PHYS506B Assignment #1 SOLUTIONS

given 21/01/2005
due 04/02/2005

1 Let A be an operator with eigenvalue a when acting on a state. Show that the operator B that
10pts satisfies

$$\left[A,B\right]=\alpha B$$

will, upon acting on the state, raise the corresponding eigenvalue of A by α.

2 Find the value of T^2 for SU(3) in the representations **6**, **8**, **15**, **15′**.
20pts

3 For SU(3), find the irreducible multiplet decomposition of $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}^*$. Can a colour singlet
30pts made of three quarks and one antiquark exist?

4 Consider the SU(n) algebra $\left[T_a,T_b\right]=i\,f_{abc}T_c \qquad a,b,c=1,2,\ldots,n^2-1$
20pts
where f_{abc} are real structure constants of the group. Consider the normalization $\mathrm{Tr}\left[T_aT_b\right]=\kappa\delta_{ab}$
where κ is real. Show this implies that f_{abc} is totally antisymmetric in all three indices.

5 Consider a Lagrangian density $\mathscr{L}$ that is form and scale invariant under a pure (infinitesimal)
20pts space-time translation

$$\delta x^\mu\left(x\right)=\delta^\mu \quad \text{(constants)} \qquad\qquad \delta\varphi_i\left(x\right)=0 \quad \text{(no mixing of fields)}$$

Show that Noether's theorem yields the continuity equation

$$\partial_\mu T^{\mu\nu}=0 \quad \text{where} \quad T^{\mu\nu}=-\mathscr{L}\cdot g^{\mu\nu}+\sum_{i=1}^{n}\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\varphi_i\right)}\partial^\nu\varphi_i$$

PHYS506B Assignment #1

6
30pts

Consider a Lagrangian density $\mathscr{L}$ that is form and scale invariant under an homogeneous (infinitesimal) Lorentz transformation

$$\delta x^{\mu}(x) = \varepsilon^{\mu\nu} x_{\nu} \qquad \text{Lorentz rotation} \qquad \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$$

$$\delta\varphi_i(x) = \varepsilon_{\mu\nu} \tfrac{1}{2} \sum_{j=1}^{n} Z_{ij}^{\mu\nu} \varphi_j(x) \qquad Z_{ij}^{\mu\nu} = -Z_{ij}^{\nu\mu}$$

Show that Noether's theorem yields the continuity equation

$$\partial_{\mu} \mathscr{M}^{\mu\alpha\beta} = 0 \quad \text{where} \quad \mathscr{M}^{\mu\alpha\beta} = -\mathscr{M}^{\mu\beta\alpha} = \left(x^{\alpha} T^{\mu\beta} - x^{\beta} T^{\mu\alpha} \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\varphi_i\right)} Z_{ij}^{\alpha\beta} \varphi_j$$

7
20pts

In the case of a scalar field, we have seen that form and scale invariance of $\mathscr{L}$ under homogeneous Lorentz transformations yield the conserved charges $M^{\alpha\beta}$. We associate

$$\vec{M} = \left(M^{23}, M^{31}, M^{12} \right)$$

to the conserved angular (in this case orbital only) momentum of the field.

Show that the conserved charges M^{0j} yield the centre of energy theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{X}_{\mathrm{CE}} = \vec{\beta}$$

where $\vec{\beta}$ is the velocity vector of the field, and $\vec{X}_{\mathrm{CE}}$ is the center of energy of the field.

