

$$< D_2^*(p', \varepsilon)|V^\mu|B(p) >= i\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_{\nu\sigma}p^\sigma(p+p')_\alpha(p-p')_\beta h(q^2) \tag{726}$$

and

$$\begin{aligned} &< D_2^*(p', \varepsilon)|A^\mu|B(p) > \\ &= \varepsilon^{*\mu\sigma}p_\sigma k(q^2) + \varepsilon^*_{\alpha\beta}p^\alpha p^\beta(p+p')^\mu b_+(q^2) + \varepsilon^*_{\alpha\beta}p^\alpha p^\beta(p-p')^\mu b_-(q^2) \end{aligned} \tag{727}$$

Form factors h, b_+, b_- and k are given by the following equations.

$$h = \frac{m_d}{2\sqrt{2}\tilde{m}_B\beta_B}\left[\frac{1}{m_c} - \frac{m_d\beta_B^2}{2\mu_-\tilde{m}_D\beta_{BD}^2}\right]F_5^{(h)} \tag{728}$$

$$k = \frac{m_d}{\sqrt{2}\beta_B}(1+\tilde{w})F_5^{(k)} \tag{729}$$

$$b_+ + b_- = \frac{m_d^2}{4\sqrt{2}m_c m_b \tilde{m}_B\beta_B}\frac{\beta_D^2}{\beta_{BD}^2}\left(1 - \frac{m_d}{2\tilde{m}_B}\frac{\beta_D^2}{\beta_{BD}^2}\right)F_5^{(b_+ + b_-)} \tag{730}$$

$$\begin{aligned} &b_+ - b_- \\ &= -\frac{m_d}{\sqrt{2}m_b\tilde{m}_D\beta_B}\left[1 - \frac{m_d m_b}{2\mu_+\tilde{m}_B}\frac{\beta_D^2}{\beta_{BD}^2} + \frac{m_d}{4m_c}\frac{\beta_D^2}{\beta_{BD}^2}\left(1 - \frac{m_d}{2\tilde{m}_B}\frac{\beta_D^2}{\beta_{BD}^2}\right)\right] \\ &\qquad F_5^{(b_+ - b_-)} \end{aligned} \tag{731}$$

I.3.7 $B \to D_1^* \ell \nu$ $(\frac{1}{2}^+)$

Form factors are

$$q_{1/2} = \sqrt{\frac{1}{6}}\frac{m_d}{\tilde{m}_D\beta_B}\left(1 - \frac{\beta_B^2\tilde{m}_B}{4m_d m_c m_b}\right)F_5^{(q_{1/2})} \tag{732}$$

$$l_{1/2} = \sqrt{\frac{2}{3}}\tilde{m}_B\beta_B\left[\frac{1}{2m_c} - \frac{3}{2m_b} + \frac{m_d\tilde{m}_D(\tilde{w}-1)}{\beta_B^2}\left(\frac{1}{m_c} - \frac{1}{2\mu_-}\frac{m_d}{\tilde{m}_D}\frac{\beta_B^2}{\beta_{BD}^2}\right)\right]F_5^{(l_{1/2})} \tag{733}$$

$$c_{1/2+} + c_{1/2-} = \frac{m_d^2\beta_D^2}{\sqrt{6}m_c\tilde{m}_B\beta_B\beta_{BD}^2}F_5^{(c_{1/2+}+c_{1/2-})} \tag{734}$$

$$c_{1/2+} - c_{1/2-} = -\sqrt{\frac{2}{3}}\frac{m_d}{2\tilde{m}_D\beta_B}\left(1 + \frac{m_d}{2m_c}\frac{\beta_D^2}{\beta_{BD}^2}\right)F_5^{(c_{1/2+}-c_{1/2-})} \tag{735}$$

I.3.8 $B \to D_1 \ell\nu$ ($\frac{3}{2}^+$)

Form factors are

$$q_{3/2} = -\frac{1}{2\sqrt{3}}\frac{m_d}{\tilde{m}_D \beta_B}\left(\frac{1+\tilde{w}}{2} + \frac{\beta_B^2 \tilde{m}_B}{2m_d m_c m_b}\right) F_5^{(q_{3/2})} \tag{736}$$

$$l_{3/2} = -\frac{2}{\sqrt{3}}\tilde{m}_B \beta_B \left[\frac{1}{m_c} + \frac{m_d \tilde{m}_D(\tilde{w}-1)}{2\beta_B^2}\left(\frac{\tilde{w}+1}{2m_c} - \frac{1}{2\mu_-}\frac{m_d}{\tilde{m}_D}\frac{\beta_B^2}{\beta_{BD}^2}\right)\right] F_5^{(l_{3/2})} \tag{737}$$

$$c_{3/2+} + c_{3/2-} = -\frac{\sqrt{3}m_d}{2\tilde{m}_B \beta_B}\left[1 - \frac{m_d}{2m_c} - \frac{m_d}{3}\frac{\beta_B^2}{\beta_{BD}^2}\left(\frac{1}{2\mu_-} - \frac{1}{\mu_+}\right)\right] F_5^{(c_{3/2+}+c_{3/2-})} \tag{738}$$

$$c_{3/2+} - c_{3/2-} = -\frac{m_d}{2\sqrt{3}\tilde{m}_D \beta_B}\left[\frac{(2-\tilde{w})\tilde{m}_D}{m_c} + \frac{m_d \beta_B^2}{\beta_{BD}^2}\left(\frac{1}{2\mu_-} - \frac{1}{\mu_+}\right)\right] F_5^{(c_{3/2+}-c_{3/2-})} \tag{739}$$

I.4 HQET model

Details of HQET model is described in [33][34]. We will summarize the part which is related to our analysis.

I.4.1 $B \to D\ell\nu$

In heavy quark limit, the form factors $h_+(w)$ and $h_-(w)$ are

$$h_+(w) = \xi(w), \;\; h_-(w) = 0 \tag{740}$$

where $\xi(w)$ is the Isgur-Wise Function.

$$\xi(w) = \xi(1)[1 - \rho_D^2(w-1)] \tag{741}$$

where ρ_D^2 is the form factor slope. $h_+(1) = \xi(1)$ can be calculated by the Lattice QCD.

I.4.2 $B \to D^*\ell\nu$

The kinematic variables in $B \to D^*\ell\nu$ decays are shown in Figure 29.

- θ_ℓ : Angle between the directions of "the ℓ in the W rest frame" and "the W in the B rest frame".
- θ_V : Angle between the directions of "the D in the D^* rest frame" and "the D^* in the B rest frame".
- χ : Azimuthal angle between the planes formed by "W-ℓ system" and "$D^* - D$ system".

Form factors $V(q^2)$, $A_1(q^2)$ and $A_2(q^2)$ are related to helicity form factors $H_0(q^2)$, $H_+(q^2)$ and $H_-(q^2)$, which correspond to the three allowed polarization state of D^* (one longitudinal and two transverse). A_1 contributes to all three helicity components, A_2 contributes only to the helicity-zero component and V contributes only to the helicity ± 1 components

$$H_+(q^2) = (m_B + m_{D^*})A_1(q^2) - \frac{\sqrt{(q^2 - m_B^2 - m_{D^*}^2)^2 - 4m_B^2 m_{D^*}^2}}{m_B + m_{D^*}} V(q^2) \tag{742}$$

$$H_-(q^2) = (m_B + m_{D^*})A_1(q^2) + \frac{\sqrt{(q^2 - m_B^2 - m_{D^*}^2)^2 - 4m_B^2 m_{D^*}^2}}{m_B + m_{D^*}} V(q^2) \tag{743}$$

$$\begin{aligned} H_0(q^2) = \frac{1}{2m_{D^*}\sqrt{q^2}} & \left[(m_B^2 - m_{D^*}^2 - q^2)(m_B + m_{D^*})A_1(q^2) \right. \\ & \left. - \frac{(q^2 - m_B^2 - m_{D^*}^2)^2 - 4m_B^2 m_{D^*}^2}{m_B + m_{D^*}} A_2(q^2) \right] \end{aligned} \tag{744}$$

Differential decay rate is given in terms of the helicity form factors,

$$\begin{aligned} \frac{d\Gamma(B \to D^*\ell\nu)}{dw d\cos\theta_V d\cos\theta_\ell d\chi} \quad = \quad & \frac{3G_F^2}{4(4\pi)^4} |V_{cb}|^2 m_B m_{D^*}^2 \sqrt{w^2 - 1}(1 - 2wr + r^2) \times \\ & [(1 - \cos\theta_\ell)^2 \sin^2\theta_V |H_+(w)|^2 \\ & + (1 + \cos\theta_\ell)^2 \sin^2\theta_V |H_-(w)|^2 \\ & + 4\sin^2\theta_\ell \cos^2\theta_V |H_0(w)|^2 \\ & - 4\sin\theta_\ell (1 - \cos\theta_\ell) \sin\theta_V \cos\theta_V \cos\chi H_+(w) H_0(w) \\ & + 4\sin\theta_\ell (1 + \cos\theta_\ell) \sin\theta_V \cos\theta_V \cos\chi H_-(w) H_0(w) \\ & - 2\sin^2\theta_\ell \sin^2\theta_V \cos 2\chi H_+(w) H_-(w)] \end{aligned} \tag{745}$$

$H_i(w)$ are

$$H_i = -m_B \frac{R^*(1 - r^2)(w + 1)}{2\sqrt{1 - 2wr + r^2}} h_{A_1}(w) \widetilde{H}_i(w) \tag{746}$$

$h_{A_1}(w)$ is

$$h_{A_1}(w) = h_{A_1}(1) \left[1 - \rho^2 (w - 1) \right] \tag{747}$$

where ρ^2 is the form factor slope, and $\widetilde{H}_i(w)$ are

$$\begin{aligned} \widetilde{H}_\pm(w) &= \frac{\sqrt{1-2wr+r^2}}{1-r} \left(1 \mp \sqrt{\frac{w-1}{w+1}} R_1(w) \right) \\ \widetilde{H}_0(w) &= 1 + \frac{w-1}{1-r} \left(1 - R_2(w) \right) \end{aligned} \tag{748}$$

where R_1 and R_2 are the form factor ratios.

$$R_1 = \frac{h_V}{h_{A_1}}, \quad R_2 = \frac{h_{A_3} + r h_{A_2}}{h_{A_1}} \tag{749}$$

