Solution to the problem 3.4

First, I need to show some identities tobe used in the following solutions.
(1) Extention of equation (3.38) From the definition of σ^μ and $\bar{\sigma}^\mu$

$$\sigma^0\sigma^2 = \sigma^2\sigma^0 \tag{1}$$

As shown in (3.38)

$$\sigma^i\sigma^2 = -\sigma^2\left(\sigma^i\right)^* \tag{2}$$

Hence

$$\sigma^\mu\sigma^2 = \sigma^2\left(\bar{\sigma}^\mu\right)^* \tag{3}$$

(2) σ^μ and $\bar{\sigma}^\mu$ contracted with a symmetric object

$$\begin{aligned}
&\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu \\
&= \textstyle\sum_\mu \sigma^\mu\bar{\sigma}^\mu\partial_\mu\partial_\mu + \sum_{\mu<\nu}(\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu + \sigma^\nu\bar{\sigma}^\mu\partial_\nu\partial_\mu) \\
&= \partial_0\partial_0 - (\sigma^1)^2\partial_1\partial_1 - (\sigma^2)^2\partial_2\partial_2 - (\sigma^3)^2\partial_3\partial_3 + \textstyle\sum_{\mu<\nu}(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)\partial_\mu\partial_\nu \\
&= \partial_0\partial_0 - \partial_1\partial_1 - \partial_2\partial_2 - \partial_3\partial_3 + (\sigma^0\bar{\sigma}^j + \sigma^j\bar{\sigma}^0)\partial_0\partial_j + \textstyle\sum_{i<j}(\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i)\partial_i\partial_j \\
&= \partial_\mu\partial^\mu + (-\sigma^j + \sigma^j)\partial_0\partial_j + \textstyle\sum_{i<j}(-\sigma^i\sigma^j - \sigma^j\sigma^i)\partial_i\partial_j \\
&= \partial_\mu\partial^\mu - \textstyle\sum_{i<j}\{\sigma^i, \sigma^j\}\partial_i\partial_j \\
&= \partial_\mu\partial^\mu - \textstyle\sum_{i<j} 2\delta_{ij}\partial_i\partial_j \\
&= \partial_\mu\partial^\mu
\end{aligned} \tag{4}$$

(3) Lorentz transformation of 2-component spinor

$$\Lambda_{\frac{1}{2}}\psi = \begin{pmatrix} \Lambda_{\frac{1}{2}L} & 0 \\ 0 & \Lambda_{\frac{1}{2}R} \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \Lambda_{\frac{1}{2}L}\psi_L \\ \Lambda_{\frac{1}{2}R}\psi_R \end{pmatrix} \tag{5}$$

From (3.37)

$$\Lambda_{\frac{1}{2}L} = 1 - i\theta\cdot\frac{\sigma}{2} - \beta\cdot\frac{\sigma}{2} \tag{6}$$

$$\Lambda_{\frac{1}{2}R} = 1 - i\theta\cdot\frac{\sigma}{2} + \beta\cdot\frac{\sigma}{2} \tag{7}$$

From (3.29)

$$\begin{aligned}
&\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu_\nu\gamma^\nu \\
&\Rightarrow \begin{pmatrix} \Lambda_{\frac{1}{2}L}^{-1} & 0 \\ 0 & \Lambda_{\frac{1}{2}R}^{-1} \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \Lambda_{\frac{1}{2}L} & 0 \\ 0 & \Lambda_{\frac{1}{2}R} \end{pmatrix} = \Lambda^\mu_\nu\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} 0 & \Lambda_{\frac{1}{2}L}^{-1}\sigma^\mu \\ \Lambda_{\frac{1}{2}R}^{-1}\bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} \Lambda_{\frac{1}{2}L} & 0 \\ 0 & \Lambda_{\frac{1}{2}R} \end{pmatrix} = \Lambda^\mu_\nu\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} 0 & \Lambda_{\frac{1}{2}L}^{-1}\sigma^\mu\Lambda_{\frac{1}{2}R} \\ \Lambda_{\frac{1}{2}R}^{-1}\bar{\sigma}^\mu\Lambda_{\frac{1}{2}L} & 0 \end{pmatrix} = \Lambda^\mu_\nu\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix}
\end{aligned} \tag{8}$$