Question 1

We are given

$$A|a\rangle = a|a\rangle \quad \text{and} \quad [A,B] = \alpha B$$

Therefore

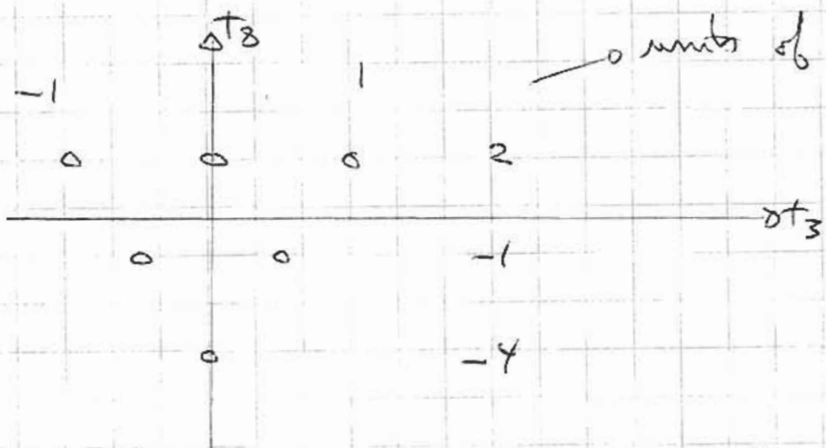
$$AB|a\rangle = \left([A,B] + BA\right)|a\rangle$$
$$= (\alpha B + BA)|a\rangle$$
$$= (\alpha + a) B|a\rangle$$

So $$A\left(B|a\rangle\right) = (\alpha+a)\left(B|a\rangle\right)$$

Therefore $$B|a\rangle \propto |a+\alpha\rangle$$

Question 2

6 $\quad p = 2 \qquad q = 0$



→ units of $\frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$

Check of centroid on t_8 : $\quad 3(2) + 2(-1) + (-4) = 0$

So

$$T^2 = \frac{8}{6}\left[2(1)^2 + 2(1/2)^2 + 2(0)^2\right]I = \frac{8}{6}\left[2 + \frac{1}{2}\right]I$$

$$= \frac{8}{6}\,\frac{5}{2}\,I = \frac{10}{3}\,I$$

or

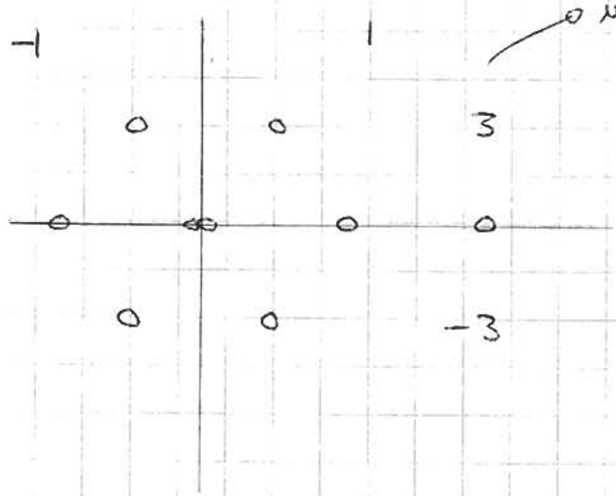
$$T^2 = \frac{8}{6}\left[3\left(\frac{1}{\sqrt{3}}\right)^2 + 2\left(\frac{1}{2\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2\right]I$$

