Peskin and Schroeder: 3.7

(a) Compute the transformation properties under P, C, and T of the antisymmetric tensor fermion bilinears, $\bar{\psi}\sigma^{\mu\nu}\psi$, with $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$.

Solution:

Applying the symmetry operators P,C, and T individually, but keeping in mind that square of each operator is the identity operator.

$$\begin{aligned}
P\bar{\psi}\sigma^{\mu\nu}\psi P &= P\bar{\psi}PP\sigma^{\mu\nu}\psi P \\
&= \eta_a^*\bar{\psi}\gamma^0\sigma^{\mu\nu}P\psi P \\
&= \eta_a^*\bar{\psi}\gamma^0\sigma^{\mu\nu}\eta_a\gamma^0\psi \\
&= \frac{i}{2}\bar{\psi}\gamma^0\left[\gamma^\mu, \gamma^\nu\right]\gamma^0\psi
\end{aligned}$$

Consider the various different cases:

Case (1): when μ or $\nu = 0$ (they cannot both be zero, or the expression will vanish). Then when we try to move γ^0 around the commutator, one minus sign will be introduced (because the γ^0 will commute with the γ^0 in the $\gamma^\nu\gamma^\mu$ while anticommute with the other).

$$\begin{aligned}
P\bar{\psi}\sigma^{\mu\nu}\psi P &= -\frac{i}{2}\bar{\psi}\left[\gamma^\mu, \gamma^\nu\right]\gamma^0\gamma^0\psi \\
&= -\bar{\psi}\sigma^{\mu\nu}\psi
\end{aligned}$$

Case (2): when neither μ nor ν is zero. Then two minus signs will be introduced as the γ^0 is moved around the commutator:

$$P\bar{\psi}\sigma^{\mu\nu}\psi P = \bar{\psi}\sigma^{\mu\nu}\psi$$

In short:

$$P\bar{\psi}\sigma^{\mu\nu}\psi P = (-1)^\mu (-1)^\nu \bar{\psi}\sigma^{\mu\nu}\psi$$

Next for T.

$$\begin{aligned}
T\bar{\psi}\sigma^{\mu\nu}\psi T &= T\bar{\psi}TT\sigma^{\mu\nu}\psi T \\
&= -\frac{i}{2}\bar{\psi}\left(-\gamma^1\gamma^3\right)T\left[\gamma^\mu, \gamma^\nu\right]\psi T \\
&= -\frac{i}{2}\bar{\psi}\left(-\gamma^1\gamma^3\right)\left[(\gamma^\mu)^*, (\gamma^\nu)^*\right]\left(\gamma^1\gamma^3\right)\psi
\end{aligned}$$

Again, split into two cases:
Case (1): Either μ or ν is 2 (can't be both 2), while the other is 0. Then:

$$[(\gamma^\mu)^*, (\gamma^\nu)^*] = -[\gamma^\mu, \gamma^\nu]$$

Furthermore, each of γ^1 and γ^3 will anticommute with the γ matrices in the commutator, introducing four minus signs in total. Hence:

$$\begin{aligned} T\bar{\psi}\sigma^{\mu\nu}\psi T &= \frac{i}{2}\bar{\psi}[\gamma^\mu, \gamma^\nu]\left(-\gamma^1\gamma^3\right)\left(\gamma^1\gamma^3\right)\psi \\ &= \frac{i}{2}\bar{\psi}[\gamma^\mu, \gamma^\nu]\left(\gamma^1\gamma^1\gamma^3\gamma^3\right)\psi \\ &= \bar{\psi}\sigma^{\mu\nu}\psi \end{aligned}$$

Case (2): One of μ and ν is 2, while the other is 1 or 3. Then:

$$[(\gamma^\mu)^*, (\gamma^\nu)^*] = -[\gamma^\mu, \gamma^\nu]$$

Moreover, either one of γ^1 or γ^3 is going to commute with *one* of the γ matrices in the commutator, and so will only introduce one less minus sign as before (therefore contributing three minus signs in total). As a result:

$$\begin{aligned} T\bar{\psi}\sigma^{\mu\nu}\psi T &= -\frac{i}{2}\bar{\psi}[\gamma^\mu, \gamma^\nu]\left(-\gamma^1\gamma^3\right)\left(\gamma^1\gamma^3\right)\psi \\ &= -\frac{i}{2}\bar{\psi}[\gamma^\mu, \gamma^\nu]\left(\gamma^1\gamma^1\gamma^3\gamma^3\right)\psi \\ &= -\bar{\psi}\sigma^{\mu\nu}\psi \end{aligned}$$

