Problem 2.2

(a) We are given the action to be:

$$S = \int (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) d^4x \tag{1}$$

Then we can easily read off the Lagrangian density to be:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{2}$$

$$\Rightarrow \mathcal{L} = \dot{\phi}^* \dot{\phi} - \partial_a \phi^* \partial^a \phi - m^2 \phi^* \phi \tag{3}$$

Then the momentum densities are:

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} \tag{4}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \tag{5}$$

The momenta can be easily obtained by integrating the momentum densities through all space.

$$p = \int \dot{\phi}^* \, d^3x \tag{6}$$

$$p^* = \int \dot{\phi} \, d^3x \tag{7}$$

The Hamiltonian density then can be calculated:

$$\begin{aligned} \mathcal{H} &= \pi \dot{\phi} + \pi^* \dot{\phi}^* - \mathcal{L} \\ \Rightarrow \mathcal{H} &= \pi\pi^* + \pi^*\pi - \pi\pi^* + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi \\ \Rightarrow \mathcal{H} &= \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi \end{aligned}$$

(Again, the Hamiltonian can be easily found by integrating $\mathcal{H}$ over all space.)

The canonical commutation relations are as follows:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{8}$$

$$[\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \tag{9}$$

whereas the rest of the commutation relations vanish because:

1. the commutation relation is between a field and a momentum of a different field; or

2. the commutation relation involves the same field but with different coordinates

The Heisenberg equation for motion is given by:

$$i\frac{\partial}{\partial t}v = [v, H] \tag{10}$$

Hence

$$i\frac{\partial}{\partial t}\phi(\mathbf{x}, t) \quad = \quad [\phi(\mathbf{x}, t), \int d^3x'\{\pi^*\pi(\mathbf{x}', t) + \nabla\phi^* \cdot \nabla\phi(\mathbf{x}', t) + m^2\phi^*\phi(\mathbf{x}', t)\}]$$

To proceed, one must pull the ϕ out from the gradient operator. To do that, recall Green's Second Identity:

Theorem 1 (Green's Second Identity) *Let $\mathcal{M}$ be a smooth 3-dimensional oriented manifold with boundary $\partial\mathcal{M}$. For smooth functions ϕ and ψ, the following holds true:*

$$\int_{\mathcal{M}} \nabla\phi \cdot \nabla\psi dV = \int_{\partial\mathcal{M}} \phi(\nabla\psi) \cdot \hat{\mathbf{n}} da - \int_{\mathcal{M}} \phi\nabla^2\psi dV \tag{11}$$

where $\hat{\mathbf{n}}$ is the outward normal, and dV and da are the volume element and the area element for $\mathcal{M}$ and $\partial\mathcal{M}$ respectively.

For fields to be physical, they must vanish at infinitely far away. Hence

$$i\frac{\partial}{\partial t}\phi(\mathbf{x}, t) \quad = \quad [\phi(\mathbf{x}, t), \int d^3x'\{\pi^*\pi(\mathbf{x}', t) - \phi(\mathbf{x}', t)\nabla^2\phi^*(\mathbf{x}', t) + m^2\phi^*\phi(\mathbf{x}', t)\}]$$

Now, note that the only quantity that does not commute with ϕ, as previously found, is π. However, to get from ϕ^* to π one applies the time derivative. Laplacian, a spatial derivative operator, acting on ϕ^* does not produce π. Therefore one can safely drop the last two terms in the integrand out of the commutation relation.

$$\begin{aligned} i\frac{\partial}{\partial t}\phi(\mathbf{x}, t) &= \int d^3x'[\phi(\mathbf{x}, t), \pi^*\pi(\mathbf{x}', t)] \\ &= \int d^3x'\Big(\pi^*(\mathbf{x}', t)[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] + [\phi(\mathbf{x}, t), \pi^*(\mathbf{x}', t)]\pi(\mathbf{x}', t)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \int d^3x' i\delta^3(\mathbf{x}-\mathbf{x}')\pi^*(\mathbf{x}',t) \\
&= i\pi^*(\mathbf{x},t) \\
i\frac{\partial}{\partial t}\pi(\mathbf{x},\mathbf{t}) &= [\pi(\mathbf{x},t), \int d^3x'\{\pi^*\pi(\mathbf{x}',t) + \nabla\phi^*\cdot\nabla\phi(\mathbf{x}',t) + m^2\phi^*\phi(\mathbf{x}',t)\}] \\
&= [\pi(\mathbf{x},t), \int d^3x'\{\pi^*\pi(\mathbf{x}',t) + \phi(\mathbf{x}',t)(-\nabla^2+m^2)\phi^*(\mathbf{x}',t)\}] \\
&= \int d^3x'(-i)\delta^3(\mathbf{x}-\mathbf{x}')(-\nabla^2+m^2)\phi^*(\mathbf{x}',t) \\
&= -i(-\nabla^2+m^2)\phi^*(\mathbf{x},t)
\end{aligned}
$$

(Green's Second Identity is used to go from the first equality to the second equality.)

The results for the complex conjugate ϕ^* and π^* are very similar:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\phi^*(\mathbf{x},t) &= i\pi(\mathbf{x},t) \\
i\frac{\partial}{\partial t}\pi^*(\mathbf{x},t) &= -i(-\nabla^2+m^2)\phi(\mathbf{x},t)
\end{aligned}
$$

Combining the above results, one obtains:

$$
\begin{aligned}
-\frac{\partial^2}{\partial t^2}\phi &= (-\nabla^2+m^2)\phi \\
-\frac{\partial^2}{\partial t^2}\phi^* &= (-\nabla^2+m^2)\phi^*
\end{aligned}
$$

Precisely the Klein-Gordon Equations.