Thus

$$\begin{aligned}\Lambda^{-1}_{\frac{1}{2}L}\sigma^\mu\Lambda_{\frac{1}{2}R} &= \Lambda^\mu_\nu\sigma^\nu \\ \Lambda^{-1}_{\frac{1}{2}R}\bar{\sigma}^\mu\Lambda_{\frac{1}{2}L} &= \Lambda^\mu_\nu\bar{\sigma}^\nu\end{aligned} \tag{9}$$

(a) Note that $\sigma^2\chi^*$ transforms like right-handed spinor

$$\begin{aligned} i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^* \quad \rightarrow \quad & i\bar{\sigma}^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu\Lambda_{\frac{1}{2}L}\chi - im\Lambda_{\frac{1}{2}R}(\sigma^2\chi^*) \\ &= \Lambda_{\frac{1}{2}R}\Lambda^{-1}_{\frac{1}{2}R}\left[i\bar{\sigma}^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu\Lambda_{\frac{1}{2}L}\chi - im\Lambda_{\frac{1}{2}R}(\sigma^2\chi^*)\right] \\ &= \Lambda_{\frac{1}{2}R}\left[i\Lambda^{-1}_{\frac{1}{2}R}\bar{\sigma}^\mu\Lambda_{\frac{1}{2}L}(\Lambda^{-1})^\nu_\mu\partial_\nu\chi - im\Lambda^{-1}_{\frac{1}{2}R}\Lambda_{\frac{1}{2}R}(\sigma^2\chi^*)\right] \\ &= \Lambda_{\frac{1}{2}R}\left[i\Lambda^\mu_\rho\bar{\sigma}^\rho(\Lambda^{-1})^\nu_\mu\partial_\nu\chi - im\sigma^2\chi^*\right] \\ &= \Lambda_{\frac{1}{2}R}\left[i\left(\Lambda\Lambda^{-1}\right)^\nu_\rho\bar{\sigma}^\rho\partial_\nu\chi - im\sigma^2\chi^*\right] \\ &= \Lambda_{\frac{1}{2}R}\left(i\delta^\nu_\rho\bar{\sigma}^\rho\partial_\nu\chi - im\sigma^2\chi^*\right) \\ &= \Lambda_{\frac{1}{2}R}\left(i\bar{\sigma}^\nu\partial_\nu\chi - im\sigma^2\chi^*\right) \\ &= 0 \end{aligned} \tag{10}$$

The second part is

$$\begin{aligned} & i\sigma^\mu\partial_\mu(i\bar{\sigma}^\nu\partial_\nu\chi - im\sigma^2\chi^*) = 0 \\ &\Rightarrow -\sigma^\mu\partial_\mu\bar{\sigma}^\nu\partial_\nu\chi + m\sigma^\mu\partial_\mu\sigma^2\chi^* = 0 \\ &\Rightarrow \sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\chi - m\sigma^\mu\sigma^2\partial_\mu\chi^* = 0 \\ &\Rightarrow \partial^\mu\partial_\mu\chi - m\sigma^2\bar{\sigma}^{\mu*}\partial_\mu\chi^* = 0 \end{aligned} \tag{11}$$

Here, by taking complex conjugate of the given equation

$$\begin{aligned} & i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^* = 0 \\ &\Rightarrow -i\bar{\sigma}^{\mu*}\partial_\mu\chi^* + im\left(\sigma^2\right)^*\chi = 0 \\ &\Rightarrow -i\bar{\sigma}^{\mu*}\partial_\mu\chi^* - im\sigma^2\chi = 0 \\ &\Rightarrow \bar{\sigma}^{\mu*}\partial_\mu\chi^* = -m\sigma^2\chi \end{aligned} \tag{12}$$

Thus

$$\begin{aligned} & \partial^\mu\partial_\mu\chi - m\sigma^2\bar{\sigma}^{\mu*}\partial_\mu\chi^* = 0 \\ &\Rightarrow \partial^\mu\partial_\mu\chi - m\sigma^2(-m\sigma^2\chi) = 0 \\ &\Rightarrow \partial^\mu\partial_\mu\chi + m^2\left(\sigma^2\right)^2\chi = 0 \\ &\Rightarrow \partial^\mu\partial_\mu\chi + m^2\chi = 0 \end{aligned} \tag{13}$$