$$= \frac{8}{6}\left[1 + \frac{1}{6} + \frac{4}{3}\right]I$$

$$= \frac{8}{6}\,\frac{15}{6}\,I = \frac{10}{3}\,I$$

8 $\quad p = q = 1$

units of $\frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$

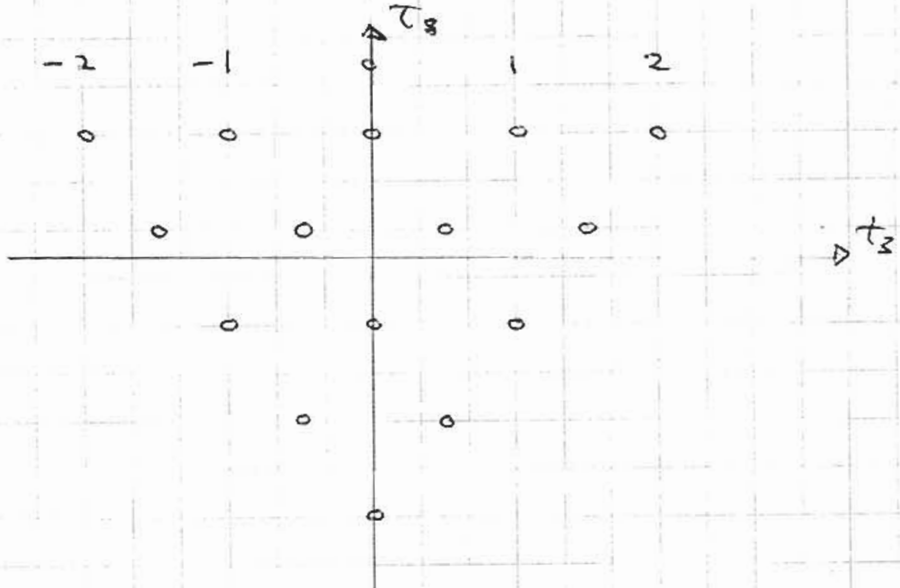


$$T^2 = \frac{8}{8}\left[2(1)^2 + 4\left(\tfrac{1}{2}\right)^2 + 2(0)^2 \right] I = (2+1) I = 3 I$$

or

$$T^2 = \frac{8}{8}\left[4\left(\frac{3}{2\sqrt{3}}\right)^2 + 4(0)^2 \right] I = 4 \cdot \frac{3}{4} I = 3 I$$

15 $\quad p = 4 \quad q = 0$



$$\frac{4\sqrt{3}}{6} = \frac{2}{\sqrt{3}}$$

$$\frac{\sqrt{3}}{6} = \frac{1}{2}\frac{1}{\sqrt{3}}$$

$$-\frac{2\sqrt{3}}{6} = -\frac{1}{\sqrt{3}}$$

$$-\frac{5\sqrt{3}}{6} = -\frac{5}{2}\frac{1}{\sqrt{3}}$$

$$-\frac{8\sqrt{3}}{6} = -\frac{4}{\sqrt{3}}$$

Check of centroid in t_8 (units of $\frac{\sqrt{3}}{6}$)

$$5(4) + 4(1) + 3(-2) + 2(-5) + 1(-8) = 0$$

So

$$T^2 = \frac{8}{15}\left[2(2)^2 + 2(3/2)^2 + 4(1)^2 + 4(1/2)^2 + 3(0)^2\right] I$$

$$= \frac{8}{15}\left[8 + \frac{9}{2} + 4 + 1\right] I = \frac{8}{15}\frac{35}{2} I = \frac{28}{3} I$$

or

$$T^2 = \frac{8}{15}\left[5\left(\frac{2}{\sqrt{3}}\right)^2 + 4\left(\frac{1}{2\sqrt{3}}\right)^2 + 3\left(\frac{1}{\sqrt{3}}\right)^2 + 2\left(\frac{5}{2\sqrt{3}}\right)^2 + \left(\frac{4}{\sqrt{3}}\right)^2\right] I$$

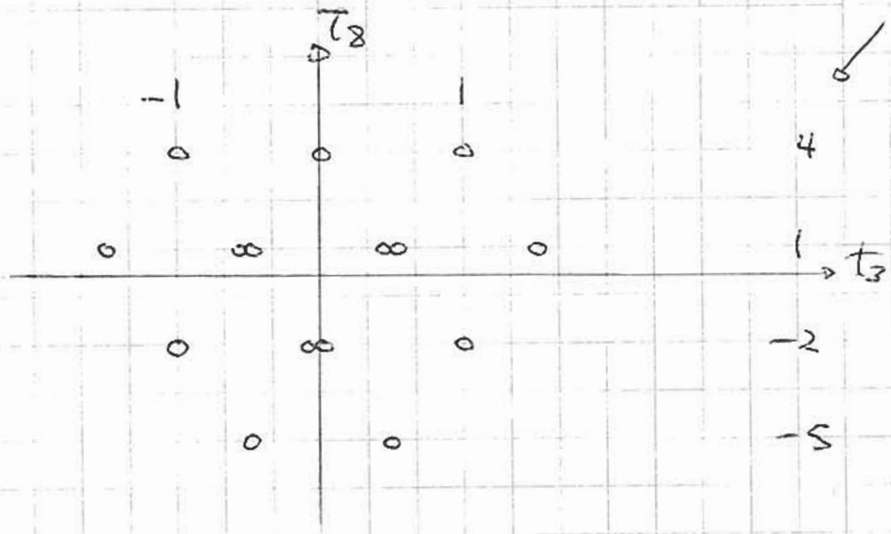
$$= \frac{8}{15}\left[\frac{5 \cdot 4}{3} + \frac{1}{3} + 1 + \frac{25}{6} + \frac{16}{3}\right] I$$