To make the notation simple, we define

$$\begin{aligned}
c_L &\equiv \cos\theta_\ell \\
c_V &\equiv \cos\theta_V \\
\tilde{h}_\pm(w) &\equiv (1-r)\widetilde{H}_\pm(w) = \sqrt{1-2wr+r^2}\left(1 \mp \sqrt{\tfrac{w-1}{w+1}} R_1\right) \\
\tilde{h}_0(w) &\equiv (1-r)\widetilde{H}_0(w) = (1-r) + (w-1)(1-R_2)
\end{aligned} \tag{750}$$

then

$$\frac{d\Gamma(B \to D^*\ell\nu)}{dw\,dc_V\,dc_L\,d\chi} = f(w) \;\; [h_{A_1}(w)]^2\, g(w) \tag{751}$$

where

$$\begin{aligned}
f(w) &\equiv \frac{3G_F^2}{4(4\pi)^4}|V_{cb}|^2 m_B m_{D^*}^2 \sqrt{w^2-1}(1-2wr+r^2) \times \\
&\qquad \left[m_B \frac{2\sqrt{m_B m_{D^*}}}{m_B+m_{D^*}} \frac{(1-r^2)(w+1)}{2\sqrt{1-2wr+r^2}} \frac{1}{(1-r)}\right]^2 \\
&= \frac{3G_F^2}{4(4\pi)^4}|V_{cb}|^2 \frac{m_B^4 m_{D^*}^3}{(m_B+m_{D^*})^2}(1+r)^2(w+1)^2\sqrt{w^2-1} \\
&= \frac{3G_F^2|V_{cb}|^2}{1024\pi^4}\frac{m_B^5 r^3}{(1+r)^2}(1+r)^2(w+1)^2\sqrt{w^2-1} \\
&= \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2} \times \frac{9}{64\pi}(w+1)^2
\end{aligned} \tag{752}$$

and

$$\begin{aligned}
g(w) \;\; = \;\; & [(1-c_L)^2(1-c_V^2)\tilde{h}_+^2 + (1+c_L)^2(1-c_V^2)\tilde{h}_-^2 \\
& +4(1-c_L^2)c_V^2\tilde{h}_0^2 - 2(1-c_L^2)(1-c_V^2)\cos 2\chi \tilde{h}_+\tilde{h}_- \\
& -4\sqrt{1-c_L^2}(1-c_L)\sqrt{1-c_V^2}c_V\cos\chi\tilde{h}_+\tilde{h}_0 \\
& +4\sqrt{1-c_L^2}(1+c_L)\sqrt{1-c_V^2}c_V\cos\chi\tilde{h}_-\tilde{h}_0]
\end{aligned} \tag{753}$$

Integrating over c_L, c_V and χ

$$\int_{-1}^{1} dc_L \int_{-1}^{1} dc_V \int_0^{2\pi} d\chi \;\; g(w) = \left[\frac{8}{3}\frac{4}{3}2\pi\tilde{h}_+^2 + \frac{8}{3}\frac{4}{3}2\pi\tilde{h}_-^2 + 4\frac{4}{3}\frac{2}{3}2\pi\tilde{h}_0^2\right] \tag{754}$$

Thus

$$\frac{d\Gamma(B \to D^*\ell\nu)}{dw} = \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2}\mathcal{J}_{D^*}(w) \tag{755}$$

where

$$\begin{aligned}
\mathcal{J}_{D^*}(w) &= \frac{9}{64\pi}(w+1)^2\,[h_{A_1}]^2\,\frac{64\pi}{9}\left[\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2\right] \\
&= (w+1)^2\,[h_{A_1}]^2\left[\tilde{h}_+^2 + \tilde{h}_-^2 + \tilde{h}_0^2\right] \\
&= (w+1)^2\,[h_{A_1}]^2\left[(1-2wr+r^2)\left(1 - \sqrt{\frac{w-1}{w+1}}\frac{h_V}{h_{A_1}}\right)^2\right.
\end{aligned}$$

$$
\begin{aligned}
&+(1-2wr+r^2)\left(1+\sqrt{\frac{w-1}{w+1}}\frac{h_V}{h_{A_1}}\right)^2 \\
&+\left\{(1-r)+(w-1)\left(1-\frac{h_{A_3}+rh_{A_2}}{h_{A_1}}\right)\right\}^2\Bigg] \\
=&\ (1-2wr+r^2)\left[\left\{(w+1)h_{A_1}-\sqrt{w^2-1}\,h_V\right\}^2\right. \\
&\left.+\left\{(w+1)h_{A_1}+\sqrt{w^2-1}\,h_V\right\}^2\right] \\
&+[(1-r)(w+1)h_{A_1} \\
&+(w^2-1)\{(h_{A_1}-(h_{A_3}+rh_{A_2})\}]^2 \\
=&\ 2(1-2wr+r^2)\left[(w+1)^2h_{A_1}^2+(w^2-1)h_V^2\right] \\
&+\left[(w-r)(w+1)h_{A_1}-(w^2-1)(h_{A_3}+rh_{A_2})\right]^2
\end{aligned}
\tag{756}
$$

Thus, we recovered equation (618).

I.5 Leibovich Ligeti Stewart Wise (LLSW) model

Details of LLSW model is described in [41][42]. We will summarize the part which is related to our analysis.
Kinematic variables are w and θ. This θ is the angle between the charged lepton and the charmed meson in the rest frame of the virtual W boson. Thus $\theta = \pi - \theta_\ell$,

$$\cos\theta = \cos(\pi - \theta_\ell) = -\cos\theta_\ell \tag{757}$$

In the paper [42], they use different approximations : Approximation A and B. In Approximation A, they treat $w-1$ as order $\Lambda_{\rm QCD}/m_Q$ and expand decay rates in these parameters. In Approximation B, they keep the known order $\Lambda_{\rm QCD}/m_Q$ contributions to form factors as well as the full w-dependence of decay rates.
In the paper some parameters are defined as follows

$$\varepsilon_b \equiv \frac{1}{2m_b}, \quad \varepsilon_c \equiv \frac{1}{2m_c} \tag{758}$$

with numerical values

$$
\begin{aligned}
m_b &= 4.8\text{GeV} \\
m_c &= 1.4\text{GeV}
\end{aligned}
\tag{759}
$$

The difference between a hadron mass and a heavy quark mass in the first order :

$$
\begin{aligned}
&\bar\Lambda \text{ for ground state doublet } (D \text{ and } D^*) \\
&\bar\Lambda' \text{ for exited } \tfrac{3}{2}^+ \text{ doublet } (D_1 \text{ and } D_2^*) \\
&\bar\Lambda^* \text{ for exited } \tfrac{1}{2}^+ \text{ doublet } (D_0^* \text{ and } D_1')
\end{aligned}
\tag{760}
$$

with numerical values

$$
\begin{aligned}
\bar{\Lambda} &= 0.4\text{GeV} \\
\bar{\Lambda}' - \bar{\Lambda} &= 0.39\text{GeV} \\
\bar{\Lambda}^* - \bar{\Lambda} &= 0.35\text{GeV}
\end{aligned} \tag{761}
$$

In the paper, they also used the branching fraction

$$\mathcal{B}(B \to D_1 e \bar{\nu}_e) = 0.0060 \tag{762}$$

I.5.1 $B \to D_1 \ell\nu$

The matrix elements are

$$\frac{< D_1(v',\varepsilon)|V^\mu|B(v) >}{\sqrt{m_{D_1} m_B}} = f_{V_1}\varepsilon^{*\mu} + (f_{V_2}v^\mu + f_{V_3}v'^\mu)(\varepsilon^* \cdot v) \tag{763}$$

$$\frac{< D_1(v',\varepsilon)|A^\mu|B(v) >}{\sqrt{m_{D_1} m_B}} = i f_A \epsilon^{\mu\alpha\beta\gamma}\varepsilon^*_\alpha v_\beta v'_\gamma \tag{764}$$

The differential decay rate is

$$
\begin{aligned}
\frac{d^2\Gamma_{D_1}}{dw\,d\cos\theta} = 3\Gamma_0 r_1^3\sqrt{w^2-1}\{&\sin^2\theta[(w-r_1)f_{V_1} + (w^2-1)(f_{V_3} + r_1 f_{V_2})]^2 \\
&+(1-2r_1 w + r_1^2)[(1+\cos^2\theta)(f_{V_1}^2 + (w^2-1)f_A^2) \\
&-4\cos\theta\sqrt{w^2-1}f_{V_1}f_A]\}
\end{aligned} \tag{765}
$$

where

$$\Gamma_0 = \frac{G_F^2|V_{cb}|^2 m_B^5}{192\pi^3} \tag{766}$$

and

$$r_1 = \frac{m_{D_1}}{m_B} \tag{767}$$

In approximation B, the form factors are given by

$$
\begin{aligned}
\sqrt{6}f_A &= -(w+1)\tau - \varepsilon_b\{(w-1)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] + (w+1)\eta_b\} \\
&\quad -\varepsilon_c[4(w\bar{\Lambda}'-\bar{\Lambda})\tau - 3(w-1)(\tau_1-\tau_2) + (w+1)(\eta_{\text{ke}} - 2\eta_1 - 3\eta_3)] \\
\sqrt{6}f_{V_1} &= (1-w^2)\tau - \varepsilon_b(w^2-1)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2 + \eta_b] \\
&\quad -\varepsilon_c[4(w+1)(w\bar{\Lambda}'-\bar{\Lambda})\tau - (w^2-1)(3\tau_1 - 3\tau_2 - \eta_{\text{ke}} + 2\eta_1 + 3\eta_3)] \\
\sqrt{6}f_{V_2} &= -3\tau - 3\varepsilon_b[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2 + \eta_b] \\
&\quad -\varepsilon_c[(4w-1)\tau_1 + 5\tau_2 + 3\eta_{\text{ke}} + 10\eta_1 + 4(w-1)\eta_2 - 5\eta_3)] \\
\sqrt{6}f_{V_3} &= (w-2)\tau + \varepsilon_b\{(2+w)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] - (2-w)\eta_b\} \\
&\quad +\varepsilon_c[4(w\bar{\Lambda}'-\bar{\Lambda})\tau + (2+w)\tau_1 + (2+3w)\tau_2 \\
&\quad +(w-2)\eta_{\text{ke}} - 2(6+w)\eta_1 - 4(w-1)\eta_2 - (3w-2)\eta_3]
\end{aligned} \tag{768}
$$

where τ is the leading $m_Q \to \infty$ Isgur-Wise function, which is assumed to be a linear form:

$$\tau(w) = \tau(1)[1 + \hat{\tau}'(w-1)] \tag{769}$$

where

$$\tau' = \frac{d\tau}{dw}, \quad \hat{\tau}' = \frac{\tau'}{\tau(1)} \tag{770}$$

The uncertainty in the $\Lambda_{\rm QCD}/m_Q$ corrections is parameterized by τ_1 and τ_2.
In Approximation B_1,

$$\begin{aligned} \tau_1 &= 0 \\ \tau_2 &= 0 \end{aligned} \tag{771}$$

In Approximation B_2,

$$\begin{aligned} \tau_1 &= \bar{\Lambda}\tau \\ \tau_2 &= -\bar{\Lambda}'\tau \end{aligned} \tag{772}$$

These are the two end values of τ_1 and τ_2. True τ_1 and τ_2 reside somewhere in between. η_i comes from the correction to the HQET Lagrangian. In Approximation B,

$$\eta_1 = \eta_2 = \eta_3 = 0 \tag{773}$$

Also, in this Approximation, $\eta_1^{(b)} = \eta_2^{(b)} = \eta_3^{(b)} = 0$. Thus

$$\eta_b \equiv \eta_{\rm ke}^{(b)} + 6\eta_1^{(b)} - 2(w-1)\eta_2^{(b)} + \eta_3^{(b)} = \eta_{\rm ke}^{(b)} \tag{774}$$