Case (3): Neither one of μ or ν is 2, and only one of μ and ν is 1 or 3. Then:

$$[(\gamma^\mu)^*, (\gamma^\nu)^*] = [\gamma^\mu, \gamma^\nu]$$

As we "push" the $\gamma^1\gamma^3$ matrices from the left to the right of the commutator, we will introduce three minus signs in total. Therefore:

$$\begin{aligned} T\bar{\psi}\sigma^{\mu\nu}\psi T &= \frac{i}{2}\bar{\psi}[\gamma^\mu, \gamma^\nu]\left(-\gamma^1\gamma^3\right)\left(\gamma^1\gamma^3\right)\psi \\ &= \frac{i}{2}\bar{\psi}[\gamma^\mu, \gamma^\nu]\left(\gamma^1\gamma^1\gamma^3\gamma^3\right)\psi \\ &= \bar{\psi}\sigma^{\mu\nu}\psi \end{aligned}$$

Case (4): $\mu = 1$ and $\nu = 3$ or vice versa. Their complex conjugates (NOT Hermitian conjugates) will then be themselves since the two γ matrices are real.
Again, bring γ^1 and γ^3 to the right of the commutator. Each of these two will commute with exactly one of the matrices in the commutator, so at the end only two minus signs will be introduced. The end result is thus:

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = \bar{\psi}\sigma^{\mu\nu}\psi$$

In short:

$$T\bar{\psi}\sigma^{\mu\nu}\psi T = -\left(-1\right)^{\mu}\left(-1\right)^{\nu}\bar{\psi}\sigma^{\mu\nu}\psi$$

Last but not least, C:

$$\begin{aligned}
C\bar{\psi}\sigma^{\mu\nu}\psi C &= C\bar{\psi}CC\sigma^{\mu\nu}\psi C \\
&= \left(-i\gamma^0\gamma^2\psi\right)^T\sigma^{\mu\nu}\left(-i\right)\left(\bar{\psi}\gamma^0\gamma^2\right)^T \\
&= -\left(\gamma^0\gamma^2\psi\right)^T\sigma^{\mu\nu}\left(\bar{\psi}\gamma^0\gamma^2\right)^T
\end{aligned}$$

Follow the advice from the textbook, we'll write this expression in indices:

$$\begin{aligned}
C\bar{\psi}\sigma^{\mu\nu}\psi C &= -\left(\gamma^0\gamma^2\psi\right)_a\sigma^{\mu\nu}_{ab}\left(\bar{\psi}\gamma^0\gamma^2\right)_b \\
&= -\frac{i}{2}\gamma^0_{ac}\gamma^2_{cd}\psi_d\left(\gamma^\mu_{ae}\gamma^\nu_{eb}-\gamma^\nu_{ae}\gamma^\mu_{eb}\right)\bar{\psi}_f\gamma^0_{fg}\gamma^2_{gb} \\
&= \frac{i}{2}\bar{\psi}_f\gamma^0_{ac}\gamma^2_{cd}\left(\gamma^\mu_{ae}\gamma^\nu_{eb}-\gamma^\nu_{ae}\gamma^\mu_{eb}\right)\gamma^0_{fg}\gamma^2_{gb}\psi_d \\
&= \frac{i}{2}\bar{\psi}_f\gamma^0_{fg}\gamma^2_{gb}\left(\gamma^\mu_{ae}\gamma^\nu_{eb}-\gamma^\nu_{ae}\gamma^\mu_{eb}\right)\gamma^0_{ac}\gamma^2_{cd}\psi_d \\
&= \frac{i}{2}\bar{\psi}_f\gamma^0_{fg}\gamma^2_{gb}\left(\gamma^\nu_{eb}\gamma^\mu_{ae}-\gamma^\mu_{eb}\gamma^\nu_{ae}\right)\gamma^0_{ac}\gamma^2_{cd}\psi_d
\end{aligned}$$