Before going into (b), I need to show more identities
(4) Complex conjugates and Hermitian conjugates

$$\begin{aligned} & \chi^\dagger = (\chi^*)^T \Rightarrow \left(\chi^\dagger\right)^* = \chi^T \\ & \sigma^{2*} = -\sigma^2 \\ & \sigma^{i\dagger} = \sigma^i \\ & \sigma^{\mu\dagger} = \sigma^\mu, \quad \bar{\sigma}^{\mu\dagger} = \bar{\sigma}^\mu \end{aligned} \tag{14}$$

(b) The given action is

$$\begin{aligned} S &= \int d^4x\left[\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \tfrac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right] \\ &= \int d^4x\left[\chi^*_a i(\bar{\sigma}^\mu)_{ab}\partial_\mu\chi_b + \tfrac{im}{2}\left(\chi_a(\sigma^2)_{ab}\chi_b - \chi^*_a(\sigma^2)_{ab}\chi^*_b\right)\right] \end{aligned} \tag{15}$$

Complex conjugate of the action is, using $(\alpha\beta)^* = \beta^*\alpha^*$

$$S^* = \int d^4x \left\{ \partial_\mu \chi_b^* (-i)(\bar{\sigma}^{\mu *})_{ab}\chi_a - \frac{im}{2}\left[\chi_b^*(\sigma^{2*})_{ab}\chi_a^* - \chi_b(\sigma^{2*})_{ab}\chi_a\right]\right\} \tag{16}$$

Here,

$$\begin{aligned} &-\int d^4x \left[\partial_\mu \chi_b^* i(\bar{\sigma}^{\mu *})_{ab}\chi_a\right] = -\int d^4x \left[i(\bar{\sigma}^{\mu *})_{ab}\partial_\mu(\chi_b^*\chi_a)\right] + \int d^4x \left[i(\bar{\sigma}^{\mu *})_{ab}\chi_b^*\partial_\mu\chi_a\right] \\ &= \int d^4x \left[i(\bar{\sigma}^{\mu *})_{ab}\chi_b^*\partial_\mu\chi_a\right] \end{aligned} \tag{17}$$

Thus

$$\begin{aligned} S^* &= \int d^4x \left\{\chi_b^* i(\bar{\sigma}^{\mu * T})_{ba}\partial_\mu\chi_a - \tfrac{im}{2}\left[\chi_b^*(\sigma^{2*T})_{ba}\chi_a^* - \chi_b(\sigma^{2*T})_{ba}\chi_a\right]\right\} \\ &= \int d^4x \left[\chi^\dagger i\bar{\sigma}^{\mu\dagger}\partial_\mu\chi - \tfrac{im}{2}\left(\chi^\dagger\sigma^{2\dagger}\chi^* - \chi^T\sigma^{2\dagger}\chi\right)\right] \\ &= \int d^4x \left[\chi^\dagger i\bar{\sigma}^{\mu}\partial_\mu\chi + \tfrac{im}{2}\left(\chi^T\sigma^{2}\chi - \chi^\dagger\sigma^{2}\chi^*\right)\right] \end{aligned} \tag{18}$$

Hence $S^* = S$

Next, take variation of S

$$\begin{aligned} \delta S \;=\; &\int d^4x \left\{\delta\chi_a^* i(\bar{\sigma}^\mu)_{ab}\partial_\mu\chi_b + \chi_a^* i(\bar{\sigma}^\mu)_{ab}\partial_\mu\delta\chi_b \right. \\ &\left. + \tfrac{im}{2}\left[\delta\chi_a(\sigma^2)_{ab}\chi_b + \chi_a(\sigma^2)_{ab}\delta\chi_b - \delta\chi_a^*(\sigma^2)_{ab}\chi_b^* - \chi_a^*(\sigma^2)_{ab}\delta\chi_b^*\right]\right\} \end{aligned} \tag{19}$$