Hence, the form factors are

$$\begin{aligned}
\sqrt{6}f_A &= -(w+1)(\tau + \varepsilon_c\eta_{\rm ke} + \varepsilon_b\eta_{\rm ke}^{(b)}) - \varepsilon_b(w-1)[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[4(w\bar{\Lambda}' - \bar{\Lambda})\tau - 3(w-1)(\tau_1 - \tau_2)] \\
\sqrt{6}f_{V_1} &= (1-w^2)(\tau + \varepsilon_c\eta_{\rm ke} + \varepsilon_b\eta_{\rm ke}^{(b)}) - \varepsilon_b(w^2-1)[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[4(w+1)(w\bar{\Lambda}' - \bar{\Lambda})\tau - 3(w^2-1)(\tau_1 - \tau_2)] \\
\sqrt{6}f_{V_2} &= -3(\tau + \varepsilon_c\eta_{\rm ke} + \varepsilon_b\eta_{\rm ke}^{(b)}) - 3\varepsilon_b[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[(4w-1)\tau_1 + 5\tau_2] \\
\sqrt{6}f_{V_3} &= (w-2)(\tau + \varepsilon_c\eta_{\rm ke} + \varepsilon_b\eta_{\rm ke}^{(b)}) + \varepsilon_b(2+w)[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad +\varepsilon_c[4(w\bar{\Lambda}' - \bar{\Lambda})\tau + (2+w)\tau_1 + (2+3w)\tau_2]
\end{aligned} \tag{775}$$

$\eta_{\rm ke}$ and $\eta_{\rm ke}^{(b)}$ can be absorbed into τ by the replacement of τ by

$$\tilde{\tau} = \tau + \varepsilon_c\eta_{\rm ke} + \varepsilon_b\eta_{\rm ke}^{(b)} \tag{776}$$

Thus, the form factors are

$$\begin{aligned}
\sqrt{6}f_A &= -(w+1)\tilde{\tau} - \varepsilon_b(w-1)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[4(w\bar{\Lambda}'-\bar{\Lambda})\tau - 3(w-1)(\tau_1-\tau_2)] \\
\sqrt{6}f_{V_1} &= (1-w^2)\tilde{\tau} - \varepsilon_b(w^2-1)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[4(w+1)(w\bar{\Lambda}'-\bar{\Lambda})\tau - 3(w^2-1)(\tau_1-\tau_2)] \\
\sqrt{6}f_{V_2} &= -3\tilde{\tau} - 3\varepsilon_b[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[(4w-1)\tau_1 + 5\tau_2] \\
\sqrt{6}f_{V_3} &= (w-2)\tilde{\tau} + \varepsilon_b(2+w)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad +\varepsilon_c[4(w\bar{\Lambda}'-\bar{\Lambda})\tau + (2+w)\tau_1 + (2+3w)\tau_2]
\end{aligned} \tag{777}$$

Other τ can also be replaced by $\tilde{\tau}$ in the same order of approximation. For simplicity, we omit $\tilde{}$, then

$$\begin{aligned}
\sqrt{6}f_A &= -(w+1)\tau - \varepsilon_b(w-1)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[4(w\bar{\Lambda}'-\bar{\Lambda})\tau - 3(w-1)(\tau_1-\tau_2)] \\
\sqrt{6}f_{V_1} &= (1-w^2)\tau - \varepsilon_b(w^2-1)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[4(w+1)(w\bar{\Lambda}'-\bar{\Lambda})\tau - 3(w^2-1)(\tau_1-\tau_2)] \\
\sqrt{6}f_{V_2} &= -3\tau - 3\varepsilon_b[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad -\varepsilon_c[(4w-1)\tau_1 + 5\tau_2] \\
\sqrt{6}f_{V_3} &= (w-2)\tau + \varepsilon_b(2+w)[(\bar{\Lambda}'+\bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\
&\quad +\varepsilon_c[4(w\bar{\Lambda}'-\bar{\Lambda})\tau + (2+w)\tau_1 + (2+3w)\tau_2]
\end{aligned} \tag{778}$$

In Approximation A, the differential decay rate can be expanded in powers of $w-1$

$$\begin{aligned}
\tfrac{d^2\Gamma_{D_1}}{dw\,d\cos\theta} &= \Gamma_0\tau^2(1)r_1^3\sqrt{w^2-1}\textstyle\sum_n(w-1)^n \\
&\quad \{\sin^2\theta s_1^{(n)} + (1-2r_1w+r_1^2)[(1+\cos^2\theta)t_1^{(n)} - 4\cos\theta\sqrt{w^2-1}u_1^{(n)}]\}
\end{aligned} \tag{779}$$

where

$$\begin{aligned}
&s_1^{(0)} = 32\varepsilon_c^2(1-r_1)^2(\bar{\Lambda}'-\bar{\Lambda})^2 + ... \\
&s_1^{(1)} = 32\varepsilon_c(1-r_1^2)(\bar{\Lambda}'-\bar{\Lambda}) + ... \\
&s_1^{(2)} = 8(1+r_1)^2 + ... \\
&t_1^{(0)} = 32\varepsilon_c^2(\bar{\Lambda}'-\bar{\Lambda})^2 + ... \\
&t_1^{(1)} = 4 + 8\varepsilon_c[4(\bar{\Lambda}'-\bar{\Lambda}) + \hat{\eta}_{\rm ke} - 2\hat{\eta}_1 - 3\hat{\eta}_3] + 8\varepsilon_b\hat{\eta}_b + ... \\
&t_1^{(2)} = 8(1+\hat{\tau}') + ... \\
&u_1^{(0)} = 8\varepsilon_c(\bar{\Lambda}'-\bar{\Lambda}) + ... \\
&u_1^{(1)} = 2 + ...
\end{aligned} \tag{780}$$

where

$$\hat{\eta}_i = \frac{\eta_i}{\tau(1)} \tag{781}$$

We use the same tricks for ηs, then

$$\begin{aligned}
s_1^{(0)} &= 32\varepsilon_c^2(1-r_1)^2(\bar{\Lambda}'-\bar{\Lambda})^2\\
s_1^{(1)} &= 32\varepsilon_c(1-r_1^2)(\bar{\Lambda}'-\bar{\Lambda})\\
s_1^{(2)} &= 8(1+r_1)^2\\
t_1^{(0)} &= 32\varepsilon_c^2(\bar{\Lambda}'-\bar{\Lambda})^2\\
t_1^{(1)} &= 4+8\varepsilon_c[4(\bar{\Lambda}'-\bar{\Lambda})]\\
t_1^{(2)} &= 8(1+\hat{\tau}')+...\\
u_1^{(0)} &= 8\varepsilon_c(\bar{\Lambda}'-\bar{\Lambda})\\
u_1^{(1)} &= 2
\end{aligned} \tag{782}$$

The differential decay rate can be integrated over $\cos\theta$ to give

$$\frac{d^2\Gamma_{D_1}}{dw} = \Gamma_0 \tau^2(1) r_1^3 \sqrt{w^2-1} \sum_n (w-1)^n \left[\frac{4}{3}s_1^{(n)} + (1-2r_1 w + r_1^2)\frac{8}{3}t_1^{(n)}\right] \tag{783}$$

$$\begin{aligned}
\frac{d^2\Gamma_{D_1}}{dw} &= \frac{G_F^2|V_{cb}|^2 m_B^5}{192\pi^3} r_1^3 (w^2-1)^{1/2}\, \tau^2(1) \sum_n (w-1)^n \left[\frac{4}{3}s_1^{(n)} + (1-2r_1 w + r_1^2)\frac{8}{3}t_1^{(n)}\right]\\
&= \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{1/2}\, \mathcal{J}_{D_1}(w)
\end{aligned} \tag{784}$$

where

$$\mathcal{J}_{D_1}(w) = \frac{1}{3}\tau^2(1) \sum_n (w-1)^n \left[s_1^{(n)} + 2(1-2r_1 w + r_1^2) t_1^{(n)}\right] \tag{785}$$

Numerical values used in the LLSW paper are

$$\begin{aligned}
\tau(1)\left[\frac{6.0\times 10^{-3}}{\mathcal{B}(B\to D_1 e\bar{\nu}_e)}\right]^{1/2} &= 0.71 \text{ for Approximation B1}\\
&= 0.75 \text{ for Approximation B2}\\
&= 0.60 \text{ for Approximation A}
\end{aligned} \tag{786}$$

and

$$\hat{\tau}' = -1.5 \tag{787}$$

I.5.2 $B \to D_2^* \ell \nu$

The matrix elements are

$$\frac{< D_2^*(v',\varepsilon)|V^\mu|B(v) >}{\sqrt{m_{D_2^*} m_B}} = i k_V \epsilon^{\mu\alpha\beta\gamma} \varepsilon^*_{\alpha\sigma} v^\sigma v_\beta v'_\gamma \tag{788}$$

$$\frac{< D_2^*(v',\varepsilon)|A^\mu|B(v) >}{\sqrt{m_{D_2^*} m_B}} = k_{A_1} \varepsilon^{*\mu\alpha} v_\alpha + (k_{A_2} v^\mu + k_{A_3} v'^\mu) \varepsilon^*_{\alpha\beta} v^\alpha v^\beta \tag{789}$$

The differential decay rate is

$$\begin{array}{l} \frac{d^2\Gamma_{D_2^*}}{dw d\cos\theta} = \frac{3}{2}\Gamma_0 r_2^3 (w^2-1)^{3/2}\{\frac{4}{3}\sin^2\theta[(w-r_2)k_{A_1} + (w^2-1)(k_{A_3}+r_2 k_{A_2})]^2 \\ \quad +(1-2r_2 w + r_2^2)[(1+\cos^2\theta)(k_{A_1}^2 + (w^2-1)k_V^2) - 4\cos\theta\sqrt{w^2-1}k_{A_1}k_V]\} \end{array} \tag{790}$$

where

$$r_2 = \frac{m_{D_2^*}}{m_B} \tag{791}$$

In Approximation B, the form factors are given by

$$\begin{aligned} k_V &= -\tau - \varepsilon_b[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2 + \eta_b] - \varepsilon_c(\tau_1 - \tau_2 + \eta_{\rm ke} - 2\eta_1 + \eta_3) \\ k_{A_1} &= -(1+w)\tau - \varepsilon_b\{(w-1)[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] + (1+w)\eta_b\} \\ &\qquad -\varepsilon_c[(w-1)(\tau_1-\tau_2) + (w+1)(\eta_{\rm ke} - 2\eta_1 + \eta_3)] \\ k_{A_2} &= -2\varepsilon_c(\tau_1 + \eta_2) \\ k_{A_3} &= \tau + \varepsilon_b[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2 + \eta_b] \\ &\qquad -\varepsilon_c(\tau_1 + \tau_2 - \eta_{\rm ke} + 2\eta_1 - 2\eta_2 - \eta_3) \end{aligned} \tag{792}$$

We use the same tricks for ηs, then

$$\begin{aligned} k_V &= -\tau - \varepsilon_b[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] - \varepsilon_c(\tau_1 - \tau_2) \\ k_{A_1} &= -(1+w)\tau - \varepsilon_b(w-1)[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] \\ &\qquad -\varepsilon_c(w-1)(\tau_1-\tau_2) \\ k_{A_2} &= -2\varepsilon_c\tau_1 \\ k_{A_3} &= \tau + \varepsilon_b[(\bar{\Lambda}' + \bar{\Lambda})\tau - (2w+1)\tau_1 - \tau_2] - \varepsilon_c(\tau_1 + \tau_2) \end{aligned} \tag{793}$$

In Approximation B_1,

$$\begin{aligned} \tau_1 &= 0 \\ \tau_2 &= 0 \end{aligned} \tag{794}$$

In Approximation B_2,

$$\begin{aligned} \tau_1 &= \bar{\Lambda}\tau \\ \tau_2 &= -\bar{\Lambda}'\tau \end{aligned} \tag{795}$$