$$= \frac{8}{15}\frac{1}{6}\left(40 + 2 + 6 + 25 + 32\right) I$$

$$= \frac{8}{15}\frac{35}{2} I = \frac{28}{3} I$$

$15'$ $\quad p = 2 \quad q = 1$

units of $\frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$



Check of centroid in T_8 (units of $\frac{\sqrt{3}}{6} = \frac{1}{2\sqrt{3}}$)

$$3(4) + 6(1) + 4(-2) + 2(-5) = 0$$

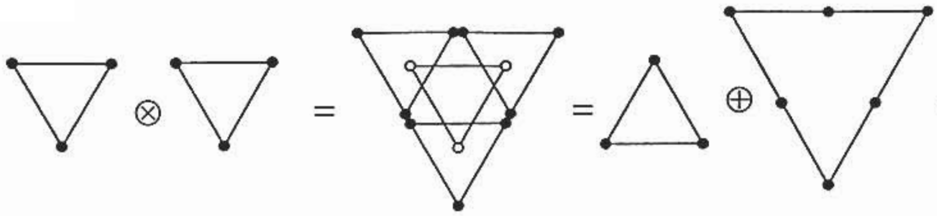
So $$T^2 = \frac{8}{15}\left[2\left(\tfrac{3}{2}\right)^2 + 4(1)^2 + 6\left(\tfrac{1}{2}\right)^2 + 3(0)^2\right] I$$
$$= \frac{8}{15}\left[\frac{9}{2} + 4 + \frac{3}{2} + 0\right] I = \frac{8}{15}\,\frac{20}{2} I = \frac{16}{3} I$$

or

$$T^2 = \frac{8}{15}\left[3\left(\frac{2}{\sqrt{3}}\right)^2 + 6\left(\frac{1}{2\sqrt{3}}\right)^2 + 4\left(\frac{1}{\sqrt{3}}\right)^2 + 2\left(\frac{5}{2\sqrt{3}}\right)^2\right] I$$
$$= \frac{8}{15}\left[4 + \frac{1}{2} + \frac{4}{3} + \frac{25}{6}\right] I = \frac{8}{15}\cdot 10 = \frac{16}{3} I$$

