# PHYS506B Assignment #2 SOLUTIONS

**given 08/02/2005**
**due 08/03/2005**

**1** 15pts

Since the action is dimensionless, $[S] = M^0$,

a) Find the dimension of the Klein-Gordon field. That is, $[\varphi] = M^n$, find $n$.

b) Find the dimension of the Dirac field. That is, $[\psi] = M^n$, find $n$.

c) Find the dimension of the Maxwell field. That is, $[A^\mu] = M^n$, find $n$.

**2** 10pts

From the Lagrangian density $\mathscr{L}_{\text{KG}} = \left(\partial_\mu \varphi\right)^* \left(\partial^\mu \varphi\right) - m^2 \varphi^* \varphi$

obtain the Klein-Gordon equation $\left(\Box + m^2\right)\varphi = 0$ and $\left(\Box + m^2\right)\varphi^* = 0$

**3** 20pts

From the Lagrangian density $\mathscr{L}_{\text{D}} = \bar{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi$

obtain the Dirac equation $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$ and $i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$

**4** 20pts

From the Lagrangian density $\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu$

obtain Maxwell's equations $\partial_\mu F^{\mu\nu} = j^\nu$

**5** 40pts

From the information given in the notes, obtain the result quoted on page 117 of the notes, U(1) Gauge Invariance, Higgs Model,

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \mu^2\sigma^2 + \tfrac{1}{2}\left(\partial_\mu \eta\right)\left(\partial^\mu \eta\right) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\left(q\text{v}\right)^2 A^\mu A_\mu + q\text{v}\left(\partial_\mu \eta\right)A^\mu + \mathscr{L}'_{\text{int}}$$

and obtain $\mathscr{L}'_{\text{int}}$ and verify that in the unitary gauge it gives the results quoted on page 119.

# PHYS506B Assignment #2

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**6** Consider the local U(1) transformation $\varphi \to \varphi' = \exp\left(-i\kappa\varepsilon\left(x\right)\right)\varphi$

15pts where $\varepsilon(x)$ is a real function, and $\kappa$ is a real constant. Defining the covariant derivative and its U(1) transformation with

$$D_\mu \equiv \partial_\mu + iq\kappa A_\mu$$
$$D_\mu\varphi \to D'_\mu\varphi' = \exp\left(-i\kappa\varepsilon\left(x\right)\right)D_\mu\varphi$$
$$D'_\mu \equiv \partial_\mu + iq\kappa A'_\mu$$

obtain the local transformation of the gauge field $A_\mu$ .

**7** Let $\mathscr{L}$ be a Lagrangian density that includes the complex scalar doublet $\varphi$ with associated mass $m_S$

30pts $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ , the Dirac doublet $\psi$ with associated mass $m_D$ $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$

and that is invariant under Poincaré transformations and under local SU(2) gauge transformations.

Let $\mathscr{L}$ be of the form $\mathscr{L} = \mathscr{L}_D + \mathscr{L}_S + \mathscr{L}_A + \mathscr{L}_{D+A} + \mathscr{L}_{S+A}$

where the terms are, respectively, pure Dirac field, pure scalar field, pure gauge field, Dirac and gauge fields interaction, and scalar and gauge fields interaction.

a) Give an expression for $\mathscr{L}$ and for each of the 5 terms above.

b) Give the gauge transformation law for each field.

c) Note that there is no term $\mathscr{L}_{S+D}$. Are any of the following terms valid? Explain why.

c.1) $\bar{\psi}\varphi_1\psi + \bar{\psi}\varphi_2\psi$

c.2) $\bar{\psi}\varphi\psi^1 + \bar{\psi}\varphi\psi^2$

c.3) $\bar{\psi}\varphi + \varphi^\dagger\psi$

**8** Consider the pure SU($n$) gauge field Lagrangian density

50pts

$$\mathscr{L}_{\mathrm{A}} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \qquad F^a_{\mu\nu} = A^a_{\mu\nu} - g f^{abc} A^b_\mu A^c_\nu \qquad A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

which is invariant under the SU($n$) gauge transformation

$$\begin{aligned} &A^a_\mu T^a \xrightarrow{\varepsilon^a(x)} A'^a_\mu T^a = U A^a_\mu T^a U^{-1} + \tfrac{1}{g} \partial_\mu \varepsilon^a T^a \\ &F^a_{\mu\nu} T^a \xrightarrow{\varepsilon^a(x)} F'^a_{\mu\nu} T^a = U F^a_{\mu\nu} T^a U^{-1} \\ &U = \exp\left(-i\varepsilon^a(x) T^a\right) \\ &\left[T^a, T^b\right] = i f^{abc} T^c \end{aligned}$$

a) Use the Euler-Lagrange equations to show that the equations of motion of the fields $A_\mu^a$ are given by

$$\partial_\mu F^{a\mu\nu} - g f^{abc} A^b_\mu F^{c\mu\nu} = 0$$

b) We have seen that the gauge current $j_A^{a\mu}$ can be obtained from $\quad \partial_\mu F^{a\mu\nu} \equiv j_A^{a\nu} \qquad \partial_\nu j_A^{a\nu} = 0$