(The third equality has a minus sign introduced due to anticomutativity of fermions). We consider the following different cases separately:
Case (1): $\mu = 0$ and $\nu = 2$ or vice versa. Then:

$$\left(\gamma^\mu\right)^T = \gamma^\mu$$

And the same applies to γ^ν. Hence:

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = \frac{i}{2}\bar{\psi}_f\gamma^0_{fg}\gamma^2_{gb}\left(\gamma^\nu_{be}\gamma^\mu_{ea}-\gamma^\mu_{be}\gamma^\nu_{ea}\right)\gamma^0_{ac}\gamma^2_{cd}\psi_d$$

$$
\begin{aligned}
&= \frac{i}{2}\bar{\psi}\gamma^0\gamma^2\left[\gamma^\nu,\gamma^\mu\right]\gamma^0\gamma^2\psi \\
&= -\frac{i}{2}\bar{\psi}\gamma^0\gamma^2\left[\gamma^\mu,\gamma^\nu\right]\gamma^0\gamma^2\psi
\end{aligned}
$$

Bringing the first $\gamma^0\gamma^2$ will introduce two minus signs from anticomutative properties of the γ matrices.

$$
\begin{aligned}
C\bar{\psi}\sigma^{\mu\nu}\psi C &= -\frac{i}{2}\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\gamma^0\gamma^2\gamma^0\gamma^2\psi \\
&= +\frac{i}{2}\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\gamma^0\gamma^0\gamma^2\gamma^2\psi \\
&= -\frac{i}{2}\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi \\
&= -\bar{\psi}\sigma^{\mu\nu}\psi
\end{aligned}
$$

Case (2): $\mu = 1$ and $\nu = 3$ or vice versa. Then the γ matrices in the commutator will be antisymmetric. The two minus signs introduced by taking the transpose of the γ matrices will then be cancelled, so up to this step the results are identical to what is given in Case (1).
The next place where minus signs can be introduced is when one tries to bring the $\gamma^0\gamma^2$ around the commutator. Each of γ^0 and γ^2 will anticommute with both γ^μ and γ^ν, introducing four minus signs in total. Again, the sign is then the same as that in Case (1).

Case (3): Neither of the above (that is one of μ or ν is 0 or 2, while the other is 1 or 3). Then one of the γ matrices in the commutator will be antisymmetric, and when we take the transpose, one minus signs will be introduced.
However, when one tries to bring the $\gamma^0\gamma^2$ around the commutator, one of which will anticommute with only one of $\gamma^\mu\gamma^\nu$, while the other anticommute with both of them. Therefore here three minus signs will be introduced.
As a consequence, four minus signs in total will be introduced, and hence the sign here is also the same as that given in Case (1).
In short:

$$
C\bar{\psi}\sigma^{\mu\nu}\psi C = -\bar{\psi}\sigma^{\mu\nu}\psi
$$

(b) Let $\phi(x)$ be a complex-valued Klein-Gordon field. Find unitary operators P, C and an antiunitary operator T (all in terms of their action on the annihilation operators a_p and b_p for the Klein-Gordon particles and antiparticles) that give the following transformation of the Klein-Gordon field:

$$\begin{aligned} P\phi(t,\mathbf{x})P &= \phi(t,-\mathbf{x}); \\ T\phi(t,\mathbf{x})T &= \phi(-t,\mathbf{x}); \\ C\phi(t,\mathbf{x})C &= \phi^*(t,\mathbf{x}); \end{aligned}$$

Find the transformation properties of the components of the current

$$J^\mu = i\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)$$

under P, C and T.

Recall that the field operator ϕ can be written in terms of the creation and annihilation operators (there are two sets of such operators because the field is complex):

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)e^{ip_\mu x^\mu}$$

We're given that:

$$\begin{aligned} P\phi(\mathbf{x},t)P &= \phi(-\mathbf{x},t) \\ \Rightarrow \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}P\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)Pe^{ip_\mu x^\mu} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)e^{ip_\mu \tilde{x}^\mu} \end{aligned}$$

(where $\tilde{x}^\mu = (t,-\mathbf{x})$).