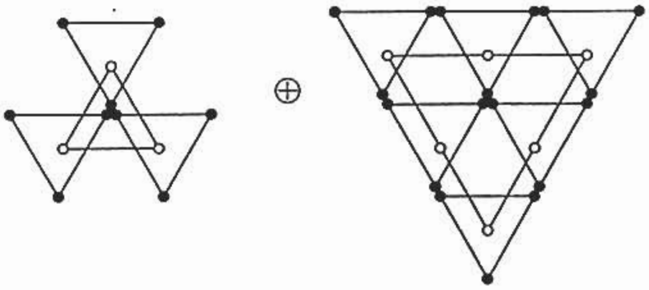
Question 3

$3 \otimes 3 =$

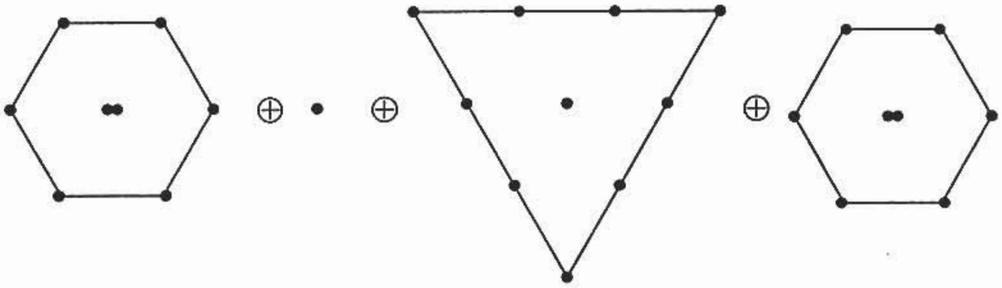


$= 3^* \oplus 6$

$3 \otimes 3 \otimes 3 = (3^* \oplus 6) \otimes 3 =$



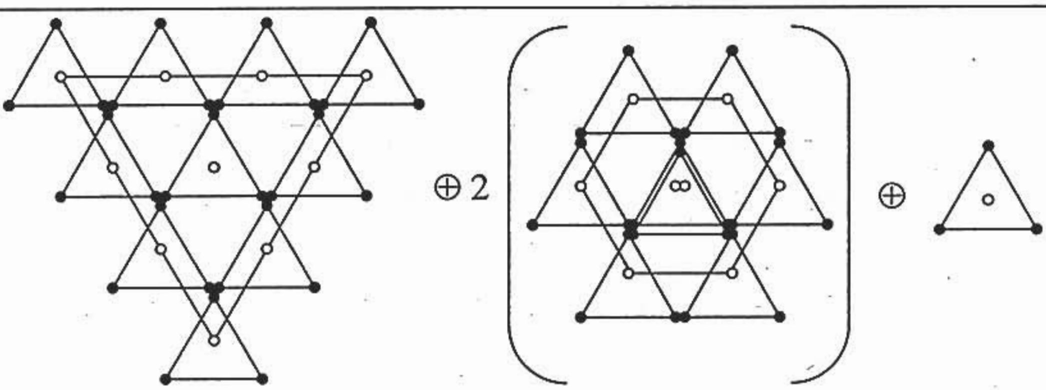
$=$



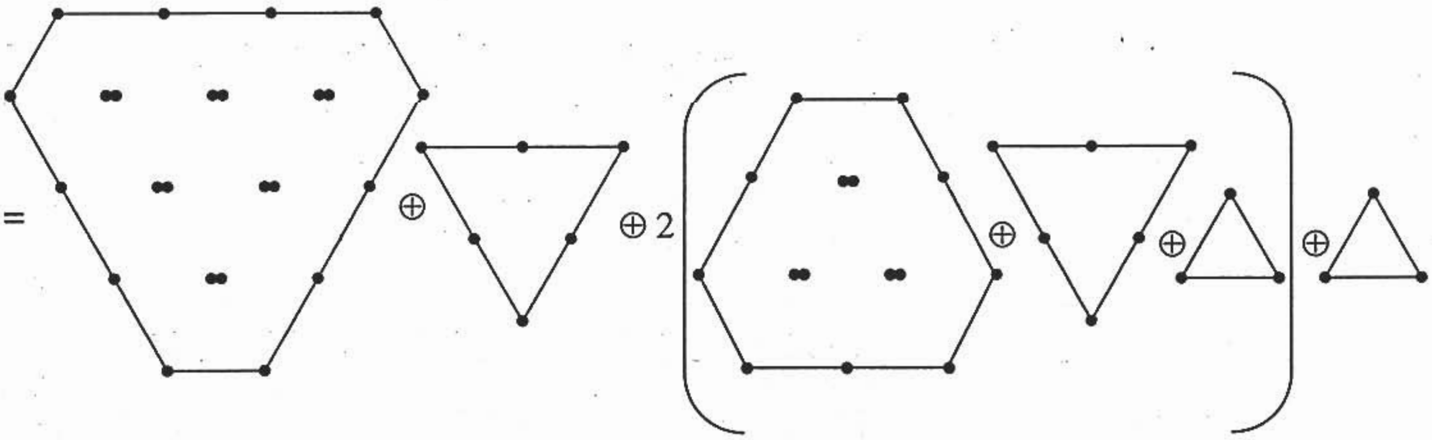
$= 10 \oplus 8 \oplus 8 \oplus 1$

$3 \otimes 3 \otimes 3 \otimes 3^* =$

$(10 \oplus 8 \oplus 8 \oplus 1) \otimes 3^* =$



$=$



$3 \otimes 3 \otimes 3 \otimes 3^* = 24 \oplus 15'^* \oplus 15'^* \oplus 6 \oplus 6 \oplus 6 \oplus 3^* \oplus 3^* \oplus 3^*$

No colour singlet exists for $qqq\bar{q}$.

