Soluiton to Problem 3.5

(a) The lagrangian is

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}\cdot\partial\chi + F^*F \tag{1}$$

The variation of this Lagrangian is

$$\delta\mathcal{L} = \partial_\mu\delta\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\delta\phi + \delta\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \chi^\dagger i\bar{\sigma}\cdot\partial\delta\chi + (\delta F^*)F + F^*\delta F \tag{2}$$

Here, as given

$$\begin{aligned}&\delta\phi = -i\epsilon^T\sigma^2\chi\\ &\delta\chi = \epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^*\\ &\delta F = -i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi\end{aligned} \tag{3}$$

Thus, using $(\alpha\beta)^* = \beta^*\alpha^*$ for Grassman numbers

$$\begin{aligned}&\delta\phi^* = i(\epsilon_a(\sigma^2)_{ab}\chi_b)^* = i\chi_b^*(\sigma^2)^*_{ab}\epsilon_a^* = i\chi_b^*(\sigma^2)^\dagger_{ba}\epsilon_a^* = i\chi^\dagger\sigma^2\epsilon^*\\ &\delta\chi^\dagger = \epsilon^\dagger F^* + \epsilon^T\sigma^{2\dagger}\partial_\mu\phi^*\sigma^{\mu\dagger} = \epsilon^\dagger F^* + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu\\ &\delta F^* = i\partial_\mu\chi^\dagger\bar{\sigma}^{\mu\dagger}\epsilon = i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon\end{aligned} \tag{4}$$

Hence

$$\begin{aligned}&\delta\mathcal{L} = \partial_\mu(i\chi^\dagger\sigma^2\epsilon^*)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu(-i\epsilon^T\sigma^2\chi) + (\epsilon^\dagger F^* + \epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu)i\bar{\sigma}\cdot\partial\chi\\ &+\chi^\dagger i\bar{\sigma}\cdot\partial(\epsilon F + \sigma\cdot\partial\phi\sigma^2\epsilon^*) + (i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon)F + F^*(-i\epsilon^\dagger\bar{\sigma}\cdot\partial\chi)\end{aligned} \tag{5}$$

As shown in Eq (4) of Solution 3.4

$$\partial_\mu A\sigma^\mu\bar{\sigma}^\nu\partial_\nu B = \partial_\mu A\partial^\mu B = \partial_\mu A\bar{\sigma}^\mu\sigma^\nu\partial_\nu B \tag{6}$$

Hence

$$\begin{aligned}&\epsilon^T\sigma^2\partial_\mu\phi^*\sigma^\mu i\bar{\sigma}^\nu\partial_\nu\chi = \epsilon^T\sigma^2\partial_\mu\phi^* i\partial^\mu\chi\\ &\int d^4x\chi^\dagger i\bar{\sigma}^\mu\partial_\mu(\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) = \int d^4x\partial_\mu(\chi^\dagger i\bar{\sigma}^\mu\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*) - \int d^4x\partial_\mu\chi^\dagger i\bar{\sigma}^\mu\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^*\\ &= -\int d^4x\partial_\mu\chi^\dagger i\bar{\sigma}^\mu\sigma^\nu\partial_\nu\phi\sigma^2\epsilon^* = -\int d^4x\partial_\mu\chi^\dagger i\partial^\mu\phi\sigma^2\epsilon^*\\ &\int d^4x\chi^\dagger i\bar{\sigma}^\mu\epsilon\partial_\mu F = \int d^4x\partial_\mu(\chi^\dagger i\bar{\sigma}^\mu\epsilon F) - \int d^4x\partial_\mu\chi^\dagger i\bar{\sigma}^\mu\epsilon F = -\int d^4x\partial_\mu\chi^\dagger i\bar{\sigma}^\mu\epsilon F\end{aligned} \tag{7}$$

