PHYS506B Assignment #1

given 21/01/2005
due 04/02/2005

1 (10pts) Let A be an operator with eigenvalue a when acting on a state. Show that the operator B that satisfies

$$[A, B] = \alpha B$$

will, upon acting on the state, raise the corresponding eigenvalue of A by α.

2 (20pts) Find the value of T^2 for SU(3) in the representations **6**, **8**, **15**, **15′**.

3 (30pts) For SU(3), find the irreducible multiplet decomposition of $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}^*$. Can a colour singlet made of three quarks and one antiquark exist?

4 (20pts) Consider the SU(n) algebra $[T_a, T_b] = i f_{abc} T_c \qquad a, b, c = 1, 2, \ldots, n^2 - 1$

where f_{abc} are real structure constants of the group. Consider the normalization $\mathrm{Tr}[T_a T_b] = \kappa \delta_{ab}$

where κ is real. Show this implies that f_{abc} is totally antisymmetric in all three indices.

5 (20pts) Consider a Lagrangian density $\mathscr{L}$ that is form and scale invariant under a pure (infinitesimal) space-time translation

$$\delta x^\mu(x) = \delta^\mu \quad \text{(constants)} \qquad \delta\varphi_i(x) = 0 \quad \text{(no mixing of fields)}$$

Show that Noether's theorem yields the continuity equation

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{where} \quad T^{\mu\nu} = -\mathscr{L} \cdot g^{\mu\nu} + \sum_{i=1}^{n} \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi_i)} \partial^\nu \varphi_i$$

6 30pts

Consider a Lagrangian density $\mathscr{L}$ that is form and scale invariant under an homogeneous (infinitesimal) Lorentz transformation

$$\delta x^{\mu}\left(x\right)=\varepsilon^{\mu\nu}x_{\nu} \qquad \text{Lorentz rotation} \qquad \varepsilon^{\mu\nu}=-\varepsilon^{\nu\mu}$$

$$\delta\varphi_i\left(x\right)=\varepsilon_{\mu\nu}\tfrac{1}{2}\sum_{j=1}^{n}Z_{ij}^{\mu\nu}\varphi_j\left(x\right) \qquad Z_{ij}^{\mu\nu}=-Z_{ij}^{\nu\mu}$$

Show that Noether's theorem yields the continuity equation

$$\partial_{\mu}\mathscr{M}^{\mu\alpha\beta}=0 \quad \text{where} \quad \mathscr{M}^{\mu\alpha\beta}=-\mathscr{M}^{\mu\beta\alpha}=\left(x^{\alpha}T^{\mu\beta}-x^{\beta}T^{\mu\alpha}\right)+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\varphi_i\right)}Z_{ij}^{\alpha\beta}\varphi_j$$

7 20pts

In the case of a scalar field, we have seen that form and scale invariance of $\mathscr{L}$ under homogeneous Lorentz transformations yield the conserved charges $M^{\alpha\beta}$. We associate

$$\vec{M}=\left(M^{23},M^{31},M^{12}\right)$$

to the conserved angular (in this case orbital only) momentum of the field.

Show that the conserved charges M^{0j} yield the centre of energy theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{X}_{\mathrm{CE}}=\vec{\beta}$$

where $\vec{\beta}$ is the velocity vector of the field, and $\vec{X}_{\mathrm{CE}}$ is the center of energy of the field.