Here

$$\begin{aligned} &\int d^4x \chi_a^* i(\bar{\sigma}^\mu)_{ab}\partial_\mu\delta\chi_b = \int d^4x i(\bar{\sigma}^\mu)_{ab}\partial_\mu(\chi_a^*\delta\chi_b) - \int d^4x i(\bar{\sigma}^\mu)_{ab}(\partial_\mu\chi_a^*)\delta\chi_b \\ &= -\int d^4x i(\bar{\sigma}^\mu)_{ab}(\partial_\mu\chi_a^*)\delta\chi_b \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\delta\chi_a^* i(\bar{\sigma}^\mu)_{ab}\partial_\mu\chi_b = -\partial_\mu\chi_b i(\bar{\sigma}^\mu)_{ab}\delta\chi_a^* = -\partial_\mu\chi_a i(\bar{\sigma}^\mu)_{ba}\delta\chi_b^* \\ &\delta\chi_a(\sigma^2)_{ab}\chi_b = -\chi_b(\sigma^2)_{ab}\delta\chi_a = -\chi_a(\sigma^2)_{ba}\delta\chi_b = -\chi_a(\sigma^{2T})_{ab}\delta\chi_b = \chi_a(\sigma^2)_{ab}\delta\chi_b \\ &\delta\chi_a^*(\sigma^2)_{ab}\chi_b^* = -\chi_b^*(\sigma^2)_{ab}\delta\chi_a^* = -\chi_a^*(\sigma^2)_{ba}\delta\chi_b^* = -\chi_a^*(\sigma^{2T})_{ab}\delta\chi_b^* = \chi_a^*(\sigma^2)_{ab}\delta\chi_b^* \end{aligned} \tag{21}$$

Thus

$$\begin{aligned} \delta S \;=\; &\int d^4x \left\{-\partial_\mu\chi_a i(\bar{\sigma}^\mu)_{ba}\delta\chi_b^* - \partial_\mu\chi_a^* i(\bar{\sigma}^\mu)_{ab}\delta\chi_b \right. \\ &\left. + \tfrac{im}{2}\left[\chi_a(\sigma^2)_{ab}\delta\chi_b + \chi_a(\sigma^2)_{ab}\delta\chi_b - \chi_a^*(\sigma^2)_{ab}\delta\chi_b^* - \chi_a^*(\sigma^2)_{ab}\delta\chi_b^*\right]\right\} \\ =\; &\int d^4x \left[-\partial_\mu\chi_a i(\bar{\sigma}^\mu)_{ba}\delta\chi_b^* - im\chi_a^*(\sigma^2)_{ab}\delta\chi_b^* - \partial_\mu\chi_a^* i(\bar{\sigma}^\mu)_{ab}\delta\chi_b + im\chi_a(\sigma^2)_{ab}\delta\chi_b\right] \end{aligned} \tag{22}$$

Here, $(\bar{\sigma}^\mu)_{ba}$ is not a simple object such as $(\bar{\sigma}^{\mu *})_{ab}$, $(\bar{\sigma}^{\mu T})_{ab}$ or $(\bar{\sigma}^{\mu\dagger})_{ab}$. Thus, we need a trick

$$\begin{aligned} &\partial_\mu\chi_a i(\bar{\sigma}^\mu)_{ba}\delta\chi_b^* = -\delta\chi_b^* i(\bar{\sigma}^\mu)_{ba}\partial_\mu\chi_a \\ &\chi_a^*(\sigma^2)_{ab}\delta\chi_b^* = -\delta\chi_b^*(\sigma^2)_{ab}\chi_a^* = -\delta\chi_b^*(\sigma^{2T})_{ba}\chi_a^* = \delta\chi_b^*(\sigma^2)_{ba}\chi_a^* \end{aligned} \tag{23}$$

Hence

$$\begin{aligned} &\delta S = \int d^4x \left[\delta\chi_b^* i(\bar{\sigma}^\mu)_{ba}\partial_\mu\chi_a - im\delta\chi_b^*(\sigma^2)_{ba}\chi_a^* - \partial_\mu\chi_a^* i(\bar{\sigma}^\mu)_{ab}\delta\chi_b + im\chi_a(\sigma^2)_{ab}\delta\chi_b\right] \\ &= \int d^4x \left[\delta\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - im\delta\chi^\dagger\sigma^2\chi^* - \partial_\mu\chi^\dagger i\bar{\sigma}^\mu\delta\chi + im\chi^T\sigma^2\delta\chi\right] \\ &= \int d^4x \left[\delta\chi^\dagger(i\bar{\sigma}^\mu\partial_\mu\chi - im\sigma^2\chi^*) + (-i\partial_\mu\chi^\dagger\bar{\sigma}^\mu + im\chi^T\sigma^2)\delta\chi\right] \end{aligned} \tag{24}$$