This yields

$$j_A^{a\nu} = g f^{abc} A^b_\mu F^{c\mu\nu}$$

Obtain this current, up to a multiplicative factor, by using Noether's theorem for the invariance of $\mathscr{L}_{\mathrm{A}}$ under the global SU($n$) transformation

$$\begin{aligned} &A^a_\mu T^a \xrightarrow{\varepsilon^a} A'^a_\mu T^a = U A^a_\mu T^a U^{-1} \\ &U = \exp\left(-i\varepsilon^a T^a\right) \\ &\left[T^a, T^b\right] = i f^{abc} T^c \end{aligned}$$

where the $\varepsilon^a$ are real constants.

Hint: first find $\delta A_\mu^a$ for the infinitesimal $\varepsilon^a$ and then apply Noether's theorem.

Question 1

a) $\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi$

but $[S] = [d^4x][\mathcal{L}] = M^{-4}[\mathcal{L}] = M^0$

So $[\mathcal{L}] = M^4$

So here $[\mathcal{L}] = [m]^2[\phi]^2 = M^2[\phi]^2 = M^4$

$\Longrightarrow [\phi] = M^1 \quad \Longrightarrow n = 1$

b) $\mathcal{L} = \bar{\Psi}[i\gamma^\mu\partial_\mu - m]\Psi$

$[\mathcal{L}] = [m][\Psi]^2 = M[\Psi]^2 = M^4$

$\Longrightarrow [\Psi] = M^{3/2} \quad \Longrightarrow n = 3/2$

c) $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$

$[\mathcal{L}] = ([x^{-1}][A])^2 = M^2[A]^2 = M^4$

$\Longrightarrow [A] = M \quad \Longrightarrow n = 1$

Question 2

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi$$
$$= g^{\alpha\mu} (\partial_\mu \phi)^* (\partial_\alpha \phi) - m^2 \phi^* \phi$$

The Euler-Lagrange equation for $\phi$ is

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

yields the equation of motion for $\phi^*$

but $\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} = g^{\alpha\mu} (\partial_\mu \phi)^* \delta_\alpha^\nu = (\partial^\nu \phi)^*$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$$

So $\Box \phi^* + m^2 \phi^* = 0$ $\qquad \Box \equiv \partial_\mu \partial^\mu$

$$(\Box + m^2) \phi^* = 0$$

The Euler-Lagrange equation for $\phi^*$ is

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0$$

yields the equation of motion for $\phi$.

but $\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi^*)} = \partial^\nu \phi$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi$$

So $\Box \phi + m^2 \phi = 0$

$$(\Box + m^2) \phi = 0$$

Question 3

$$\mathcal{L}_D = \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi$$
$$= \sum_{j=1}^{4}\sum_{k=1}^{4} \bar{\Psi}_j\left[i\left(\gamma^\mu\right)_{jk}\partial_\mu - m\,\delta_{jk}\right]\Psi_k$$

The Euler-Lagrange equation for $\bar{\Psi}_j$ yields the equation of motion for $\Psi$:

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \bar{\Psi}_\ell)}\right) - \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_\ell} = 0 \qquad \ell = 1,2,3,4$$

but $$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \bar{\Psi}_\ell)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}_\ell} = \sum_{j=1}^{4}\sum_{k=1}^{4} \frac{\partial}{\partial \bar{\Psi}_\ell}\left\{\bar{\Psi}_j\left[i\left(\gamma^\mu\right)_{jk}\partial_\mu - m\,\delta_{jk}\right]\Psi_k\right\}$$
$$= \sum_{j=1}^{4}\sum_{k=1}^{4}\left\{\delta_{\ell j}\left[i\left(\gamma^\mu\right)_{jk}\partial_\mu - m\,\delta_{jk}\right]\Psi_k\right\}$$
$$= \sum_{k=1}^{4}\left[i\left(\gamma^\mu\right)_{\ell k}\partial_\mu - m\,\delta_{\ell k}\right]\Psi_k$$

Therefore
$$\sum_{k=1}^{4}\left[i\left(\gamma^\mu\right)_{\ell k}\partial_\mu - m\,\delta_{\ell k}\right]\Psi_k = 0$$

In Matrix notation, this reads
$$\left[\left(i\gamma^\mu\partial_\mu - m\right)\Psi\right]_\ell = 0 \qquad \ell = 1,2,3,4.$$

or, more compactly
$$\left(i\gamma^\mu\partial_\mu - m\right)\Psi = 0$$
which really is 4-equations.