Thus

$$\begin{aligned}&\delta\mathcal{L} = i\partial_\mu\chi^\dagger\sigma^2\epsilon^*\partial^\mu\phi + \partial_\mu\phi^*(-i)\epsilon^T\sigma^2\partial^\mu\chi + \epsilon^T\sigma^2\partial_\mu\phi^* i\partial^\mu\chi - \partial_\mu\chi^\dagger i\partial^\mu\phi\sigma^2\epsilon^*\\ &+F^*\epsilon^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - \partial_\mu\chi^\dagger i\bar{\sigma}^\mu\epsilon F + i\partial_\mu\chi^\dagger\bar{\sigma}^\mu\epsilon F - iF^*\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi\\ &= 0\end{aligned} \tag{8}$$

(b) The additional part is

$$\Delta\mathcal{L} = m\phi F + \frac{1}{2}im\chi^T\sigma^2\chi + \text{c.c.} \tag{9}$$

The variation of this is

$$\delta(\Delta\mathcal{L}) = m\delta\phi F + m\phi\delta F + \frac{1}{2}im(\delta\chi^T\sigma^2\chi + \chi^T\sigma^2\delta\chi) \tag{10}$$

Here

$$\delta\chi^T\sigma^2\chi = \delta\chi_a(\sigma^2)_{ab}\chi_b = -\chi_b(\sigma^2)_{ab}\delta\chi_a = -\chi_b(\sigma^{2T})_{ba}\delta\chi_a = \chi_b(\sigma^2)_{ba}\delta\chi_a = \chi^T\sigma^2\delta\chi \tag{11}$$

Thus

$$\begin{aligned}&\delta(\Delta\mathcal{L}) = m(-i\epsilon^T\sigma^2\chi)F + m\phi(-i\epsilon^T\bar{\sigma}^\mu\partial_\mu\chi) + im\chi^T\sigma^2(\epsilon F + \sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*)\\ &= -im\epsilon^T\sigma^2\chi F - im\phi\epsilon^T\bar{\sigma}^\mu\partial_\mu\chi + im\chi^T\sigma^2\epsilon F + im\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^*\end{aligned} \tag{12}$$

Here

$$\chi^T\sigma^2\epsilon = \chi_a(\sigma^2)_{ab}\epsilon_b = -\epsilon_b(\sigma^2)_{ab}\chi_a = -\epsilon_b(\sigma^{2T})_{ba}\chi_a = \epsilon_b(\sigma^2)_{ba}\chi_a = \epsilon^T\sigma^2\chi \tag{13}$$

and

$$\begin{aligned}&im\chi^T\sigma^2\sigma^\mu\partial_\mu\phi\sigma^2\epsilon^* = im\partial_\mu\phi\chi^T\sigma^2\sigma^\mu\sigma^2\epsilon^* = im\partial_\mu\phi\chi^T\sigma^2\sigma^2\bar{\sigma}^{\mu*}\epsilon^* = im\partial_\mu\phi\chi^T\bar{\sigma}^{\mu*}\epsilon^*\\ &= im\partial_\mu\phi\chi_a(\bar{\sigma}^{\mu*})_{ab}\epsilon^*_b = -im\partial_\mu\phi\epsilon^*_b(\bar{\sigma}^{\mu*})_{ab}\chi_a = -im\partial_\mu\phi\epsilon^*_b(\bar{\sigma}^{\mu\dagger})_{ba}\chi_a\\ &= -im\partial_\mu\phi\epsilon^\dagger\bar{\sigma}^{\mu\dagger}\chi = -im\partial_\mu\phi\epsilon^\dagger\bar{\sigma}^\mu\chi\end{aligned} \tag{14}$$

and also

$$-\int d^4xim\partial_\mu\phi\epsilon^\dagger\bar{\sigma}^\mu\chi = -\int d^4xim\partial_\mu(\phi\epsilon^\dagger\bar{\sigma}^\mu\chi + \int d^4xim\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi = \int d^4xim\phi\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi \tag{15}$$

Threfore

$$\begin{aligned}&\delta(\Delta\mathcal{L}) = -im\epsilon^T\sigma^2\chi F - im\phi\epsilon^T\bar{\sigma}^\mu\partial_\mu\chi + im\epsilon^T\sigma^2\chi F + im\phi\epsilon^T\bar{\sigma}^\mu\partial_\mu\chi\\ &= 0\end{aligned} \tag{16}$$

