Peskin and Schroeder: 3.1

(a) We are given the following commutation relation:

$$[J^{\mu\nu}, J^{\sigma\rho}] = i\left(g^{\nu\sigma}J^{\mu\rho} - g^{\mu\sigma}J^{\nu\rho} - g^{\nu\rho}J^{\mu\sigma} + g^{\mu\rho}J^{\nu\sigma}\right) \tag{1}$$

As well, we are given the following relations:

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \tag{2}$$

$$K^i = J^{0i} \tag{3}$$

Hence:

$$\begin{aligned}
[L^i, L^j] &= \frac{1}{4}[\epsilon^{ikl}J^{kl}, \epsilon^{jmn}J^{mn}] \\
&= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn}[J^{kl}, J^{mn}] \\
&= \frac{i}{4}\epsilon^{ikl}\epsilon^{jmn}\left(g^{lm}J^{kn} - g^{km}J^{ln} - g^{ln}J^{km} + g^{kn}J^{lm}\right) \\
&= \frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jmn}g^{km}J^{ln} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km} + \epsilon^{ikl}\epsilon^{jmn}g^{kn}J^{lm}\right)
\end{aligned}$$

For the second and last terms, interchange the roles of k and l

$$\begin{aligned}
[L^i, L^j] &= \frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ilk}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km} + \epsilon^{ilk}\epsilon^{jmn}g^{ln}J^{km}\right) \\
&= \frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km} - \epsilon^{ikl}\epsilon^{jmn}g^{ln}J^{km}\right)
\end{aligned}$$

For the third and last terms, interchange the roles of m and n.

$$\begin{aligned}
[L^i, L^j] &= \frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jnm}g^{lm}J^{kn} - \epsilon^{ikl}\epsilon^{jnm}g^{lm}J^{kn}\right) \\
&= \frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} + \epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn}\right) \\
&= i\epsilon^{ikl}\epsilon^{jmn}g^{lm}J^{kn} \\
&= -i\epsilon^{ikl}\epsilon^{jln}J^{kn} \\
&= i\epsilon^{ikl}\epsilon^{jnl}J^{kn} \\
&= i\left(\delta^{ij}\delta^{kn} - \delta^{in}\delta^{kj}\right)J^{kn} \\
&= i\left(\delta^{ij}J^{nn} - J^{ji}\right) \\
&= -iJ^{ji}
\end{aligned}$$

The next task is to write J^{ji} in terms of L^i and L^j

$$\begin{aligned}
L^i &= \frac{1}{2}\epsilon^{ijk}J^{jk} \\
\Rightarrow \epsilon^{ilm}L^i &= \frac{1}{2}\epsilon^{ijk}\epsilon^{ilm}J^{jk} \\
&= \frac{1}{2}\left(\delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}\right)J^{jk} \\
&= \frac{1}{2}\left(J^{lm} - J^{ml}\right) \\
&= J^{lm} \\
\Rightarrow \epsilon^{kji}L^k &= J^{ji}
\end{aligned}$$

Hence we have:

$$[L^i, L^j] = -i\epsilon^{kji}L^k = i\epsilon^{ijk}L^k \tag{4}$$

The commutation relation for K can be calculated also:

$$\begin{aligned}
[K^i, K^j] &= [J^{0i}, J^{0j}] \\
&= i(g^{i0}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}) \\
&= -iJ^{ij} \\
&= -i\epsilon^{ijk}L^k
\end{aligned} \tag{5}$$

The commutation between L and K:

$$\begin{aligned}
[K^i, L^j] &= [J^{0i}, \frac{1}{2}\epsilon^{jkl}J^{kl}] \\
&= \frac{i}{2}\epsilon^{jkl}\left(g^{ik}J^{0l} - g^{il}J^{0k}\right) \\
&= \frac{i}{2}\left(-\epsilon^{jil}J^{0l} + \epsilon^{jki}J^{0k}\right) \\
&= \frac{i}{2}\left(-\epsilon^{jik}J^{0k} + \epsilon^{jki}J^{0k}\right) \\
&= i\epsilon^{ijk}K^k
\end{aligned} \tag{6}$$