The Euler - Lagrange equation for $\psi_j$ yields the equation of motion for $\bar{\psi}$ :

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi_\ell)} \right) - \frac{\partial \mathcal{L}}{\partial \psi_\ell} = 0 \qquad \ell = 1, 2, 3, 4$$

but

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi_\ell)} = \frac{\partial}{\partial(\partial_\nu \psi_\ell)} \sum_{j=1}^{4} \sum_{k=1}^{4} \bar{\psi}_j \left[ i (\gamma^\mu)_{jk} \partial_\mu - m \delta_{jk} \right] \psi_k$$

$$= \sum_{j=1}^{4} \sum_{k=1}^{4} \bar{\psi}_j \, i (\gamma^\mu)_{jk} \delta^\nu_\mu \delta_{\ell k}$$

$$= \sum_{j=1}^{4} \bar{\psi}_j \, i (\gamma^\nu)_{j\ell}$$

$$\frac{\partial \mathcal{L}}{\partial \psi_\ell} = \sum_{j=1}^{4} \sum_{k=1}^{4} \frac{\partial}{\partial \psi_\ell} \left\{ \bar{\psi}_j \left[ i (\gamma^\mu)_{jk} \partial_\mu - m \delta_{jk} \right] \psi_k \right\}$$

$$= -m \sum_{j=1}^{4} \sum_{k=1}^{4} \bar{\psi}_j \delta_{jk} \delta_{k\ell}$$

$$= -m \sum_{j=1}^{4} \bar{\psi}_j \delta_{j\ell}$$

$$= -m \bar{\psi}_\ell$$

Therefore

$$\partial_\nu \sum_{j=1}^{4} \bar{\psi}_j \, i (\gamma^\nu)_{j\ell} + m \sum_{j=1}^{4} \bar{\psi}_j \delta_{j\ell} = 0$$

$$\sum_{j=1}^{4} \left[ i \partial_\nu \bar{\psi}_j (\gamma^\nu)_{j\ell} + m \delta_{j\ell} \bar{\psi}_j \right] = 0$$

In Matrix notation, This reads

$$\left( i \partial_\nu \bar{\psi} \gamma^\nu + m \bar{\psi} \right)_\ell = 0 \qquad \ell = 1, 2, 3; 4$$

or

$$i \partial_\nu \bar{\psi} \gamma^\nu + m \bar{\psi} = 0$$

Question 4

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^{\mu} A_{\mu}$$

where $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$

The Euler – Lagrange equation for $A^{\mu}$ yields The equation of motion

$$\partial_{\mu} \left( \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\nu})} \right) - \frac{\partial \mathcal{L}}{\partial A_{\nu}} = 0 \qquad \nu = 0, 1, 2, 3$$

but $\dfrac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\nu})} = -\dfrac{1}{4} \dfrac{\partial}{\partial(\partial_{\mu} A_{\nu})} \left( F^{\alpha\beta} F_{\alpha\beta} \right)$

Consider $\dfrac{\partial}{\partial A_{\mu}} \left( B_{\nu} B^{\nu} \right) = \dfrac{\partial}{\partial A_{\mu}} \left( g^{\alpha\nu} B_{\nu} B_{\alpha} \right)$

$$= g^{\alpha\nu} \left( B_{\nu} \frac{\partial B_{\alpha}}{\partial A_{\mu}} + B_{\alpha} \frac{\partial B_{\nu}}{\partial A_{\mu}} \right)$$

$$= B^{\nu} \frac{\partial B_{\nu}}{\partial A_{\mu}} + B^{\nu} \frac{\partial B_{\nu}}{\partial A_{\mu}}$$

$$= 2 B^{\nu} \frac{\partial B_{\nu}}{\partial A_{\mu}}$$

Therefore $\dfrac{\partial}{\partial(\partial_{\mu} A_{\nu})} \left( F^{\alpha\beta} F_{\alpha\beta} \right) = 2 F^{\alpha\beta} \dfrac{\partial F_{\alpha\beta}}{\partial(\partial_{\mu} A_{\nu})}$

but $\dfrac{\partial F_{\alpha\beta}}{\partial(\partial_{\mu} A_{\nu})} = \dfrac{\partial}{\partial(\partial_{\mu} A_{\nu})} \left[ \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} \right]$

$$= \delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} - \delta_{\beta}^{\mu} \delta_{\alpha}^{\nu}$$