The full lagrangian is

$$\mathcal{L} = \partial^\mu\phi\partial_\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + F^*F + m\phi F + \frac{1}{2}im\chi^T\sigma^2\chi + m\phi^*F^* - \frac{1}{2}im\chi^\dagger\sigma^2\chi^* \tag{17}$$

Then, field equation (Euler-Lagrangian equation $\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$) is

$$\begin{aligned}&\tfrac{\partial\mathcal{L}}{\partial F^*} = F + m\phi^* = 0 \Rightarrow \quad F = -m\phi^*\\ &\tfrac{\partial\mathcal{L}}{\partial F} = F^* + m\phi = 0 \Rightarrow \quad F^* = -m\phi\end{aligned} \tag{18}$$

Hence

$$\begin{aligned}&\mathcal{L} = \partial^\mu\phi\partial_\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + (-m\phi)(-m\phi^*)\\ &+m\phi(-m\phi^*) + \tfrac{1}{2}im\chi^T\sigma^2\chi + m\phi^*(-m\phi) - \tfrac{1}{2}im\chi^\dagger\sigma^2\chi^*\\ &= \partial^\mu\phi\partial_\mu\phi - m^2\phi^*\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \tfrac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\\ &= (\text{complex K-G boson with mass } m) + (\text{Majorana fermion with mass } m)\end{aligned} \tag{19}$$

(c) The additional Lagramgian is

$$\Delta\mathcal{L} = F_i\frac{\partial W}{\partial\phi_i} + \frac{i}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j \tag{20}$$

The variation of this Lagrangian is

$$\delta(\Delta\mathcal{L}) = \delta F_i \frac{\partial W}{\partial\phi_i} + F_i \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\delta\phi_j + i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\delta\chi_j + \frac{i}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_i^T\sigma^2\chi_j\delta\phi_k \tag{21}$$

where we used

$$\delta\chi_i^T\sigma^2\chi_j = \chi_i^T\sigma^2\delta\chi_j \tag{22}$$

Thus

$$\begin{aligned}&\delta(\Delta\mathcal{L}) = \tfrac{\partial W}{\partial\phi_i}(-i\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi_i) + F_i\tfrac{\partial^2 W}{\partial\phi_i\partial\phi_j}(-i\epsilon^T\sigma^2\chi_j)\\ &+i\tfrac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2(\epsilon F_j + \sigma^\mu\partial_\mu\phi_j\sigma^2\epsilon^*) + \tfrac{i}{2}\tfrac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_i^T\sigma^2\chi_j(-i\epsilon^T\sigma^2\chi_k)\end{aligned} \tag{23}$$

Here

$$\chi_i^T\sigma^2\epsilon = \epsilon^T\sigma^2\chi_i \tag{24}$$

then, second ant third terms of $\delta(\Delta\mathcal{L})$ cancel. And also

$$\chi_i^T\sigma^2\sigma^\mu\partial_\mu\phi_j\sigma^2\epsilon^* = \chi_i^T\sigma^2\sigma^2\bar\sigma^{\mu*}\partial_\mu\phi_j\epsilon^* = \chi_i^T\bar\sigma^{\mu*}\epsilon^*\partial_\mu\phi_j = -\epsilon^\dagger\bar\sigma^\mu\chi_i\partial_\mu\phi_j \tag{25}$$

then

$$\begin{aligned}&\textstyle\int d^4x i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\sigma^\mu\partial_\mu\phi_j\sigma^2\epsilon^* = -\int d^4x i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\epsilon^\dagger\bar\sigma^\mu\chi_i\frac{\partial\phi_j}{\partial x^\mu} = -\int d^4x i\frac{\partial^2 W}{\partial\phi_i\partial x^\mu}\epsilon^\dagger\bar\sigma^\mu\chi_i\\ &\textstyle= -\int d^4x i\left(\partial_\mu\frac{\partial W}{\partial\phi_i}\right)\epsilon^\dagger\bar\sigma^\mu\chi_i = -\int d^4x i\partial_\mu\left(\frac{\partial W}{\partial\phi_i}\epsilon^\dagger\bar\sigma^\mu\chi_i\right) + \int d^4x i\frac{\partial W}{\partial\phi_i}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi_i\\ &\textstyle= \int d^4x i\frac{\partial W}{\partial\phi_i}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi_i\end{aligned} \tag{26}$$