Equations of motion are

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0 \tag{25}$$
$$-i\partial\chi^\dagger\cdot\bar{\sigma} + im\chi^T\sigma^2 = 0 \tag{26}$$

(c) Spinor is

$$\begin{aligned}
&\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} \\
&\psi^\dagger = \left(\chi_1^\dagger, -i\chi_2^{*\dagger}\sigma^{2\dagger}\right) = \left(\chi_1^\dagger, -i\chi_2^T\sigma^2\right) \\
&\bar{\psi} = \psi^\dagger\gamma^0 = \left(\chi_1^\dagger, -i\chi_2^T\sigma^2\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(-i\chi_2^T\sigma^2, \chi_1^\dagger\right)
\end{aligned} \tag{27}$$

Lagrangian is

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\
&= \left(-i\chi_2^T\sigma^2, \chi_1^\dagger\right)\left[i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\partial_\mu - m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} \\
&= \left(-i\chi_2^T\sigma^2, \chi_1^\dagger\right)\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix}\begin{pmatrix} \chi_1 \\ i\sigma^2\chi_2^* \end{pmatrix} \\
&= \left(-i\chi_2^T\sigma^2, \chi_1^\dagger\right)\begin{pmatrix} -m\chi_1 - \sigma^\mu\sigma^2\partial_\mu\chi_2^* \\ i\bar{\sigma}^\mu\partial_\mu\chi_1 - im\sigma^2\chi_2^* \end{pmatrix} \\
&= im\chi_2^T\sigma^2\chi_1 - i\chi_2^T\sigma^2\sigma^\mu\sigma^2\partial_\mu\chi_2^* + i\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 - im\chi_1^\dagger\sigma^2\chi_2^*
\end{aligned} \tag{28}$$

Here,

$$\sigma^2\sigma^\mu\sigma^2 = \bar{\sigma}^{\mu*}\sigma^2\sigma^2 = \bar{\sigma}^{\mu*} \tag{29}$$

Thus

$$\mathcal{L} = \chi_1^\dagger i\bar{\sigma}^\mu\partial_\mu\chi_1 + \chi_2^T i\bar{\sigma}^{\mu*}\partial_\mu\chi_2^* + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*) \tag{30}$$

Here,

$$\begin{aligned}
&\int d^4x\chi_2^T i\bar{\sigma}^{\mu*}\partial_\mu\chi_2^* = \int d^4x\chi_{2a}i(\bar{\sigma}^{\mu*})_{ab}\partial_\mu\chi_{2b}^* \\
&= \int d^4xi(\bar{\sigma}^{\mu*})_{ab}\partial_\mu(\chi_{2a}\chi_{2b}^*) - \int d^4xi(\bar{\sigma}^{\mu*})_{ab}(\partial_\mu\chi_{2a})\chi_{2b}^* \\
&= \int d^4x\chi_{2b}^* i(\bar{\sigma}^{\mu*})_{ab}\partial_\mu\chi_{2a} \\
&= \int d^4x\chi_{2b}^* i(\bar{\sigma}^{\mu*T})_{ba}\partial_\mu\chi_{2a} \\
&= \int d^4x\chi_{2b}^* i(\bar{\sigma}^{\mu})_{ba}\partial_\mu\chi_{2a} \\
&= \int d^4x\chi_2^\dagger i\bar{\sigma}^\mu\partial_\mu\chi_2
\end{aligned} \tag{31}$$

Thus

$$\mathcal{L} = \chi_1^\dagger i\bar{\sigma}^\mu\partial_\mu\chi_1 + \chi_2^\dagger i\bar{\sigma}^\mu\partial_\mu\chi_2 + im(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*) \tag{32}$$