These are the two end values of τ_1 and τ_2. True τ_1 and τ_2 reside somewhere in between. In Approximation A, the differential decay rate can be expanded in powers of $w-1$

$$\begin{array}{l} \frac{d^2\Gamma_{D_2^*}}{dw d\cos\theta} = \frac{3}{2}\Gamma_0\tau^2(1) r_2^3 (w^2-1)^{3/2}\sum_n (w-1)^n \\ \quad \{\frac{4}{3}\sin^2\theta s_2^{(n)} + (1-2r_2 w + r_2^2)[(1+\cos^2\theta)t_2^{(n)} - 4\cos\theta\sqrt{w^2-1}u_2^{(n)}]\} \end{array} \tag{796}$$

where

$$
\begin{aligned}
s_2^{(0)} &= 4(1-r_2)^2[1+2\varepsilon_b\hat{\eta}_b + 2\varepsilon_c(\hat{\eta}_{\rm ke} - 2\hat{\eta}_1 + \hat{\eta}_3)] + ... \\
s_2^{(1)} &= 4(1-r_2)^2(1+2\hat{\tau}') + ... \\
t_2^{(0)} &= 4 + 8\varepsilon_b\hat{\eta}_b + 8\varepsilon_c(\hat{\eta}_{\rm ke} - 2\hat{\eta}_1 + \hat{\eta}_3) + ... \\
t_2^{(1)} &= 2(3+4\hat{\tau}') + ... \\
u_2^{(0)} &= 2 + ...
\end{aligned}
\tag{797}
$$

We use the same tricks for ηs, then

$$
\begin{aligned}
s_2^{(0)} &= 4(1-r_2)^2 \\
s_2^{(1)} &= 4(1-r_2)^2(1+2\hat{\tau}') \\
t_2^{(0)} &= 4 \\
t_2^{(1)} &= 2(3+4\hat{\tau}') \\
u_2^{(0)} &= 2
\end{aligned}
\tag{798}
$$

The differential decay rate can be integrated over $\cos\theta$

$$
\frac{d^2\Gamma_{D_2^*}}{dw} = \frac{3}{2}\Gamma_0\tau^2(1)r_2^3(w^2-1)^{3/2}\sum_n (w-1)^n\left[\frac{4}{3}\frac{4}{3}s_2^{(n)} + (1-2r_2w+r_2^2)\frac{8}{3}t_2^{(n)}\right] \tag{799}
$$

Thus,

$$
\begin{aligned}
\frac{d^2\Gamma_{D_2^*}}{dw} &= \frac{3}{2}\frac{G_F^2|V_{cb}|^2m_B^5}{192\pi^3}r_2^3(w^2-1)^{3/2}\tau^2(1) \\
&\quad \sum_n (w-1)^n\left[\frac{16}{9}s_2^{(n)} + (1-2r_2w+r_2^2)\frac{8}{3}t_2^{(n)}\right] \\
&= \frac{G_F^2|V_{cb}|^2m_B^5}{48\pi^3}r^3(w^2-1)^{3/2}\,\mathcal{J}_{D_2^*}(w)
\end{aligned}
\tag{800}
$$

where

$$
\mathcal{J}_{D_2^*}(w) = \frac{1}{3}\tau^2(1)\sum_n (w-1)^n\left[2s_2^{(n)} + 3(1-2r_2w+r_2^2)t_2^{(n)}\right] \tag{801}
$$

Numerical values used in the LLSW paper are

$$
\begin{aligned}
\tau(1)\left[\frac{6.0\times10^{-3}}{\mathcal{B}(B\to D_1 e\bar{\nu}_e)}\right]^{1/2} &= 0.71 \text{ for Approximation B1} \\
&= 0.75 \text{ for Approximation B2} \\
&= 0.60 \text{ for Approximation A}
\end{aligned}
\tag{802}
$$

and

$$
\hat{\tau}' = -1.5 \tag{803}
$$

I.5.3 $B \to D_0^*\ell\nu$

The matrix elements are

$$\frac{< D_0^*(v',\varepsilon)|V^\mu|B(v)>}{\sqrt{m_{D_0^*}m_B}} = 0 \tag{804}$$

$$\frac{< D_0^*(v',\varepsilon)|A^\mu|B(v)>}{\sqrt{m_{D_0^*}m_B}} = g_+(v^\mu + v'^\mu) + g_-(v^\mu - v'^\mu) \tag{805}$$

The differential decay rate is

$$\frac{d^2\Gamma_{D_0^*}}{dw\,d\cos\theta} = 3\Gamma_0 r_0^{*3}(w^2-1)^{3/2}\sin^2\theta[(1+r_0^*)g_+ - (1-r_0^*)g_-]^2 \tag{806}$$

where

$$r_0^* = \frac{m_{D_0^*}}{m_B} \tag{807}$$

In Approximation B, the form factors are given by

$$\begin{aligned} g_+ &= \varepsilon_c\left[2(w-1)\zeta_1 - 3\zeta\tfrac{w\bar\Lambda^*-\bar\Lambda}{w+1}\right] - \varepsilon_b\left[\tfrac{\bar\Lambda^*(2w+1)-\bar\Lambda(w+2)}{w+1}\zeta - 2(w-1)\zeta_1\right] \\ g_- &= \zeta + \varepsilon_c\left[\chi_{\rm ke} + 6\chi_1 - 2(w+1)\chi_2\right] + \varepsilon_b\chi_b \end{aligned} \tag{808}$$

We can use the same tricks for χs as ηs, then

$$\begin{aligned} g_+ &= \varepsilon_c\left[2(w-1)\zeta_1 - 3\zeta\tfrac{w\bar\Lambda^*-\bar\Lambda}{w+1}\right] - \varepsilon_b\left[\tfrac{\bar\Lambda^*(2w+1)-\bar\Lambda(w+2)}{w+1}\zeta - 2(w-1)\zeta_1\right] \\ g_- &= \zeta \end{aligned} \tag{809}$$

where

$$\zeta(w) = \frac{w+1}{\sqrt{3}}\tau(w) \tag{810}$$

Thus,

$$\begin{aligned} \zeta(1)[1+\hat\zeta'(w-1)] &= \tfrac{2+(w-1)}{\sqrt3}\tau(1)[1+\hat\tau'(w-1)] \\ &\simeq \tfrac{1}{\sqrt3}\tau(1)[2+2\hat\tau'(w-1)+(w-1)] \\ &= \tfrac{2}{\sqrt3}\tau(1)\left[1+\left(\hat\tau'+\tfrac12\right)(w-1)\right] \end{aligned} \tag{811}$$

Hence

$$\begin{aligned} \zeta(1) &= \tfrac{2}{\sqrt3}\tau(1) \\ \hat\zeta' &= \tfrac12 + \hat\tau' = -1 \end{aligned} \tag{812}$$

In Approximation B$_1$,

$$\begin{aligned} \zeta_1 &= 0 \\ \zeta_2 &= 0 \end{aligned} \tag{813}$$

In Approximation B$_2$,

$$\begin{aligned} \zeta_1 &= \bar\Lambda\zeta \\ \zeta_2 &= -\bar\Lambda^*\zeta \end{aligned} \tag{814}$$

These are the two end values of ζ_1 and ζ_2. True ζ_1 and ζ_2 reside somewhere in between. In Approximation A, the differential decay rate can be expanded in powers of $w-1$

$$\frac{d^2\Gamma_{D_0^*}}{dw\,d\cos\theta} = 3\Gamma_0 \zeta^2(1) r_0^{*3} (w^2-1)^{3/2} \sin^2\theta \sum_n (w-1)^n s_0^{(n)} \tag{815}$$

where

$$\begin{aligned} s_0^{(0)} &= (1-r_0^*)^2[1 + 2\varepsilon_c(\hat{\chi}_{\text{ke}} + 6\hat{\chi}_1 - 4\hat{\chi}_2) + 4\varepsilon_b\hat{\chi}_b] \\ &\qquad +3(\varepsilon_c+\varepsilon_b)(1-r_0^{*2})(\bar{\Lambda}^* - \bar{\Lambda}) + ... \\ s_0^{(1)} &= 2(1-r_0^*)^2\hat{\zeta}' + ... \end{aligned} \tag{816}$$

We can use the same tricks for χs, then

$$\begin{aligned} s_0^{(0)} &= (1-r_0^*)^2 + 3(\varepsilon_c+\varepsilon_b)(1-r_0^{*2})(\bar{\Lambda}^* - \bar{\Lambda}) \\ s_0^{(1)} &= 2(1-r_0^*)^2\hat{\zeta}' \end{aligned} \tag{817}$$

The differential decay rate can be integrated over $\cos\theta$

$$\frac{d^2\Gamma_{D_0^*}}{dw} = 3\Gamma_0 \zeta^2(1) r_0^{*3} (w^2-1)^{3/2} \frac{4}{3} \sum_n (w-1)^n s_0^{(n)} \tag{818}$$

Thus

$$\begin{aligned} \frac{d^2\Gamma_{D_0^*}}{dw} &= 3\frac{G_F^2|V_{cb}|^2 m_B^5}{192\pi^3} r_0^{*3} (w^2-1)^{3/2} \zeta^2(1) \frac{4}{3} \sum_n (w-1)^n s_0^{(n)} \\ &= \frac{G_F^2|V_{cb}|^2 m_B^5}{48\pi^3} r^3 (w^2-1)^{3/2}\, \mathcal{J}_{D_0^*}(w) \end{aligned} \tag{819}$$

where

$$\mathcal{J}_{D_0^*}(w) = \zeta^2(1) \sum_n (w-1)^n s_0^{(n)} \tag{820}$$

Numerical values used in the LLSW paper are

$$\begin{aligned} \tau(1)\left[\frac{6.0\times10^{-3}}{\mathcal{B}(B\to D_1 e\bar{\nu}_e)}\right]^{1/2} &= 0.71 \text{ for Approximation B1} \\ &= 0.75 \text{ for Approximation B2} \\ &= 0.60 \text{ for Approximation A} \end{aligned} \tag{821}$$

and

$$\hat{\tau}' = -1.5 \tag{822}$$

I.5.4 $B \to D_1^*\ell\nu$

The matrix elements are

$$\frac{< D_1^*(v',\varepsilon)|V^\mu|B(v) >}{\sqrt{m_{D_1^*}m_B}} = g_{V_1}\varepsilon^{*\mu} + (g_{V_2}v^\mu + g_{V_3}v'^\mu)(\varepsilon^* \cdot v) \tag{823}$$

$$\frac{< D_1^*(v',\varepsilon)|A^\mu|B(v) >}{\sqrt{m_{D_1^*}m_B}} = ig_A\epsilon^{\mu\alpha\beta\gamma}\varepsilon^*_\alpha v_\beta v'_\gamma \tag{824}$$