With these relations derived, one can now determine the commutation relation between $\mathbf{J}_+$ and $\mathbf{J}_-$ with ease:

$$\begin{aligned}
[J_+^i, J_-^j] &= \left[\frac{1}{2}\left(L^i + iK^i\right), \frac{1}{2}\left(L^j - iK^j\right)\right] \\
&= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] - i[L^i, K^j] + [K^i, K^j]\right) \\
&= \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k - \epsilon^{jik}K^k - i\epsilon^{ijk}L^k\right) = 0
\end{aligned} \tag{7}$$

$$
\begin{aligned}
[J_+^i, J_+^j] &= \left[\frac{1}{2}\left(L^i + iK^i\right), \frac{1}{2}\left(L^j + iK^j\right)\right] \\
&= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] + i[L^i, K^j] - [K^i, K^j]\right) \\
&= \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k + \epsilon^{jik}K^k + i\epsilon^{ijk}L^k\right) \\
&= \frac{1}{2}\left(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k\right) \\
&= i\epsilon^{ijk}J_+^k
\end{aligned} \tag{8}
$$

Similarly,

$$[J_-^i, J_-^j] = i\epsilon^{ijk}J_-^k \tag{9}$$

(b) Rewrite $\mathbf{K}$ and $\mathbf{L}$ in terms of $\mathbf{J}_+$ and $\mathbf{J}_-$:

$$\mathbf{L} = \mathbf{J}_+ + \mathbf{J}_- \tag{10}$$

$$i\mathbf{K} = \mathbf{J}_+ - \mathbf{J}_- \tag{11}$$

Rewriting the transformation, we have

$$
\begin{aligned}
\Phi &\rightarrow (1 - i\theta \cdot (\mathbf{J}_+ + \mathbf{J}_-) + \beta \cdot (\mathbf{J}_+ - \mathbf{J}_-))\,\Phi \\
&\rightarrow (1 + (\beta - i\theta) \cdot \mathbf{J}_+ + (-\beta - i\theta) \cdot \mathbf{J}_-)\,\Phi
\end{aligned} \tag{12}
$$

The $\left(\frac{1}{2}, 0\right)$ representation transforms as:

$$\Phi \rightarrow \left(1 + \frac{1}{2}\beta \cdot \sigma - \frac{i}{2}\theta \cdot \sigma\right)\Phi$$

which means that Φ transforms like a right-handed Weyl spinor as indicated in (3.37) in Peskin and Schroeder.
Similarly, for $\left(0, \frac{1}{2}\right)$ representation, the transformation rule is:

$$\Phi \rightarrow \left(1 - \frac{1}{2}\beta \cdot \sigma - \frac{i}{2}\theta \cdot \sigma\right)\Phi$$

which is identical to the left-handed Weyl spinor transformation rule.
(c) The parameterization of the 2x2 matrix is given to be:

$$\begin{pmatrix} V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & V_0 - V_3 \end{pmatrix} = V_\mu \sigma^\mu$$

Now we apply the transformation as described:

$$\begin{aligned}
V_\mu \sigma^\mu &\rightarrow \left(1 - i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2}\right) V_\mu \sigma^\mu \left(1 + i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2}\right) \\
&\rightarrow \left(1 + (\beta - i\theta) \cdot \frac{\sigma}{2}\right) V_\mu \sigma^\mu \left(1 + (\beta + i\theta) \cdot \frac{\sigma}{2}\right) \\
&\rightarrow V_\mu \sigma^\mu + (\beta - i\theta) \cdot \frac{\sigma}{2} V_\mu \sigma^\mu + V_\mu \sigma^\mu (\beta + i\theta) \cdot \frac{\sigma}{2} + (\beta - i\theta) \cdot \frac{\sigma}{2} V_\mu \sigma^\mu (\beta + i\theta) \cdot \frac{\sigma}{2}
\end{aligned}$$

The last term is quadratic in the infinitessimal parameters, and hence should be omitted to be consistent. (The above expansion is based on expanding the exponential to the linear order)

$$\begin{aligned}
\Rightarrow V_\mu \sigma^\mu &\rightarrow V_\mu \sigma^\mu + \frac{V_\mu}{2} (\beta_i - i\theta_i) \sigma^i \sigma^\mu + \frac{V_\mu}{2} (\beta_i + i\theta_i) \sigma^\mu \sigma^i \\
&\rightarrow V_\mu \sigma^\mu + \frac{V_\mu \beta_i}{2} \left\{\sigma^i, \sigma^\mu\right\} - \frac{i V_\mu \theta_i}{2} \left[\sigma^i, \sigma^\mu\right] \\
&\rightarrow V_\mu \sigma^\mu + \frac{V_0 \beta_i}{2} \left\{\sigma^i, I\right\} + \frac{V_j \beta_i}{2} \left\{\sigma^i, \sigma^j\right\} - \frac{i V_0 \theta_i}{2} \left[\sigma^i, I\right] - \frac{i V_j \theta_i}{2} \left[\sigma^i, \sigma^j\right]
\end{aligned}$$