We define

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \tag{33}$$

Then, using identities

$$\begin{aligned}
&\chi_1^T\sigma^2\chi_2 = \chi_{1a}(\sigma^2)_{ab}\chi_{2b} = -\chi_{2b}(\sigma^2)_{ab}\chi_{1a} = -\chi_{2b}(\sigma^{2T})_{ba}\chi_{1a} = \chi_{2b}(\sigma^2)_{ba}\chi_{1a} = \chi_2^T\sigma^2\chi_1 \\
&\chi_1^\dagger\sigma^2\chi_2^* = \chi_{1a}^*(\sigma^2)_{ab}\chi_{2b}^* = -\chi_{2b}^*(\sigma^2)_{ab}\chi_{1a}^* = -\chi_{2b}^*(\sigma^{2T})_{ba}\chi_{1a}^* = \chi_{2b}^*(\sigma^2)_{ba}\chi_{1a}^* = \chi_2^\dagger\sigma^2\chi_1^*
\end{aligned} \tag{34}$$

We get

$$\chi^\dagger i\bar{\sigma}\cdot\partial\chi = \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi = (\chi_1^\dagger,\chi_2^\dagger)i\bar{\sigma}^\mu\partial_\mu\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = \chi_1^\dagger i\bar{\sigma}^\mu\partial_\mu\chi_1 + \chi_2^\dagger i\bar{\sigma}^\mu\partial_\mu\chi_2 \tag{35}$$

and

$$\begin{aligned}&\chi^T\sigma^2 M\chi = (\chi_1^T,\chi_2^T)\sigma^2\begin{pmatrix}0 & m\\ m & 0\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = (\chi_1^T,\chi_2^T)\sigma^2\begin{pmatrix}m\chi_2\\ m\chi_1\end{pmatrix}\\ &= \chi_1^T\sigma^2 m\chi_2 + \chi_2^T\sigma^2 m\chi_1 = 2\chi_1^T\sigma^2 m\chi_2\end{aligned} \tag{36}$$

and also

$$\begin{aligned}&\chi^\dagger\sigma^2 M\chi^* = (\chi_1^\dagger,\chi_2^\dagger)\sigma^2\begin{pmatrix}0 & m\\ m & 0\end{pmatrix}\begin{pmatrix}\chi_1^*\\ \chi_2^*\end{pmatrix} = (\chi_1^\dagger,\chi_2^\dagger)\sigma^2\begin{pmatrix}m\chi_2^*\\ m\chi_1^*\end{pmatrix}\\ &= \chi_1^\dagger\sigma^2 m\chi_2^* + \chi_2^\dagger\sigma^2 m\chi_1^* = 2\chi_1^\dagger\sigma^2 m\chi_2^*\end{aligned} \tag{37}$$

Therefore, the Lagrangian is

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{i}{2}(\chi^T\sigma^2 M\chi - \chi^\dagger\sigma^2 M\chi^*) \tag{38}$$

(d) Consider transformation

$$\begin{aligned}&\chi_1 \to e^{i\alpha}\chi_1, \quad \chi_2 \to e^{-i\alpha}\chi_2\\ &\chi_1^T \to \chi_1^T e^{i\alpha}, \quad \chi_2^T \to \chi_2^T e^{-i\alpha}\\ &\chi_1^\dagger \to \chi_1^\dagger e^{-i\alpha}, \quad \chi_2^\dagger \to \chi_2^\dagger e^{i\alpha}\end{aligned} \tag{39}$$

In the case of (c)

$$\begin{aligned}&\mathcal{L} \to\\ &\mathcal{L}' = \chi_1^\dagger e^{-i\alpha} i\bar{\sigma}^\mu\partial_\mu e^{i\alpha}\chi_1 + \chi_2^\dagger e^{i\alpha} i\bar{\sigma}^\mu\partial_\mu e^{-i\alpha}\chi_2 + im(\chi_2^T e^{-i\alpha}\sigma^2 e^{i\alpha}\chi_1 - \chi_1^\dagger e^{-i\alpha}\sigma^2 e^{i\alpha}\chi_2^*)\\ &= \mathcal{L}\end{aligned} \tag{40}$$

To show $\partial_\mu J^\mu$, we need equations of motion. First,

$$i\bar{\sigma}^\mu\partial_\mu\chi - i\sigma^2 M\chi^* = 0 \to i\bar{\sigma}^\mu\partial_\mu\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} - i\sigma^2\begin{pmatrix}0 & m\\ m & 0\end{pmatrix}\begin{pmatrix}\chi_1^*\\ \chi_2^*\end{pmatrix} = 0 \tag{41}$$