The differential decay rate is

$$\begin{aligned}\frac{d^2\Gamma_{D_1^*}}{dw\,d\cos\theta} &= 3\Gamma_0 r_1^{*3}\sqrt{w^2-1}\{\sin^2\theta[(w-r_1^*)g_{V_1} + (w^2-1)(g_{V_3} + r_1^* g_{V_2})]^2 \\ &+(1-2r_1^*w + r_1^{*2})[(1+\cos^2\theta)(g_{V_1}^2 + (w^2-1)g_A^2) - 4\cos\theta\sqrt{w^2-1}g_{V_1}g_A]\}\end{aligned} \tag{825}$$

where

$$r_1^* = \frac{m_{D_1^*}}{m_B} \tag{826}$$

In Approximation B, the form factors are given by

$$\begin{aligned}
g_A &= \zeta + \varepsilon_c\left[\tfrac{w\bar{\Lambda}^* - \bar{\Lambda}}{w+1}\zeta + \chi_{\rm ke} - 2\chi_1\right] - \varepsilon_b\left[\tfrac{\bar{\Lambda}^*(2w+1) - \bar{\Lambda}(w+2)}{w+1}\zeta - 2(w-1)\zeta_1 - \chi_b\right] \\
g_{V_1} &= (w-1)\zeta + \varepsilon_c\left[(w\bar{\Lambda}^* - \bar{\Lambda})\zeta + (w-1)(\chi_{\rm ke} - 2\chi_1)\right] \\
&\quad -\varepsilon_b\left[(\bar{\Lambda}^*(2w+1) - \bar{\Lambda}(w+2))\zeta - 2(w^2-1)\zeta_1 - (w-1)\chi_b\right] \\
g_{V_2} &= 2\varepsilon_c(\zeta_1 - \chi_2) \\
g_{V_3} &= -\zeta - \varepsilon_c\left[\tfrac{w\bar{\Lambda}^* - \bar{\Lambda}}{w+1}\zeta + 2\zeta_1 + \chi_{\rm ke} - 2\chi_1 + 2\chi_2\right] \\
&\quad +\varepsilon_b\left[\tfrac{\bar{\Lambda}^*(2w+1) - \bar{\Lambda}(w+2)}{w+1}\zeta - 2(w-1)\zeta_1 - \chi_b\right]
\end{aligned} \tag{827}$$

where

$$\zeta(w) = \frac{w+1}{\sqrt{3}}\tau(w) \tag{828}$$

We use the same tricks for χs, then

$$\begin{aligned}
g_A &= \zeta + \varepsilon_c\left[\tfrac{w\bar{\Lambda}^* - \bar{\Lambda}}{w+1}\zeta\right] - \varepsilon_b\left[\tfrac{\bar{\Lambda}^*(2w+1) - \bar{\Lambda}(w+2)}{w+1}\zeta - 2(w-1)\zeta_1\right] \\
g_{V_1} &= (w-1)\zeta + \varepsilon_c(w\bar{\Lambda}^* - \bar{\Lambda})\zeta \\
&\quad -\varepsilon_b\left[(\bar{\Lambda}^*(2w+1) - \bar{\Lambda}(w+2))\zeta - 2(w^2-1)\zeta_1\right] \\
g_{V_2} &= 2\varepsilon_c\zeta_1 \\
g_{V_3} &= -\zeta - \varepsilon_c\left[\tfrac{w\bar{\Lambda}^* - \bar{\Lambda}}{w+1}\zeta + 2\zeta_1\right] \\
&\quad +\varepsilon_b\left[\tfrac{\bar{\Lambda}^*(2w+1) - \bar{\Lambda}(w+2)}{w+1}\zeta - 2(w-1)\zeta_1\right]
\end{aligned} \tag{829}$$

In Approximation B_1,

$$\begin{aligned}\zeta_1 &= 0 \\ \zeta_2 &= 0\end{aligned} \tag{830}$$

In Approximation B_2,

$$\begin{aligned}\zeta_1 &= \bar{\Lambda}\zeta \\ \zeta_2 &= -\bar{\Lambda}^*\zeta\end{aligned} \tag{831}$$

These are the two end values of ζ_1 and ζ_2. True ζ_1 and ζ_2 reside somewhere in between. In Approximation A, the differential decay rate can be expanded in powers of $w-1$

$$\begin{aligned}&\tfrac{d^2\Gamma_{D_1^*}}{dwd\cos\theta} = 3\Gamma_0\zeta^2(1)r_1^{*3}\sqrt{w^2-1}\textstyle\sum_n(w-1)^n \\ &\{\sin^2\theta s_1^{(n)} + (1-2r_1^*w+r_1^{*2})[(1+\cos^2\theta)t_1^{(n)} - 4\cos\theta\sqrt{w^2-1}u_1^{(n)}]\}\end{aligned} \tag{832}$$

where

$$\begin{aligned}s_1^{(0)} &= (\varepsilon_c-3\varepsilon_b)^2(1-r_1^*)^2(\bar{\Lambda}^*-\bar{\Lambda})^2+... \\ s_1^{(1)} &= -2(\varepsilon_c-3\varepsilon_b)(1-r_1^{*2})(\bar{\Lambda}^*-\bar{\Lambda})+... \\ s_1^{(2)} &= (1+r_1^*)^2+... \\ t_1^{(0)} &= (\varepsilon_c-3\varepsilon_b)^2(\bar{\Lambda}^*-\bar{\Lambda})^2+... \\ t_1^{(1)} &= 2+4(\varepsilon_c-3\varepsilon_b)(\bar{\Lambda}^*-\bar{\Lambda})+4\varepsilon_c(\hat{\chi}_{\rm ke}-2\hat{\chi}_1)+4\varepsilon_b\hat{\chi}_b+... \\ t_1^{(2)} &= 2(1+2\hat{\zeta}')+... \\ u_1^{(0)} &= (\varepsilon_c-3\varepsilon_b)(\bar{\Lambda}^*-\bar{\Lambda})+... \\ u_1^{(1)} &= 1+...\end{aligned} \tag{833}$$

We use the same tricks for χs, then

$$\begin{aligned}s_1^{(0)} &= (\varepsilon_c-3\varepsilon_b)^2(1-r_1^*)^2(\bar{\Lambda}^*-\bar{\Lambda})^2 \\ s_1^{(1)} &= -2(\varepsilon_c-3\varepsilon_b)(1-r_1^{*2})(\bar{\Lambda}^*-\bar{\Lambda}) \\ s_1^{(2)} &= (1+r_1^*)^2 \\ t_1^{(0)} &= (\varepsilon_c-3\varepsilon_b)^2(\bar{\Lambda}^*-\bar{\Lambda})^2 \\ t_1^{(1)} &= 2+4(\varepsilon_c-3\varepsilon_b)(\bar{\Lambda}^*-\bar{\Lambda}) \\ t_1^{(2)} &= 2(1+2\hat{\zeta}') \\ u_1^{(0)} &= (\varepsilon_c-3\varepsilon_b)(\bar{\Lambda}^*-\bar{\Lambda}) \\ u_1^{(1)} &= 1\end{aligned} \tag{834}$$

The differential decay rate can be integrated over $\cos\theta$

$$\frac{d^2\Gamma_{D_1^*}}{dw} = 3\Gamma_0\zeta^2(1)r_1^{*3}\sqrt{w^2-1}\sum_n(w-1)^n\left[\frac{4}{3}s_1^{(n)} + (1-2r_1^*w+r_1^{*2})\frac{8}{3}t_1^{(n)}\right] \tag{835}$$

Thus

$$\begin{aligned}\frac{d^2\Gamma_{D_1^*}}{dw} &= \frac{G_F^2|V_{cb}|^2m_B^5}{192\pi^3}r_1^{*3}(w^2-1)^{1/2}\zeta^2(1) \\ &\quad \sum_n(w-1)^n\left[4s_1^{(n)}+8(1-2r_1^*w+r_1^{*2})t_1^{(n)}\right] \\ &= \frac{G_F^2|V_{cb}|^2m_B^5}{48\pi^3}r^3(w^2-1)^{1/2}\,\mathcal{J}_{D_1'}(w)\end{aligned} \tag{836}$$

where

$$\mathcal{J}_{D_1'}(w) = \zeta^2(1)\sum_n (w-1)^n \left[s_1^{(n)} + 2(1-2r_1^* w + r_1^{*2})t_1^{(n)}\right] \tag{837}$$

Numerical values used in the LLSW paper are

$$\begin{aligned}\tau(1)\left[\frac{6.0\times 10^{-3}}{\mathcal{B}(B\to D_1 e\bar{\nu}_e)}\right]^{1/2} &= 0.71 \text{ for Approximation B1}\\ &= 0.75 \text{ for Approximation B2}\\ &= 0.60 \text{ for Approximation A}\end{aligned} \tag{838}$$

and

$$\hat{\tau}' = -1.5 \tag{839}$$

I.6 GoityRoberts model

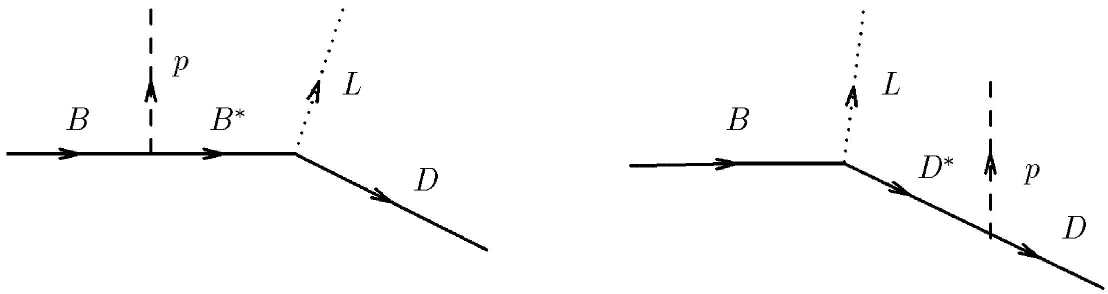
In GoiryRoberts model [44], the "soft-pion" domain was studied in the framework of the chiral Lagrangian approach combined with the $1/m_Q$ expansion.

I.6.1 $B \to D\pi\ell\nu$

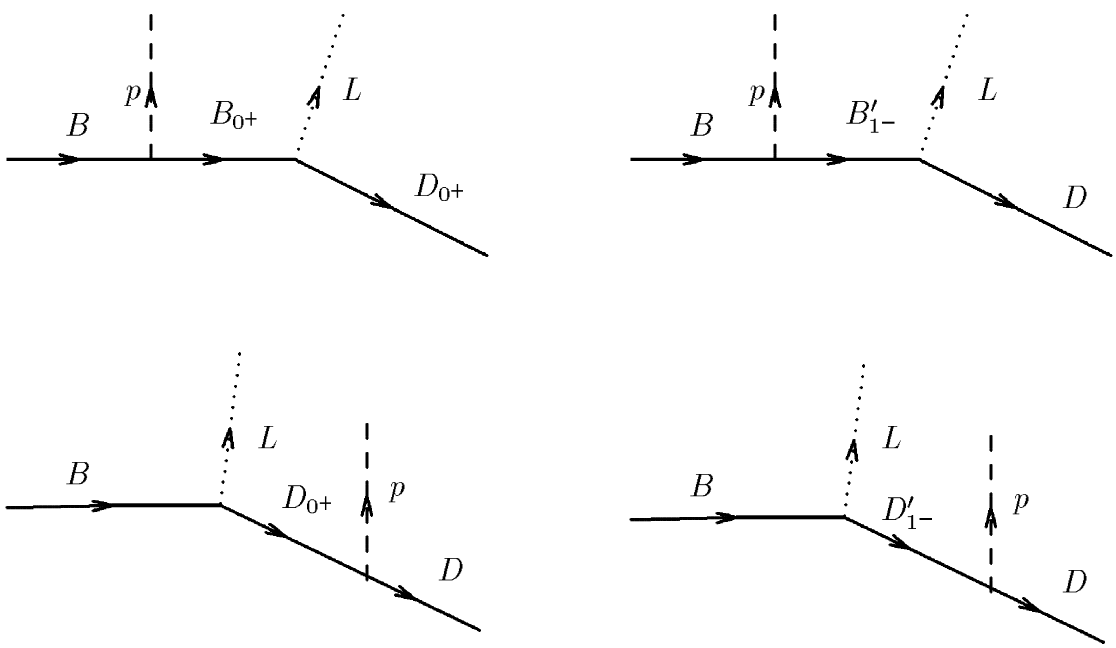
The amplitude for this process has the general form

$$T = \kappa j_\mu \Omega^\mu \tag{840}$$

Here j_μ is the V-A charged leptonic current and the Ω^μ is the hadronic part of the amplitude. To evaluate Ω^μ, these six diagrams are considered to get the amplitude (matrix element)