(b) Since the field is no longer purely real, the coefficients before $e^{ip_\mu x^\mu}$ and $e^{-ip_\mu x^\mu}$ need not to be equal.

Introduce lowering and raising operators $a_{\mathbf{p}}$ and $b_{\mathbf{p}}^\dagger$ for the field ϕ such that:

$$
\begin{aligned}
\phi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}}e^{ip_\mu x^\mu} + b_{\mathbf{p}}^\dagger e^{-ip_\mu x^\mu}) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger)e^{ip_\mu x^\mu} \qquad (12)
\end{aligned}
$$

The second term in the integration is changed via the transformation $p_\mu \to -p_\mu$, and the relation $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}}$ is employed.

Then its complex conjugate ϕ^* is:

$$\begin{aligned}
\phi^*(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (b_{\mathbf{p}} e^{ip_\mu x^\mu} + a_{\mathbf{p}}^\dagger e^{-ip_\mu x^\mu}) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (b_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) e^{ip_\mu x^\mu} \qquad (13)
\end{aligned}$$

From earlier results, $\pi = \frac{\partial}{\partial t}\phi^*$ and $\pi^* = \frac{\partial}{\partial t}\phi$. Recall also that $\frac{\partial}{\partial t} e^{ip_\mu x^\mu} = \frac{\partial}{\partial t} e^{i(\omega_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{x})} = i\omega_{\mathbf{p}} e^{ip_\mu x^\mu}$. Hence:

$$\begin{aligned}
\pi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (b_{\mathbf{p}} e^{ip_\mu x^\mu} - a_{\mathbf{p}}^\dagger e^{-ip_\mu x^\mu}) \\
&= \int \frac{d^3p}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (b_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) e^{ip_\mu x^\mu} \\
\pi^*(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{ip_\mu x^\mu} - b_{\mathbf{p}}^\dagger e^{-ip_\mu x^\mu}) \\
&= \int \frac{d^3p}{(2\pi)^3} i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - b_{-\mathbf{p}}^\dagger) e^{ip_\mu x^\mu} \qquad (14)
\end{aligned}$$

Hence the Hamiltonian density can be written as follows:

$$\begin{aligned}
\mathcal{H} &= \int \frac{d^3p d^3p'}{(2\pi)^6} \Big(- \frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{2} (a_{\mathbf{p}} - b_{-\mathbf{p}}^\dagger)(b_{\mathbf{p}'} - a_{-\mathbf{p}'}^\dagger) e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} + \\
&\quad \frac{-\mathbf{p}'\cdot\mathbf{p} + m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (b_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{p}'} + b_{-\mathbf{p}'}^\dagger) e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \Big) \qquad (15)
\end{aligned}$$

The Hamiltonian is just $\mathcal{H}$ integrated over all space. However, when doing that, only the exponential gets integrated, and the result yields a Dirac Delta function $(2\pi)^3\delta^3(\mathbf{p}+\mathbf{p}')$. Then integrate over p':

$$\begin{aligned}
H &= \int \frac{d^3p}{(2\pi)^3} \Big(- \frac{\sqrt{\omega_{\mathbf{p}}\omega_{-\mathbf{p}}}}{2} (a_{\mathbf{p}} - b_{-\mathbf{p}}^\dagger)(b_{-\mathbf{p}} - a_{\mathbf{p}}^\dagger) \\
&\quad + \frac{\mathbf{p}\cdot\mathbf{p} + m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{-\mathbf{p}}}} (b_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{-\mathbf{p}} + b_{\mathbf{p}}^\dagger) \Big) \\
&= \int \frac{d^3p}{(2\pi)^3} \Big(- \frac{\omega_{\mathbf{p}}}{2} (a_{\mathbf{p}} b_{-\mathbf{p}} - a_{\mathbf{p}} a_{\mathbf{p}}^\dagger - b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} + b_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger) \\
&\quad + \frac{\omega_{\mathbf{p}}^2}{2\omega_{\mathbf{p}}} (b_{\mathbf{p}} a_{-\mathbf{p}} + b_{\mathbf{p}} b_{\mathbf{p}}^\dagger + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} + a_{-\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger) \Big) \qquad (16)
\end{aligned}$$

where we have used $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}} = \sqrt{\mathbf{p}\cdot\mathbf{p} + m^2}$.

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3}\Big(\frac{\omega_{\mathbf{p}}}{2}(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}) + \frac{\omega_{\mathbf{p}}}{2}(b_{\mathbf{p}}b^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}})\Big) \\ &= \int \frac{d^3p}{(2\pi)^3}\Big(\frac{\omega_{\mathbf{p}}}{2}([a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}] + 2a^\dagger_{\mathbf{p}}a_{\mathbf{p}}) + \frac{\omega_{\mathbf{p}}}{2}([b_{\mathbf{p}}, b^\dagger_{\mathbf{p}}] + 2b^\dagger_{\mathbf{p}}b_{\mathbf{p}})\Big) \qquad (17) \end{aligned}$$

This Hamiltonian gives rise to two particles because of the two distinct ladder operators, both with mass m.