If we change p^μ to $\tilde{p}^\mu$, where $\tilde{p}^\mu = (p^0,-\mathbf{p})$, we discover that $x^\mu p_\mu = \tilde{x}^\mu\tilde{p}_\mu$. To change from p^μ to $\tilde{p}^\mu$, we merely need to make a change of variable from **p** to **-p**, the rest remains the same for the RHS. Hence:

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}P\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)Pe^{ip_\mu x^\mu} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{-\mathbf{p}} + b^\dagger_{\mathbf{p}}\right)e^{ip_\mu x^\mu}$$

So the transformation rules for P are $Pa_{\mathbf{p}}P = a_{-\mathbf{p}}$ and $Pb_{\mathbf{p}}P = b_{-\mathbf{p}}$.

Now for the transformation rule for T. We're given this time that:

$$\begin{aligned} T\phi(\mathbf{x},t)T &= \phi(\mathbf{x},-t) \\ \Rightarrow \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}T\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)e^{ip_\mu x^\mu}T &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)e^{-ip_\mu \tilde{x}^\mu} \\ \Rightarrow \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}T\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)Te^{-ip_\mu x^\mu} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)e^{-ip_\mu \tilde{x}^\mu} \end{aligned}$$

The rest will be very similar to the case where we calculate the transformation of the creation and annihilation operators under P. Hence the transformation rules for T are $Ta_{\mathbf{p}}T = a_{-\mathbf{p}}$ and $Tb_{\mathbf{p}}T = b_{-\mathbf{p}}$. This is expected because the creation and annihilation operators are independent from spatial coordinates. While P flips the direction for both the momentum and space, T flips only the direction of momentum, which is the same thing for the creation and annihilation operators.
However, one important difference between P and T is that P is a *unitary* operator whereas T is an *antiunitary* operator.
Last but not least, the C operator:

$$
\begin{aligned}
C\phi\left(\mathbf{x},t\right)C &= \phi^*\left(\mathbf{x},t\right) \\
\Rightarrow \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}C\left(a_{\mathbf{p}} + b^\dagger_{-\mathbf{p}}\right)e^{ip_\mu x^\mu}C &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(b_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)e^{ip_\mu x^\mu}
\end{aligned}
$$

At this point, we can compare both sides and conclude that the transformation rule is $Ca_{\mathbf{p}}C = b_{\mathbf{p}}$ and $Cb_{\mathbf{p}}C = a_{\mathbf{p}}$.
For the transformation of the charge current:

$$
\begin{aligned}
PJ^\mu P &= iP\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)P \\
&= i\left(\phi^*\left(t,-\mathbf{x}\right)\partial^\mu\phi\left(t,-\mathbf{x}\right) - \partial^\mu\phi^*\left(t,-\mathbf{x}\right)\phi\left(t,-\mathbf{x}\right)\right) \\
&= (-1)^\mu\, i\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)
\end{aligned}
$$

where we employ the shorthand as in the textbook: $(-1)^\mu \equiv -1$ if $\mu = 1,2,3$ and 1 if otherwise.
For T, since it is antiunitary, pushing it through the "i" at the beginning would introduce a minus sign:

$$
\begin{aligned}
TJ^\mu T &= -iT\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)T \\
&= -i\left(\phi^*\left(-t,\mathbf{x}\right)\partial^\mu\phi\left(-t,\mathbf{x}\right) - \partial^\mu\phi^*\left(-t,\mathbf{x}\right)\phi\left(-t,\mathbf{x}\right)\right) \\
&= (-1)^\mu\, i\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)
\end{aligned}
$$

Last but not least, for C:

$$
\begin{aligned}
CJ^\mu C &= iC\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)C \\
&= i\left(\phi\partial^\mu\phi^* - \partial^\mu\phi\phi^*\right) \\
&= -J^\mu
\end{aligned}
$$

With CPT combined, J^μ turns out to be odd.

(c) Show that any Hermitian Lorentz-scalar local operator built from $\psi(x)$, $\phi(x)$ and their conjugates has CPT = +1.