Question 4

Consider $[T_a, T_b] = i f_{abc} T_c \longrightarrow f_{abc} = -f_{bac}$
and the normalization

$$\mathrm{Tr}[T_a T_b] = \kappa \delta_{ab}$$

Then

$$\mathrm{Tr}\{[T_a, T_b] T_m\} = i f_{abc} \mathrm{Tr}\{T_c T_m\} = i f_{abm} \kappa.$$

Therefore

$$f_{abm} = -\frac{i}{\kappa} \mathrm{Tr}\{[T_a, T_b] T_m\}$$

But, using the cyclic property of traces, we find

$$\begin{aligned}
\mathrm{Tr}\{[T_a, T_b] T_m\} &= \mathrm{Tr}\{T_a T_b T_m\} - \mathrm{Tr}\{T_b T_a T_m\} \\
&= \mathrm{Tr}\{T_b T_m T_a\} - \mathrm{Tr}\{T_m T_b T_a\} \\
&= \mathrm{Tr}\{[T_b, T_m] T_a\}
\end{aligned}$$

Therefore, using the fact that $f_{abc} = -f_{bac}$, we obtain

$$\begin{aligned}
\mathrm{Tr}\{[T_a, T_b] T_m\} &= -\mathrm{Tr}\{[T_m, T_b] T_a\} \\
&= +\mathrm{Tr}\{[T_m, T_a], T_b\} \\
&= -\mathrm{Tr}\{[T_a, T_m], T_b\}
\end{aligned}$$

and we see that f_{abc} is antisymmetric on all 3 indices

$$\begin{aligned}
\mathrm{Tr}\{[T_a, T_b] T_m\} &= -\mathrm{Tr}\{[T_m, T_b] T_a\} \\
&= -\mathrm{Tr}\{[T_a, T_m] T_b\} \\
&= -\mathrm{Tr}\{[T_b, T_a] T_m\}.
\end{aligned}$$

or

$$\begin{aligned}
f_{abm} &= -f_{mba} \\
&= -f_{amb} \\
&= -f_{bam}
\end{aligned}$$

Question 5

Consider $\mathcal{L}$ that is form and scale invariant under a pure space-time translation

$$\delta x^\mu(x) = \delta^\mu \qquad \text{constants}$$
$$\delta \varphi_i(x) = 0 \qquad \text{no mixing of fields}$$

Noether's theorem yields

$$\partial_\mu f^\mu(x) = 0$$

where
$$f^\mu(x) = \mathcal{L}\,\delta x^\mu + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\left[\delta\varphi_i - \partial_\nu \varphi_i\,\delta x^\nu\right]$$

So
$$f^\mu(x) = \mathcal{L}\,\delta^\mu + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\left[-\partial_\nu \varphi_i\,\delta^\nu\right]$$
$$= -\delta_\nu \left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\partial^\nu \varphi_i - \mathcal{L} g^{\mu\nu}\right]$$

Let
$$T^{\mu\nu} \equiv -\mathcal{L} g^{\mu\nu} + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)}\partial^\nu \varphi_i$$

Then
$$\partial_\mu f^\mu(x) = -\partial_\mu\left(T^{\mu\nu}\delta_\nu\right) = \delta_\nu \partial_\mu T^{\mu\nu} = 0$$

Since the δ_ν are arbitrary, then

$$\partial_\mu T^{\mu\nu} = 0$$

Question 6

Consider $\mathcal{L}$ that is form and scale invariant under an homogeneous (infinitesimal) Lorentz transformation

$$\delta x^\mu(x) = \varepsilon^{\mu\nu} x_\nu \qquad \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$$

$$\delta \varphi_i(x) = \varepsilon_{\mu\nu} \frac{1}{2} \sum_j Z_{ij}^{\mu\nu} \varphi_j(x) \qquad Z_{ij}^{\mu\nu} = -Z_{ij}^{\nu\mu}$$

Noether's Theorem yields

$$\partial_\mu j^\mu(x) = 0$$

where
$$j^\mu(x) = \mathcal{L}\,\delta x^\mu + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \left[\delta\varphi_i - \partial_\nu \varphi_i \delta x^\nu \right]$$