(a)



(b)

(a) shows two non-resonant decays , (b) shows four resonant decays. The non-resonant portion of Ω^μ is

$$
\begin{aligned}
\Omega_\mu^{NR} \;=\;& \tfrac{g}{F_0}\xi(\nu)p_\nu\Big\{\tfrac{\Theta^{\nu\rho}(v)}{-2(p\cdot v-\delta m_B)+i\varepsilon}\left[iv^\alpha v'^\beta\epsilon_{\mu\alpha\beta\rho}+g_{\mu\rho}(1+\nu)-v_\mu v'_\rho\right] \\
&+\tfrac{\Theta^{\nu\rho}(v')}{2(p\cdot v'-\delta m_D)+i\varepsilon}\left[iv^\alpha v'^\beta\epsilon_{\mu\alpha\beta\rho}+g_{\mu\rho}(1+\nu)-v_\rho v'_\mu\right]\Big\}
\end{aligned}
\qquad (841)
$$

where $\delta m_D = m_{D^*} - m_D$ and $\delta m_B = m_{B^*} - m_B$, and

$$
\Theta^{\mu\nu}(v) = g^{\mu\nu} - v^\mu v^\nu \qquad (842)
$$

The resonant portion of Ω^μ is

$$\begin{aligned}
\Omega^R_\mu &= \frac{\alpha_1}{F_0}\rho_1(\nu)(v-v')_\mu\left[-\frac{p\cdot v}{2(-p\cdot v-\delta\tilde m_1)}+\frac{p\cdot v'}{2(p\cdot v'-\delta\tilde m_1)}\right]\\
&+\frac{\alpha_2}{3F_0}\rho_2(\nu)p_\rho\Big\{\frac{\Theta^{\nu\rho}(v)}{2(-p\cdot v-\delta\tilde m_2)}\left[i\epsilon_{\mu\nu\alpha\beta}v^\alpha v'^\beta(\nu-1)+g_{\mu\nu}(\nu^2-1)\right.\\
&\qquad\left.-v'_\nu\left(v_\mu(2+\nu)-3v'_\mu\right)\right]\\
&+\frac{\Theta^{\nu\rho}(v')}{2(p\cdot v'-\delta\tilde m_2)}\left[i\epsilon_{\mu\nu\alpha\beta}v^\alpha v'^\beta(\nu-1)+g_{\mu\nu}(\nu^2-1)\right.\\
&\qquad\left.-v_\nu\left(v'_\mu(2+\nu)-3v_\mu\right)\right]\Big\}\\
&+\frac{\alpha_3}{F_0}\xi^{(1)}(\nu)p_\nu\Big\{\frac{\Theta^{\nu\rho}(v)}{-2p\cdot v-\delta\tilde m_3}\left[i\epsilon_{\mu\alpha\beta\rho}v^\alpha v'^\beta+g_{\mu\rho}(\nu+1)-v_\mu v'_\rho\right]\\
&+\frac{\Theta^{\nu\rho}(v')}{2p\cdot v'-\delta\tilde m_3}\left[i\epsilon_{\mu\alpha\beta\rho}v^\alpha v'^\beta+g_{\mu\rho}(\nu+1)-v_\rho v'_\mu\right]\Big\}
\end{aligned}\qquad(843)$$

Here we denote $\delta\tilde m - j = \delta m_j - i\Gamma_j/2$, where Γ_j is the total width of the resonance. Contribution from other resonances are small. Isgur-Wise functions are given by

$$\begin{aligned}
&\xi(\nu)=\exp\left[\tfrac{\bar\Lambda^2}{4\beta^2}(\nu^2-1)\right]\\
&\xi^{(1)}(\nu)=-\sqrt{\tfrac{2}{3}}\left[\tfrac{\bar\Lambda^2}{4\beta^2}(\nu^2-1)\right]\exp\left[\tfrac{\bar\Lambda^2}{4\beta^2}(\nu^2-1)\right]\\
&\rho_1(\nu)=\tfrac{1}{\sqrt2}\tfrac{\bar\Lambda}{\beta}\left(\tfrac{2\beta\beta'}{\beta^2+\beta'^2}\right)^{5/2}\exp\left[\tfrac{\bar\Lambda^2}{2(\beta^2+\beta'^2)}(\nu^2-1)\right]\\
&\rho_2(\nu)=\tfrac{1}{2\sqrt2}\left(\tfrac{\bar\Lambda}{\beta}\right)^2\left(\tfrac{2\beta\beta'}{\beta^2+\beta'^2}\right)^{7/2}\exp\left[\tfrac{\bar\Lambda^2}{2(\beta^2+\beta'^2)}(\nu^2-1)\right]
\end{aligned}\qquad(844)$$

where β and β' are the harmonic oscillator parameters of the ground and excited states, respectively. $\bar\Lambda$ is defined by

$$M_{(0^-,1^-)}=m_Q+\bar\Lambda \qquad (845)$$

Ω_μ can be, in general, expressed as

$$\Omega_\mu=-ihm_Bm_D\epsilon_{\mu\nu\rho\sigma}v^\nu v'^\rho p^\sigma_\pi+A_1p_{\pi\mu}+A_2m_Bv_\mu+A_3m_Dv'_{mu} \qquad (846)$$

where h, A_1, A_2 and A_3 are the form factors and p_π is the 4-momentum of the pion. The non-resonant contribution to the form factors are

$$\begin{aligned}
&h_{NR}=\tfrac{g}{2F_0}\tfrac{\xi(\nu)}{m_Bm_D}\left(\tfrac{1}{p_\pi\cdot v+\delta m_B-i\varepsilon}-\tfrac{1}{p_\pi\cdot v'-\delta m_D+i\varepsilon}\right)\\
&A_{1NR}=-\tfrac{g}{2F_0}\xi(\nu)(1+\nu)\left(\tfrac{1}{p_\pi\cdot v+\delta m_B-i\varepsilon}-\tfrac{1}{p_\pi\cdot v'-\delta m_D+i\varepsilon}\right)\\
&A_{2NR}=\tfrac{g}{2F_0}\tfrac{\xi(\nu)}{m_B}\left(\tfrac{p_\pi\cdot v+p_\pi\cdot v'}{p_\pi\cdot v+\delta m_B-i\varepsilon}\right)\\
&A_{3NR}=-\tfrac{g}{2F_0}\tfrac{\xi(\nu)}{m_D}\left(\tfrac{p_\pi\cdot v+p_\pi\cdot v'}{p_\pi\cdot v'-\delta m_D+i\varepsilon}\right)
\end{aligned}\qquad(847)$$

In SP5/SP6, the contribution from $B \to D^*\ell\nu \to D\pi\ell\nu$ was removed. Hence

$$\begin{array}{l} h_{NR} = \frac{g}{2F_0}\frac{\xi(\nu)}{m_B m_D}\left(\frac{1}{p_\pi\cdot v+\delta m_B - i\varepsilon}\right) \\ A_{1NR} = -\frac{g}{2F_0}\xi(\nu)(1+\nu)\left(\frac{1}{p_\pi\cdot v+\delta m_B - i\varepsilon}\right) \\ A_{2NR} = \frac{g}{2F_0}\frac{\xi(\nu)}{m_B}\left(\frac{p_\pi\cdot v + p_\pi\cdot v'}{p_\pi\cdot v+\delta m_B - i\varepsilon}\right) \\ A_{3NR} = 0 \end{array} \tag{848}$$

The resonant contribution to the form factors are

$$\begin{aligned} h_R &= \frac{\alpha_2\rho_2(\nu)}{6F_0 m_B m_D}(\nu-1)\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) \\ &\quad +\frac{\alpha_3\xi^{(1)}(\nu)}{2F_0 m_B m_D}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \end{aligned} \tag{849}$$

$$\begin{aligned} A_{1R} &= -\frac{\alpha_2\rho_2(\nu)}{6F_0}(\nu^2-1)\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) \\ &\quad -\frac{\alpha_3\xi^{(1)}(\nu)}{2F_0}(1+\nu)\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \end{aligned} \tag{850}$$

$$\begin{aligned} A_{2R} &= \frac{\alpha_1\rho_1(\nu)}{2F_0 m_B}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_1} + \frac{p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) \\ &\quad +\frac{\alpha_2\rho_2(\nu)}{F_0 m_B}\left\{\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2}\left[\frac{1}{6}(\nu p_\pi\cdot v' - p_\pi\cdot v) + \frac{1}{3}(p_\pi\cdot v' - \nu p_\pi\cdot v)\right]\right. \\ &\quad \left.+\frac{1}{2}\frac{1}{p_\pi\cdot v'-\delta\tilde{m}_2}(p_\pi\cdot v - \nu p_\pi\cdot v')\right\} \\ &\quad +\frac{\alpha_3\xi^{(1)}(\nu)}{2F_0 m_B}\left(\frac{p_\pi\cdot(v+v')}{p_\pi\cdot v+\delta\tilde{m}_3}\right) \end{aligned} \tag{851}$$

$$\begin{aligned} A_{3R} &= -\frac{\alpha_1\rho_1(\nu)}{2F_0 m_D}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_1} + \frac{p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_1}\right) \\ &\quad -\frac{\alpha_2\rho_2(\nu)}{F_0 m_D}\left\{\frac{1}{p_\pi\cdot v'-\delta\tilde{m}_2}\left[\frac{1}{6}(\nu p_\pi\cdot v - p_\pi\cdot v') + \frac{1}{3}(p_\pi\cdot v - \nu p_\pi\cdot v')\right]\right. \\ &\quad \left.+\frac{1}{2}\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2}(p_\pi\cdot v' - \nu p_\pi\cdot v)\right\} \\ &\quad -\frac{\alpha_3\xi^{(1)}(\nu)}{2F_0 m_D}\left(\frac{p_\pi\cdot(v+v')}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \end{aligned} \tag{852}$$