This gives

$$\begin{aligned}&i\bar{\sigma}^\mu\partial_\mu\chi_1 - i\sigma^2 m\chi_2^* = 0\\ &i\bar{\sigma}^\mu\partial_\mu\chi_2 - i\sigma^2 m\chi_1^* = 0\end{aligned} \tag{42}$$

Second,

$$-i\partial_\mu\chi^\dagger\bar{\sigma}^\mu + i\chi^T M\sigma^2 = 0 \to -i\partial_\mu(\chi_1^\dagger,\chi_2^\dagger)\bar{\sigma}^\mu + i(\chi_1^T,\chi_2^T)\begin{pmatrix}0 & m\\ m & 0\end{pmatrix}\sigma^2 = 0 \tag{43}$$

This gives

$$\begin{aligned}&-i\partial_\mu\chi_1^\dagger\bar{\sigma}^\mu + i\chi_2^T m\sigma^2 = 0\\ &-i\partial_\mu\chi_2^\dagger\bar{\sigma}^\mu + i\chi_1^T m\sigma^2 = 0\end{aligned} \tag{44}$$

Thus

$$\begin{aligned}
&\partial_\mu J^\mu = \partial_\mu(\chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2) \\
&= (\partial_\mu\chi_1^\dagger)\bar{\sigma}^\mu\chi_1 + \chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 - (\partial_\mu\chi_2^\dagger)\bar{\sigma}^\mu\chi_2 - \chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2 \\
&= \chi_2^T m\sigma^2\chi_1 + \chi_1^\dagger\sigma^2 m\chi_2^* - \chi_1^T m\sigma^2\chi_2 - \chi_2^\dagger\sigma^2 m\chi_1^* \\
&= 0
\end{aligned} \tag{45}$$

Here we used the identity (34).
In case of (b), Lagrangian is

$$\begin{aligned}
&\mathcal{L} = \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \\
&= \chi_1^\dagger i\bar{\sigma}\cdot\partial\chi_1 + \chi_2^\dagger i\bar{\sigma}\cdot\partial\chi_2 + \tfrac{im}{2}(\chi_1^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_1^* + \chi_2^T\sigma^2\chi_2 - \chi_2^\dagger\sigma^2\chi_2^*)
\end{aligned} \tag{46}$$

This is not symmetric under the $U(1)$ phase transformation except for $m = 0$. The current is

$$\partial_\mu J^\mu = \partial_\mu(\chi^\dagger\bar{\sigma}^\mu\chi) = (\partial_\mu\chi^\dagger)\bar{\sigma}^\mu\chi + \chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi = m\chi^T\sigma^2\chi + \chi^\dagger m\sigma^2\chi^* \neq 0 \tag{47}$$

The current is not conserved except for $m = 0$.
The last part of (d): Lagrangian is

$$\mathcal{L} = \chi_a^\dagger i\bar{\sigma}\cdot\partial\chi_a + \frac{im}{2}(\chi_a^T\sigma^2\chi_a - \chi_a^\dagger\sigma^2\chi_a^*) \tag{48}$$

where $a = 1, 2, ..., N$. The rotation is

$$\chi_a \rightarrow \chi_b = R_{ba}\chi_a \tag{49}$$

Since this is a rotation, a length-like quantity xy must be invariant under the transformation

$$xy = x_a y_a \rightarrow x_a R_{ab} R_{ba} y_a = x_a y_a \tag{50}$$

Thus

$$R_{ab}R_{ba} = 1 \;\;\Rightarrow\;\; R^T R = 1 \tag{51}$$

The lagrangian is

$$\begin{aligned}
&\mathcal{L} \rightarrow \mathcal{L}' = \chi^\dagger R^T i\bar{\sigma}\cdot\partial R\chi + \tfrac{im}{2}(\chi^T R^T\sigma^2 R\chi - \chi^\dagger R^T\sigma^2 R\chi^*) \\
&= \chi^\dagger i\bar{\sigma}\cdot\partial\chi + \tfrac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*) \\
&= \mathcal{L}
\end{aligned} \tag{52}$$

Hence, it is invariant under the rotation.

$$\tag{53}$$

$$\tag{54}$$

$$\tag{55}$$