So
$$j^\mu(x) = \mathcal{L}\,\varepsilon^{\mu\alpha} x_\alpha + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \left[\varepsilon_{\alpha\beta} \frac{1}{2} \sum_j Z_{ij}^{\alpha\beta} \varphi_j - \partial_\nu \varphi_i \varepsilon^{\nu\alpha} x_\alpha \right]$$

$$= -\left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial_\nu \varphi_i \varepsilon^{\nu\alpha} x_\alpha - \mathcal{L}\,\varepsilon^{\mu\alpha} x_\alpha \right]$$

$$+ \sum_i \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \varepsilon_{\alpha\beta} \frac{1}{2} Z_{ij}^{\alpha\beta} \varphi_j$$

$$= -\varepsilon^{\alpha\beta} \left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial_\alpha \varphi_i - \mathcal{L}\,\delta^\mu_\alpha \right] x_\beta$$

$$+ \varepsilon_{\alpha\beta} \sum_i \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \frac{1}{2} Z_{ij}^{\alpha\beta} \varphi_j$$

$$= -\varepsilon_{\alpha\beta} \left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial^\alpha \varphi_i - \mathcal{L}\,g^{\mu\alpha} \right] x^\beta$$

$$+ \varepsilon_{\alpha\beta} \sum_i \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \frac{1}{2} Z_{ij}^{\alpha\beta} \varphi_j$$

$$= -\varepsilon_{\alpha\beta} T^{\mu\alpha} x^\beta + \varepsilon_{\alpha\beta} \sum_i \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \frac{1}{2} Z_{ij}^{\alpha\beta} \varphi_j$$

Since $\varepsilon_{\alpha\beta}$ is antisymmetric, we can write

$$J^{\mu}(x) = \frac{1}{2}\varepsilon_{\alpha\beta}\left[\sum_{i,j}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} Z_{ij}^{\alpha\beta}\varphi_j + \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}\right)\right]$$

Now consider

$$\mathcal{M}^{\mu\alpha\beta} = \sum_{i,j}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} Z_{ij}^{\alpha\beta}\varphi_j + \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}\right)$$

Since the $\varepsilon_{\alpha\beta}$ are arbitrary, we have

$$\partial_\mu \mathcal{M}^{\mu\alpha\beta} = 0 \qquad \mathcal{M}^{\mu\alpha\beta} = -\mathcal{M}^{\mu\beta\alpha}$$

from $\partial^\mu J_\mu(x) = \partial_\mu \frac{1}{2}\varepsilon_{\alpha\beta}\mathcal{M}^{\mu\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta}\partial_\mu \mathcal{M}^{\mu\alpha\beta} = 0$

Question 7

For a scalar field,

$$M^{ij} = \int dV \left\{ x^i T^{0j} - x^j T^{0i} \right\}$$

$$M^{0j} = \int dV \left\{ x^0 T^{0j} - x^j T^{00} \right\}, \quad \text{but } x^0 = \tau$$

$$= \tau \int dV\, T^{0j} - \int dV\, x^j T^{00}, \quad \text{but } \int dV\, T^{0j} = P^j$$

$$= \tau P^j - \int dV\, x^j T^{00}$$

We know that $\frac{dM^{0j}}{d\tau} = 0$ and $\frac{dP^j}{d\tau} = 0$

Therefore $\frac{dM^{0j}}{d\tau} = P^j - \frac{d}{d\tau} \int dV\, x^j T^{00} = 0$

We also know that $E = P^0 = \int dV\, T^{00}$ and $\frac{dE}{d\tau} = 0$

Therefore $$\frac{d}{d\tau} \left(\frac{\int dV\, x^j T^{00}}{\int dV\, T^{00}} \right) = \frac{P^j}{\int dV\, T^{00}}$$

Also $$\frac{P^j}{\int dV\, T^{00}} = \frac{P^j}{E} = \beta^j$$

Let $$X^j_{CE} \equiv \frac{\int dV\, x^j T^{00}}{\int dV\, T^{00}} = \text{centre of energy}$$

Then we obtain

$$\frac{d}{dT} \vec{X}_{CE} = \vec{\beta}$$