In SP5/SP6 contributions from D^{**} was removed

$$
\begin{aligned}
h_R &= \tfrac{\alpha_2 \rho_2(\nu)}{6F_0 m_B m_D}(\nu - 1)\left(\tfrac{1}{p_\pi \cdot v + \delta\tilde{m}_2}\right) + \tfrac{\alpha_3 \xi^{(1)}(\nu)}{2F_0 m_B m_D}\left(\tfrac{1}{p_\pi \cdot v + \delta\tilde{m}_3}\right) \\
A_{1R} &= -\tfrac{\alpha_2 \rho_2(\nu)}{6F_0}(\nu^2 - 1)\left(\tfrac{1}{p_\pi \cdot v + \delta\tilde{m}_2}\right) - \tfrac{\alpha_3 \xi^{(1)}(\nu)}{2F_0}(1+\nu)\left(\tfrac{1}{p_\pi \cdot v + \delta\tilde{m}_3}\right) \\
A_{2R} &= \tfrac{\alpha_1 \rho_1(\nu)}{2F_0 m_B}\left(\tfrac{p_\pi \cdot v}{p_\pi \cdot v + \delta\tilde{m}_1}\right) \\
&\quad + \tfrac{\alpha_2 \rho_2(\nu)}{F_0 m_B}\left\{\tfrac{1}{p_\pi \cdot v + \delta\tilde{m}_2}\left[\tfrac{1}{6}(\nu p_\pi \cdot v' - p_\pi \cdot v) + \tfrac{1}{3}(p_\pi \cdot v' - \nu p_\pi \cdot v)\right]\right\} \\
&\quad + \tfrac{\alpha_3 \xi^{(1)}(\nu)}{2F_0 m_B}\left(\tfrac{p_\pi \cdot (v+v')}{p_\pi \cdot v + \delta\tilde{m}_3}\right) \\
A_{3R} &= -\tfrac{\alpha_1 \rho_1(\nu)}{2F_0 m_D}\left(\tfrac{p_\pi \cdot v}{p_\pi \cdot v + \delta\tilde{m}_1}\right) \\
&\quad - \tfrac{\alpha_2 \rho_2(\nu)}{F_0 m_D}\left\{\tfrac{1}{2}\tfrac{1}{p_\pi \cdot v + \delta\tilde{m}_2}(p_\pi \cdot v' - \nu p_\pi \cdot v)\right\}
\end{aligned}
\tag{853}
$$

Numerical values used in SP5/SP6 are

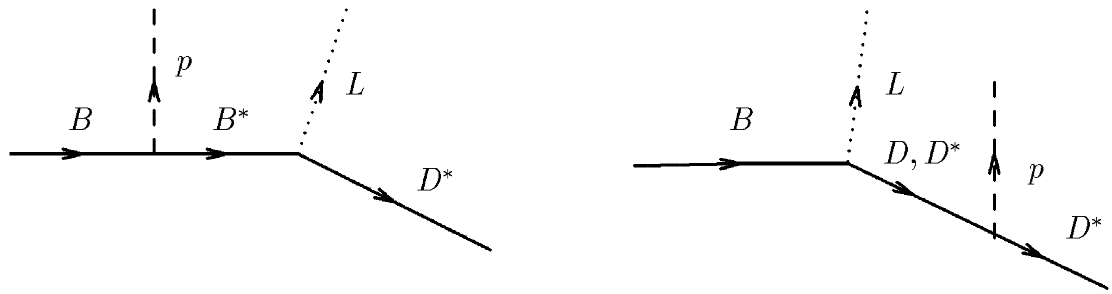
$$
\begin{aligned}
&g = 0.5 \\
&\alpha_1 = -1.430 \\
&\alpha_2 = -0.140 \\
&\alpha_3 = 0.690 \\
&F_0 = 0.093 \\
&\delta m_B = 0.0460 - \tfrac{0.00001}{2} i \\
&\delta m_D = 0.1421 - \tfrac{0.00006}{2} i \\
&\delta\tilde{m}_1 = 0.392 - \tfrac{1.040}{2} i \\
&\delta\tilde{m}_2 = 0.709 - \tfrac{0.405}{2} i \\
&\delta\tilde{m}_3 = 0.563 - \tfrac{0.191}{2} i \\
&\beta = 0.285 \\
&\beta' = 0.280 \quad \text{for} \quad \rho_1(\nu) \\
&\beta' = 0.260 \quad \text{for} \quad \rho_2(\nu) \\
&\bar{\Lambda} = 0.750
\end{aligned}
\tag{854}
$$

I.6.2 $B \to D^* \pi \ell \nu$

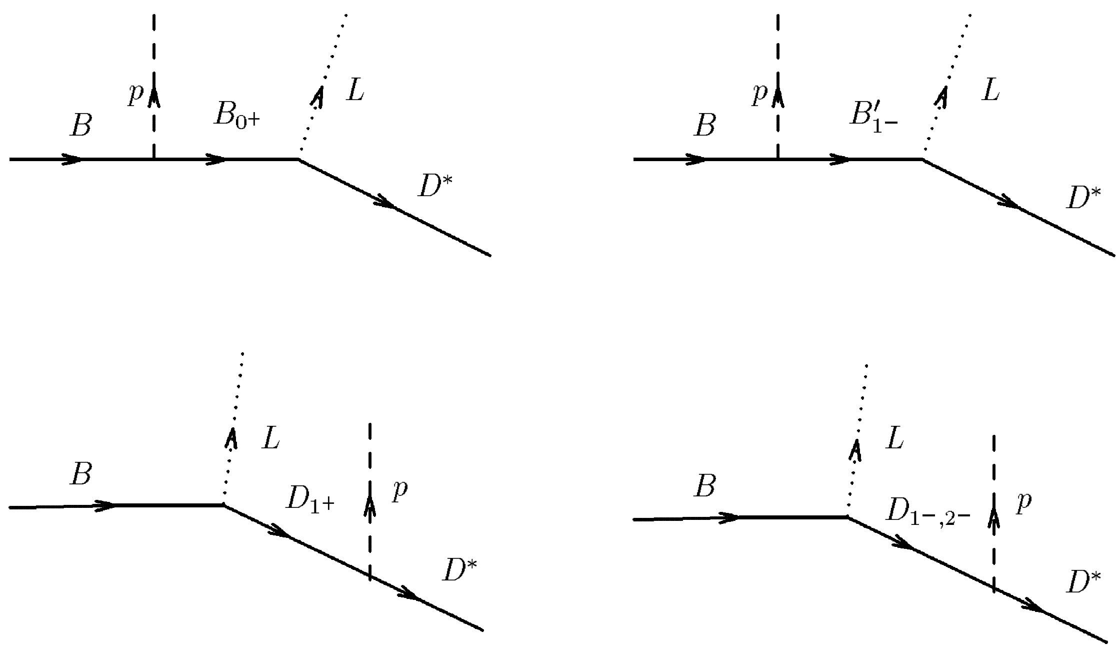
The amplitude for this process has the general form

$$
T = \kappa j_\mu \Omega^{\mu\nu} \varepsilon_{D^* \nu} \tag{855}
$$

where $\varepsilon_{D^* \nu}$ is the polarization vector of the D^*.



(a)



(b)

(a) shows two non-resonant decays , (b) shows four resonant decays. The non-resonant portion of $\Omega_{\mu\nu}$ is

$$\begin{aligned}
\Omega^{NR}_{\mu\nu} = & \tfrac{g}{2F_0}\xi(\nu)\Big\{(v+v')_\mu \tfrac{p_\nu}{p\cdot v'+\delta m_D + i\varepsilon} \\
& -p^\rho \tfrac{\Theta_{\rho\sigma}(v)}{-(p\cdot v+\delta m_B)+i\varepsilon}\left[g_{\nu\sigma}(v+v')_\mu - g_{\mu\sigma}v_\nu - g_{\mu\nu}v'_\sigma + i\epsilon_{\mu\alpha\sigma\nu}(v+v')^\alpha\right] \\
& +\tfrac{\Theta_{\rho\delta}(v')}{p\cdot v'+i\varepsilon}\left[-i\epsilon_{\mu\rho\alpha\beta}v'^\alpha v^\beta + g_{\mu\rho}(1+\nu) - v_\rho v'_\mu\right] i\epsilon_{\omega\nu\gamma\delta}p^\gamma v'^\omega\Big\}
\end{aligned} \tag{856}$$

The resonant portion of $\Omega_{\mu\nu}$ is

$$
\begin{aligned}
\Omega^R_{\mu\nu} \;=\;& \tfrac{\alpha_1}{F_0}\rho_1(\nu)\left[-g_{\mu\nu}(\nu-1)+v'_\mu v_\nu - i\epsilon_{\mu\nu\alpha\beta}v^\alpha v'^\beta\right]\left[-\tfrac{p\cdot v}{2(-p\cdot v-\delta\tilde m_1)}+\tfrac{p\cdot v'}{2(p\cdot v'-\delta\tilde m_1)}\right]\\
&-\tfrac{\alpha_2}{3F_0}\rho_2(\nu)\Big\{p_\rho\tfrac{\Theta^{\rho\sigma}(v)}{2(-p\cdot v-\delta\tilde m_2)}\left[-g_{\nu\sigma}(v+v')_\mu(\nu-1)+3v_\nu v'_\mu v'_\sigma-2g_{\mu\nu}v'_\sigma(\nu-1)\right.\\
&\left.+g_{\mu\sigma}v_\nu(\nu-1)+i\epsilon_{\mu\sigma\alpha\nu}(v-v')^\alpha(1+\nu)+2i\epsilon_{\sigma\nu\alpha\beta}v'^\alpha v^\beta v'_\mu+i\epsilon_{\mu\nu\alpha\beta}v'_\sigma v'^\alpha v^\beta\right]\\
&+3p^\gamma\tfrac{\Theta^{\sigma\rho}_{\gamma\nu}(v')}{2(p\cdot v'-\delta\tilde m_2)+i\varepsilon}\left[v_\delta g_{\rho\mu}(\nu-1)-v_\delta v_\rho v'_\mu+i\epsilon_{\mu\delta\alpha\beta}v_\rho v^\alpha v'^\beta\right]\\
&+\tfrac{i}{2}\tfrac{\Theta^{\sigma\rho}(v)}{2(p\cdot v'-\delta\tilde m_2)}\epsilon_{\rho\delta\gamma\nu}p^\delta_\pi v'^\gamma\\
&\quad\times\left[i\epsilon_{\mu\sigma\alpha\beta}v^\alpha v'^\beta(\nu-1)+g_{\mu\sigma}(\nu^2-1)-v_\alpha\left(v'_\mu(2+\nu)-3v_\mu\right)\right]\Big\}\\
&+\tfrac{\alpha_3}{F_0}\xi^{(1)}(\nu)\Big\{(v+v')_\mu\tfrac{p_\nu}{2(p\cdot v'-\delta\tilde m_3)}\\
&+p^\rho\tfrac{\Theta_{\rho\sigma}(v)}{-2(-p\cdot v-\delta\tilde m_3)}\left[g_{\nu\sigma}(v+v')_\mu-g_{\mu\sigma}v_\nu-g_{\mu\nu}v'_\sigma+i\epsilon_{\mu\alpha\sigma\nu}(v+v')^\alpha\right]\\
&+\tfrac{\Theta_{\rho\delta}(v')}{2(p\cdot v'-\delta\tilde m_3)}\left[-i\epsilon_{\mu\rho\alpha\beta}v'^\alpha v^\beta+g_{\mu\rho}(1+\nu)-v_\rho v'_\mu\right]i\epsilon_{\omega\nu\gamma\delta}p^\gamma v'^\omega\Big\}
\end{aligned}
\tag{857}
$$