Recall that $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma_k$, then:

$$\begin{aligned}
\left\{\sigma^i, \sigma^j\right\} &= \sigma^i \sigma^j + \sigma^j \sigma^i \\
&= \delta^{ij} + \delta^{ji} \\
&= 2\delta^{ij} \\
\left[\sigma^i, \sigma^j\right] &= \sigma^i \sigma^j - \sigma^j \sigma^i \\
&= 2i\epsilon^{ijk}\sigma_k
\end{aligned}$$

And $[\sigma^i, I] = 0$ and $\{\sigma^i, I\} = 2\sigma^i$. Hence:

$$\begin{aligned}
\Rightarrow V_\mu \sigma^\mu &\rightarrow V_\mu \sigma^\mu + V_0 \beta_i \sigma^i + V_j \beta_i \delta^{ij} + V_j \theta_i \epsilon^{ijk} \sigma_k \\
&\rightarrow V_\mu \sigma^\mu + V_0 \beta_i \sigma^i - V_i \beta^i + V_j \theta_i \epsilon^{ijk} \sigma_k
\end{aligned}$$

Define antisymmetric tensor $\omega_{\alpha\beta}$ such that $\omega_{0i} = \beta_i$ and $\omega_{ij} = \epsilon_{ijk}\theta^k$. Noting that $\epsilon^{ijk} V_j \theta_i \sigma_k = \epsilon^{kij} V_i \sigma_j \theta_k = \epsilon^{ijk} V_i \sigma_j \theta_k$, one can rewrite the above transformation as:

$$\begin{aligned}
V_\mu \sigma^\mu &\rightarrow V_\mu \sigma^\mu + V_0 \omega_{0i} \sigma^i - V^i \omega_{0i} + \omega_{ij} V^i \sigma^j \\
&\rightarrow V_\mu \sigma^\mu + V_0 \omega_{0\mu} \sigma^\mu + \omega_{i0} V^i \sigma^0 + \omega_{ij} V^i \sigma^j \\
&\rightarrow V_\mu \sigma^\mu + V_0 \omega_{0\mu} \sigma^\mu + \omega_{i0} V^i \sigma^0 + \omega_{i\mu} V^i \sigma^\mu - \omega_{i0} V^i \sigma^0
\end{aligned}$$

$$
\begin{aligned}
&\rightarrow \quad V_\mu \sigma^\mu + V^0 \omega_{0\mu} \sigma^\mu + \omega_{i\mu} V^i \sigma^\mu \\
&\rightarrow \quad V_\mu \sigma^\mu + \omega_{\nu\mu} V^\nu \sigma^\mu \\
&\rightarrow \quad \left(\delta^\nu_\mu + \omega^\nu{}_\mu\right) V_\nu \sigma^\mu
\end{aligned}
$$

Claim that this is identical to what is given in 3.19 in Peskin and Schroeder. Eq. 3.19 states that for a four-vector V^μ, the transformation is as follows:

$$
V^\alpha \rightarrow \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\alpha_\beta\right) V^\beta
$$

where $\left(\mathcal{J}^{\mu\nu}\right)_{\alpha\beta}$ is given as:

$$
\left(\mathcal{J}^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)
$$

The proof is as follows:

$$
\begin{aligned}
\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\alpha_\beta &= \delta^\alpha_\beta + \frac{\omega_{\mu\nu}}{2}\left(\delta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta \delta^{\nu\alpha}\right) \\
&= \delta^\alpha_\beta + \frac{1}{2}\left(\omega^\alpha{}_\beta - \omega_\beta{}^\alpha\right) \\
&= \delta^\alpha_\beta + \frac{1}{2}\left(\omega^\alpha{}_\beta - g^{\alpha\gamma}\omega_{\beta\gamma}\right) \\
&= \delta^\alpha_\beta + \frac{1}{2}\left(\omega^\alpha{}_\beta + g^{\alpha\gamma}\omega_{\gamma\beta}\right) \\
&= \delta^\alpha_\beta + \frac{1}{2}\left(\omega^\alpha{}_\beta + \omega^\alpha{}_\beta\right) \\
&= \delta^\alpha_\beta + \omega^\alpha{}_\beta
\end{aligned}
$$