Hence, the first and 4th terms cancell. Thus, we are left with

$$\delta(\Delta\mathcal{L}) = \frac{i}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}\chi_i^T\sigma^2\chi_j(-i\epsilon^T\sigma^2\chi_k) \tag{27}$$

The $\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}$ is aymmetric in (i, j, k). The rest is

$$\begin{aligned}&F_{(i.j,k)} \equiv \chi_i^T\sigma^2\chi_j\epsilon^T\sigma^2\chi_k = -i(\chi_{i1}\chi_{j2} - \chi_{i2}\chi_{j1})i(\epsilon_1\chi_{k2} - \epsilon_2\chi_{k1})\\ &= -\epsilon_1\chi_{i1}\chi_{j2}\chi_{k2} + \epsilon_1\chi_{i2}\chi_{j1}\chi_{k2} + \epsilon_2\chi_{i1}\chi_{j2}\chi_{k1} - \epsilon_2\chi_{i2}\chi_{j1}\chi_{k1}\end{aligned} \tag{28}$$

If you exchange i and j

$$\begin{aligned}&F_{(j,i,k)} = -\epsilon_1\chi_{j1}\chi_{i2}\chi_{k2} + \epsilon_1\chi_{j2}\chi_{i1}\chi_{k2} + \epsilon_2\chi_{j1}\chi_{i2}\chi_{k1} - \epsilon_2\chi_{j2}\chi_{i1}\chi_{k1}\\ &= +\epsilon_1\chi_{i2}\chi_{j1}\chi_{k2} - \epsilon_1\chi_{i1}\chi_{j2}\chi_{k2} - \epsilon_2\chi_{i2}\chi_{j1}\chi_{k1} + \epsilon_2\chi_{i1}\chi_{j2}\chi_{k1}\\ &= F_{(i,j,k)}\end{aligned} \tag{29}$$

If you excahnge j and k

$$\begin{aligned}&F_{(i,k,j)} = -\epsilon_1\chi_{i1}\chi_{k2}\chi_{j2} + \epsilon_1\chi_{i2}\chi_{k1}\chi_{j2} + \epsilon_2\chi_{i1}\chi_{k2}\chi_{j1} - \epsilon_2\chi_{i2}\chi_{k1}\chi_{j1}\\ &= +\epsilon_1\chi_{i1}\chi_{j2}\chi_{k2} - \epsilon_1\chi_{i2}\chi_{j2}\chi_{k1} - \epsilon_2\chi_{i1}\chi_{j1}\chi_{k2} + \epsilon_2\chi_{i2}\chi_{j1}\chi_{k1}\end{aligned} \tag{30}$$

Huum... How can I proceed from here?

Thus, $\delta(\Delta\mathcal{L}) = 0$. Then, $\delta\mathcal{L} = 0$. This means the Lagrangian does not change under supersymmetry transformation. In other words, the Lagrangian is supersymmetric.
For $n = 1$ and $W = \frac{1}{3}g\phi^3$,

$$\frac{\partial W}{\partial \phi} = g\phi^2, \quad \frac{\partial^2 W}{\partial \phi^2} = 2g\phi \tag{31}$$

Thus

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + F^*F + Fg\phi^2 + ig\phi\chi^T\sigma^2\chi + F^*g\phi^{*2} - ig\phi^*\chi^\dagger\sigma^2\chi^* \tag{32}$$

Euler-Lagrange equations of F and F^* give

$$\begin{array}{l} \frac{\partial\mathcal{L}}{\partial F} = 0 \Rightarrow \quad F^* = -g\phi^2 \\ \frac{\partial\mathcal{L}}{\partial F^*} = 0 \Rightarrow \quad F = -g\phi^{*2} \end{array} \tag{33}$$