The tensor $\Omega_{\mu\nu}$ can be expressed as

$$
\begin{aligned}
\Omega_{\mu\nu} \;=\;& \tfrac{i}{2}\epsilon_{\mu\nu\rho\sigma}\left[h_1 m_B m_D v^\rho v'^\sigma+h_2 m_B v^\rho p^\sigma_\pi+h_3 m_D v'^\rho p^\sigma_\pi\right]\\
&+f_1 m_B v_\mu p_{\pi\nu}+f_2 m_D v'_\mu p_{\pi\nu}+f_3 p_{\pi\mu}p_{\pi\nu}+f_4 m_B^2 v_\mu v_\nu\\
&+f_5 m_B m_D v'_\mu v_\nu+f_6 m_B p_{\pi\mu}v_\nu\\
&+k g_{\mu\nu}\\
&+\tfrac{i}{2}\epsilon_{\mu\delta\rho\sigma}v^\delta v'^\rho p^\sigma_\pi(g_1 p_{\pi\nu}+g_2 m_B v_\nu)\\
&+\tfrac{i}{2}\epsilon_{\nu\delta\rho\sigma}v^\delta v'^\rho p^\sigma_\pi(g_3 m_B v_\mu+g_4 m_D v'_\mu+g_5 p_{\pi\mu})
\end{aligned}
\tag{858}
$$

The non-resonant contribution to the form factors are

$$
\begin{aligned}
h_{1NR} &= -\tfrac{g\xi(\nu)}{F_0 m_B m_D} \tfrac{p_\pi \cdot v}{p_\pi \cdot v + \delta m_B - i\varepsilon} \\
h_{2NR} &= -\tfrac{g\xi(\nu)}{F_0 m_B} \tfrac{1}{p_\pi \cdot v + \delta m_B - i\varepsilon} \\
h_{3NR} &= -\tfrac{g\xi(\nu)}{F_0 m_D} \left(\tfrac{1}{p_\pi \cdot v + \delta m_B - i\varepsilon} - \tfrac{1+\nu}{p_\pi \cdot v' + i\varepsilon} \right) \\
f_{1NR} &= -\tfrac{g\xi(\nu)}{2F_0 m_B} \left(\tfrac{1}{p_\pi \cdot v + \delta m_B - i\varepsilon} - \tfrac{1}{p_\pi \cdot v' + \delta m_D + i\varepsilon} \right) \\
f_{2NR} &= \tfrac{m_B}{m_D} f_{1NR} \\
f_{3NR} &= 0 \\
f_{4NR} &= 0 \\
f_{5NR} &= -\tfrac{g\xi(\nu)}{2F_0 m_B m_D} \left(1 + \tfrac{p_\pi \cdot v}{p_\pi \cdot v + \delta m_B - i\varepsilon} \right) \\
f_{6NR} &= -\tfrac{g\xi(\nu)}{2F_0 m_B} \left(\tfrac{1}{p_\pi \cdot v + \delta m_B - i\varepsilon} - \tfrac{1}{p_\pi \cdot v' + i\varepsilon} \right) \\
k_{NR} &= \tfrac{g\xi(\nu)}{2F_0} \left(\tfrac{p_\pi \cdot v' - \nu p_\pi \cdot v}{p_\pi \cdot v + \delta m_B - i\varepsilon} + \tfrac{p_\pi \cdot v - \nu p_\pi \cdot v'}{p_\pi \cdot v' + i\varepsilon} \right) \\
g_{1NR} &= 0 \\
g_{2NR} &= 0 \\
g_{3NR} &= 0 \\
g_{4NR} &= \tfrac{g\xi(\nu)}{F_0 m_D} \tfrac{1}{p_\pi \cdot v' + i\varepsilon} \\
g_{5NR} &= 0
\end{aligned}
\tag{859}
$$

The resonant contribution to the form factors are

$$
\begin{aligned}
h_{1R} &= \frac{\alpha_1\rho_1(\nu)}{F_0 m_B m_D}\left(\frac{p_\pi\cdot v}{-p_\pi\cdot v-\delta\tilde{m}_1} - \frac{p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_1}\right) \\
&\quad + \frac{\alpha_2\rho_2(\nu)}{3F_0 m_B m_D}\left(\frac{p_\pi\cdot v(1+2\nu)-p_\pi\cdot v'}{p_\pi\cdot v+\delta\tilde{m}_2} + \frac{3}{2}\frac{\nu p_\pi\cdot v'-p_\pi\cdot v}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) \\
&\quad - \frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_B m_D}\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_3} \\
h_{2R} &= \frac{\alpha_2(1+\nu)\rho_2(\nu)}{3F_0 m_B}\frac{1}{-p_\pi\cdot v-\delta\tilde{m}_2} - \frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_B}\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3} \\
h_{3R} &= \frac{\alpha_2\rho_2(\nu)}{3F_0 m_D}\left(\frac{1+\nu}{p_\pi\cdot v+\delta\tilde{m}_2} - \frac{\nu^2-1}{2(p_\pi\cdot v'-\delta\tilde{m}_2)}\right) - \frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_D}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3} - \frac{1+\nu}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \\
f_{1R} &= -\frac{\alpha_2\rho_2(\nu)(\nu-1)}{6F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) - \frac{\alpha_3\xi^{(1)}(\nu)}{2F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \\
f_{2R} &= \frac{m_B}{m_D} f_{1R} \\
f_{3R} &= 0 \\
f_{4R} &= 0 \\
f_{5R} &= -\frac{\alpha_1\rho_1(\nu)}{2F_0 m_B m_D}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_1} + \frac{p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_1}\right) \\
&\quad + \frac{\alpha_2\rho_2(\nu)}{2F_0 m_B m_D}\left(\frac{p_\pi\cdot v'-\frac{1}{3}p_\pi\cdot v(1+2\nu)}{p_\pi\cdot v+\delta\tilde{m}_2} + \frac{p_\pi\cdot v-\frac{1}{3}p_\pi\cdot v'(1+2\nu)}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) \\
&\quad + \frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_B m_D}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_3} + \frac{p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \\
f_{6R} &= \frac{\alpha_2\rho_2(\nu)(\nu-1)}{6F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) + \frac{\alpha_3\xi^{(1)}(\nu)}{2F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3} - \frac{1}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \\
k_R &= -\frac{\alpha_1\rho_1(\nu)(\nu-1)}{2F_0}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_1} + \frac{p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_1}\right) \\
&\quad - \frac{\alpha_2\rho_2(\nu)(\nu-1)}{3F_0}\left(\frac{p_\pi\cdot v'-\nu p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_2} + \frac{p_\pi\cdot v-\nu p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) \\
&\quad + \frac{\alpha_3\xi^{(1)}(\nu)}{2F_0}\left(\frac{p_\pi\cdot v'-\nu p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_3} + \frac{p_\pi\cdot v-\nu p_\pi\cdot v'}{p_\pi\cdot v'-\delta\tilde{m}_3}\right) \\
g_{1R} &= 0 \\
g_{2R} &= -\frac{\alpha_2\rho_2(\nu)}{2F_0 m_B}\frac{1}{p_\pi\cdot v'-\delta\tilde{m}_2} \\
g_{3R} &= -g_{2R} \\
g_{4R} &= \frac{\alpha_2\rho_2(\nu)}{3F_0 m_D}\left(\frac{2}{p_\pi\cdot v+\delta\tilde{m}_2} - \frac{1+\frac{1}{2}\nu}{p_\pi\cdot v'-\delta\tilde{m}_2}\right) + \frac{\alpha_3\xi^{(1)}(\nu)}{F_0}\frac{1}{p_\pi\cdot v'-\delta\tilde{m}_3} \\
g_{5R} &= 0
\end{aligned}
\tag{860}
$$

In SP5/SP6 contributions from D^{**} was removed

$$
\begin{aligned}
h_{1R} &= \frac{\alpha_1\rho_1(\nu)}{F_0 m_B m_D}\left(\frac{p_\pi\cdot v}{-p_\pi\cdot v-\delta\tilde{m}_1}\right)+\frac{\alpha_2\rho_2(\nu)}{3F_0 m_B m_D}\left(\frac{p_\pi\cdot v(1+2\nu)-p_\pi\cdot v'}{p_\pi\cdot v+\delta\tilde{m}_2}\right)\\
&\quad -\frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_B m_D}\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_3}\\
h_{2R} &= \frac{\alpha_2(1+\nu)\rho_2(\nu)}{3F_0 m_B}\frac{1}{-p_\pi\cdot v-\delta\tilde{m}_2}-\frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_B}\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3}\\
h_{3R} &= \frac{\alpha_2\rho_2(\nu)}{3F_0 m_D}\left(\frac{1+\nu}{p_\pi\cdot v+\delta\tilde{m}_2}\right)-\frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_D}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3}\right)\\
f_{1R} &= -\frac{\alpha_2\rho_2(\nu)(\nu-1)}{6F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2}\right)-\frac{\alpha_3\xi^{(1)}(\nu)}{2F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3}\right)\\
f_{2R} &= \frac{m_B}{m_D}f_{1R}\\
f_{3R} &= 0\\
f_{4R} &= 0\\
f_{5R} &= -\frac{\alpha_1\rho_1(\nu)}{2F_0 m_B m_D}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_1}\right)+\frac{\alpha_2\rho_2(\nu)}{2F_0 m_B m_D}\left(\frac{p_\pi\cdot v'-\frac{1}{3}p_\pi\cdot v(1+2\nu)}{p_\pi\cdot v+\delta\tilde{m}_2}\right)\\
&\quad +\frac{\alpha_3\xi^{(1)}(\nu)}{F_0 m_B m_D}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_3}\right)\\
f_{6R} &= \frac{\alpha_2\rho_2(\nu)(\nu-1)}{6F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_2}\right)+\frac{\alpha_3\xi^{(1)}(\nu)}{2F_0 m_B}\left(\frac{1}{p_\pi\cdot v+\delta\tilde{m}_3}\right)\\
k_R &= -\frac{\alpha_1\rho_1(\nu)(\nu-1)}{2F_0}\left(\frac{p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_1}\right)-\frac{\alpha_2\rho_2(\nu)(\nu-1)}{3F_0}\left(\frac{p_\pi\cdot v'-\nu p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_2}\right)\\
&\quad +\frac{\alpha_3\xi^{(1)}(\nu)}{2F_0}\left(\frac{p_\pi\cdot v'-\nu p_\pi\cdot v}{p_\pi\cdot v+\delta\tilde{m}_3}\right)\\
g_{1R} &= 0\\
g_{2R} &= 0\\
g_{3R} &= -g_{2R}\\
g_{4R} &= \frac{\alpha_2\rho_2(\nu)}{3F_0 m_D}\left(\frac{2}{p_\pi\cdot v+\delta\tilde{m}_2}\right)\\
g_{5R} &= 0
\end{aligned}
\tag{861}
$$

Numerical values used in SP5/SP6 are

$$
\begin{aligned}
&g = 0.5 \\
&\alpha_1 = -1.430 \\
&\alpha_2 = -0.140 \\
&\alpha_3 = 0.690 \\
&F_0 = 0.093 \\
&\delta m_B = 0.0460 - \tfrac{0.00001}{2} i \\
&\delta m_D = 0.1421 - \tfrac{0.00006}{2} i \\
&\delta \tilde{m}_1 = 0.392 - \tfrac{1.040}{2} i \\
&\delta \tilde{m}_2 = 0.709 - \tfrac{0.405}{2} i \\
&\delta \tilde{m}_3 = 0.563 - \tfrac{0.191}{2} i \\
&\beta = 0.285 \\
&\beta' = 0.280 \quad \text{for} \quad \rho_1(\nu) \\
&\beta' = 0.260 \quad \text{for} \quad \rho_2(\nu) \\
&\bar{\Lambda} = 0.750
\end{aligned}
\tag{862}
$$

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