PHYS506B Assignment #2

given 08/02/2005
due 08/03/2005

1 (15pts) Since the action is dimensionless, $[S] = M^0$,

a) Find the dimension of the Klein-Gordon field. That is, $[\varphi] = M^n$, find n.

b) Find the dimension of the Dirac field. That is, $[\psi] = M^n$, find n.

c) Find the dimension of the Maxwell field. That is, $[A^\mu] = M^n$, find n.

2 (10pts) From the Lagrangian density $\mathscr{L}_{\text{KG}} = \left(\partial_\mu \varphi\right)^* \left(\partial^\mu \varphi\right) - m^2 \varphi^* \varphi$

obtain the Klein-Gordon equation $\left(\Box + m^2\right)\varphi = 0$ and $\left(\Box + m^2\right)\varphi^* = 0$

3 (20pts) From the Lagrangian density $\mathscr{L}_{\text{D}} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi$

obtain the Dirac equation $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$ and $i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$

4 (20pts) From the Lagrangian density $\mathscr{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu$

obtain Maxwell's equations $\partial_\mu F^{\mu\nu} = j^\nu$

5 (40pts) From the information given in the notes, obtain the result quoted on page 117 of the notes, U(1) Gauge Invariance, Higgs Model,

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \mu^2 \sigma^2 + \tfrac{1}{2}\left(\partial_\mu \eta\right)\left(\partial^\mu \eta\right) - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \tfrac{1}{2}\left(q\text{v}\right)^2 A^\mu A_\mu + q\text{v}\left(\partial_\mu \eta\right) A^\mu + \mathscr{L}'_{\text{int}}$$

and obtain $\mathscr{L}'_{\text{int}}$ and verify that in the unitary gauge it gives the results quoted on page 119.

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6 15pts Consider the local U(1) transformation $\varphi \to \varphi' = \exp\left(-i\kappa\varepsilon\left(x\right)\right)\varphi$
where $\varepsilon(x)$ is a real function, and κ is a real constant. Defining the covariant derivative and its U(1) transformation with

$$D_\mu \equiv \partial_\mu + iq\kappa A_\mu$$
$$D_\mu\varphi \to D'_\mu\varphi' = \exp\left(-i\kappa\varepsilon\left(x\right)\right)D_\mu\varphi$$
$$D'_\mu \equiv \partial_\mu + iq\kappa A'_\mu$$

obtain the local transformation of the gauge field A_μ .

7 30pts Let $\mathscr{L}$ be a Lagrangian density that includes the complex scalar doublet φ with associated mass m_S

$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$, the Dirac doublet ψ with associated mass m_D $\quad \psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}$

and that is invariant under Poincaré transformations and under local SU(2) gauge transformations.

Let $\mathscr{L}$ be of the form $\quad \mathscr{L} = \mathscr{L}_D + \mathscr{L}_S + \mathscr{L}_A + \mathscr{L}_{D+A} + \mathscr{L}_{S+A}$

where the terms are, respectively, pure Dirac field, pure scalar field, pure gauge field, Dirac and gauge fields interaction, and scalar and gauge fields interaction.

a) Give an expression for $\mathscr{L}$ and for each of the 5 terms above.

b) Give the gauge transformation law for each field.

c) Note that there is no term $\mathscr{L}_{S+D}$. Are any of the following terms valid? Explain why.

c.1) $\bar{\psi}\varphi_1\psi + \bar{\psi}\varphi_2\psi$

c.2) $\bar{\psi}\varphi\psi^1 + \bar{\psi}\varphi\psi^2$

c.3) $\bar{\psi}\varphi + \varphi^\dagger\psi$

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8 Consider the pure SU(n) gauge field Lagrangian density
50pts

$$\mathscr{L}_A = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \qquad F^a_{\mu\nu} = A^a_{\mu\nu} - g f^{abc} A^b_\mu A^c_\nu \qquad A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

which is invariant under the SU(n) gauge transformation

$$A^a_\mu T^a \xrightarrow{\varepsilon^a(x)} A'^a_\mu T^a = U A^a_\mu T^a U^{-1} + \tfrac{1}{g} \partial_\mu \varepsilon^a T^a$$
$$F^a_{\mu\nu} T^a \xrightarrow{\varepsilon^a(x)} F'^a_{\mu\nu} T^a = U F^a_{\mu\nu} T^a U^{-1}$$
$$U = \exp\left(-i\varepsilon^a(x) T^a\right)$$
$$\left[T^a, T^b\right] = i f^{abc} T^c$$

a) Use the Euler-Lagrange equations to show that the equations of motion of the fields A_μ^a are given by

$$\partial_\mu F^{a\mu\nu} - g f^{abc} A^b_\mu F^{c\mu\nu} = 0$$

b) We have seen that the gauge current $j_A^{a\mu}$ can be obtained from $\partial_\mu F^{a\mu\nu} \equiv j^{a\nu}_A \qquad \partial_\nu j^{a\nu}_A = 0$

This yields
$$j^{a\nu}_A = g f^{abc} A^b_\mu F^{c\mu\nu}$$

Obtain this current, up to a multiplicative factor, by using Noether's theorem for the invariance of $\mathscr{L}_A$ under the global SU(n) transformation

$$A^a_\mu T^a \xrightarrow{\varepsilon^a} A'^a_\mu T^a = U A^a_\mu T^a U^{-1}$$
$$U = \exp\left(-i\varepsilon^a T^a\right)$$
$$\left[T^a, T^b\right] = i f^{abc} T^c$$

where the ε^a are real constants.

Hint: first find δA_μ^a for the infinitesimal ε^a and then apply Noether's theorem.