Hence

$$\begin{array}{l} \mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + (-g\phi^2)(-g\phi^{*2}) \\ +(-g\phi^{*2})g\phi^2 + ig\phi\chi^T\sigma^2\chi + (-g\phi^2)g\phi^{*2} - ig\phi^*\chi^\dagger\sigma^2\chi^* \\ = \partial_\mu\phi^*\partial^\mu\phi - g^2\phi^{*2}\phi^2 + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + ig(\phi\chi^T\sigma^2\chi - \phi^*\chi^\dagger\sigma^2\chi^*) \end{array} \tag{34}$$

Euler-Lagrange equation of ϕ

$$\begin{array}{l} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \\ \Rightarrow \quad \partial_\mu(\partial^\mu\phi^*) + 2g^2\phi^{*2}\phi - ig\chi^T\sigma^2\chi = 0 \\ \Rightarrow \quad \partial^2\phi^* + 2g^2\phi\phi^{*2} - ig\chi^T\sigma^2\chi = 0 \end{array} \tag{35}$$

Euler-Lagrange equation of ϕ^*

$$\begin{array}{l} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} - \frac{\partial\mathcal{L}}{\partial\phi^*} = 0 \\ \Rightarrow \quad \partial_\mu(\partial^\mu\phi) + 2g^2\phi^*\phi^2 + ig\chi^\dagger\sigma^2\chi^* = 0 \\ \Rightarrow \quad \partial^2\phi + 2g^2\phi^*\phi^2 + ig\chi^\dagger\sigma^2\chi^* = 0 \end{array} \tag{36}$$

The χ and χ^* related part of Lagrangin is

$$\mathcal{L} = \chi_a^* i(\bar{\sigma}^\mu)_{ab}\partial_\mu\chi_b + ig(\phi\chi_a(\sigma^2)_{ab}\chi_b - \phi^*\chi_a^*(\sigma^2)_{ab}\chi_b^*) \tag{37}$$

The differentiation of Grassman numbers are tricky.

$$\frac{\partial}{\partial\alpha_i}(\alpha_1\alpha_2) = -\delta_{i1}\alpha_2 + \alpha_1\delta_{i2} \tag{38}$$

Thus, Euler-Lagrange equation of χ_c

$$\begin{array}{l} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_c)} - \frac{\partial\mathcal{L}}{\partial\chi_c} = 0 \\ \Rightarrow \quad \partial_\mu(\chi_a^* i(\bar{\sigma}^\mu)_{ac}) - ig(-\phi(\sigma^2)_{cb}\chi_b + \phi\chi_a(\sigma^2)_{ac}) = 0 \\ \Rightarrow \quad \partial_\mu\chi_a^* i(\bar{\sigma}^\mu)_{ac} - ig(-\phi\chi_b(\sigma^2)_{cb} + \phi\chi_a(\sigma^2)_{ac}) = 0 \\ \Rightarrow \quad \partial_\mu\chi_a^* i(\bar{\sigma}^\mu)_{ac} - ig(-\phi\chi_b(\sigma^{2T})_{bc} + \phi\chi_a(\sigma^2)_{ac}) = 0 \\ \partial_\mu\chi_a^* i(\bar{\sigma}^\mu)_{ac} - ig(\phi\chi_b(\sigma^2)_{bc} + \phi\chi_a(\sigma^2)_{ac}) = 0 \\ \Rightarrow \quad \partial_\mu\chi_a^* i(\bar{\sigma}^\mu)_{ac} - 2ig\phi\chi_a(\sigma^2)_{ac} = 0 \\ \Rightarrow \quad i\partial_\mu\chi^\dagger\bar{\sigma}^\mu - 2ig\phi\chi^T\sigma^2 = 0 \end{array} \tag{39}$$

Similarly for χ^*

$$\begin{array}{l} 0 - i\bar{\sigma}^\mu\partial_\mu\chi + 2ig\phi^*\sigma^2\chi^* = 0 \\ \Rightarrow \quad i\bar{\sigma}^\mu\partial_\mu\chi - 2ig\phi^*\sigma^2\chi^* = 0 \end{array} \tag{40}$$