So $\dfrac{\partial}{\partial(\partial_{\mu} A_{\nu})} \left( F^{\alpha\beta} F_{\alpha\beta} \right) = 2 F^{\alpha\beta} \left[ \delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} - \delta_{\beta}^{\mu} \delta_{\alpha}^{\nu} \right]$

$$= 2 \left[ F^{\mu\nu} - F^{\nu\mu} \right]$$

$$= 4 F^{\mu\nu}$$

Therefore $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$ $\longrightarrow$ useful result.

Also $\dfrac{\partial \mathcal{L}}{\partial A_\nu} = -\dfrac{\partial}{\partial A_\nu} J^\mu A_\mu = -J^\mu \delta_\mu^\nu = -J^\nu$

Finally we obtain

$$-\partial_\mu F^{\mu\nu} + J^\nu = 0$$

or $$\partial_\mu F^{\mu\nu} = J^\nu \qquad \nu = 0, 1, 2, 3$$

Question 5

$$\mathcal{L} = (D_\mu \varphi)^* (D^\mu \phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - V(\phi)$$

where $F^{\mu\nu}(x) = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$V(\varphi) = -\mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2 \qquad \lambda > 0$$

$$D^\mu = \partial^\mu + iq A^\mu$$

With $\mu^2 > 0$ we have $\phi(x) = \frac{1}{\sqrt{2}}\left[v + \sigma(x) + i\eta(x)\right]$

So
$$\mathcal{L} = (\partial^\mu \phi)^* (\partial_\mu \phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - V(\phi) + \mathcal{L}_{INT}$$

where $\mathcal{L}_{INT} = -iq\left[\phi^*(\partial^\mu \phi) - (\partial^\mu \phi)^* \phi\right] A_\mu + q^2 A^\mu A_\mu \phi^* \phi$

Now, $(\partial^\mu \varphi)^* (\partial_\mu \varphi) = \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta)$

$$\phi^* \phi = \frac{1}{2}(v + \sigma - i\eta)(v + \sigma + i\eta)$$
$$= \frac{1}{2}(v^2 + 2v\sigma + \sigma^2 + \eta^2)$$

$$(\phi^* \phi)^2 = \frac{1}{4}\left[v^4 + 4v^2\sigma^2 + (\sigma^2+\eta^2)^2 + 4v^3\sigma + 2v^2(\sigma^2+\eta^2) + 4v\sigma(\sigma^2+\eta^2)\right]$$

but recall $v^2 = \frac{\mu^2}{\lambda}$.

We can gather the powers of the $\sigma, \eta$ fields in $\mathcal{L} - \mathcal{L}_{INT}$:

(0) $\mu^2 \frac{v^2}{2} - \lambda \frac{v^4}{4} = \mu^2 \frac{v^2}{2} - \frac{\mu^2 v^2}{4} = \frac{\mu^2 v^2}{4}$

(1) $\mu^2 v\sigma - \lambda v^3 \sigma = \mu^2 v\sigma - \mu^2 v\sigma = 0$

(2) $\frac{\mu^2}{2}(\sigma^2+\eta^2) - \lambda\left(v^2\sigma^2 + \frac{1}{2}v^2(\sigma^2+\eta^2)\right)$
$= (\sigma^2+\eta^2)\left[\frac{\mu^2}{2} - \frac{\mu^2}{2}\right] - \lambda v^2\sigma^2 = -\lambda v^2\sigma^2 = -\mu^2\sigma^2$

(3) $-\lambda v\sigma(\sigma^2+\eta^2)$

(4) $-\frac{\lambda}{4}(\sigma^2+\eta^2)^2$

So

$$\mathcal{L} - \mathcal{L}_{INT} = \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$
$$+ \frac{\mu^2 v^2}{4} - \mu^2 \sigma^2 - \lambda v \sigma (\sigma^2 + \eta^2) - \frac{\lambda}{4}(\sigma^2 + \eta^2)^2$$

where

$$\mathcal{L}_{INT} = -iq \Big[ \frac{1}{2}(v + \sigma - i\eta)(\partial_\mu \sigma + i\partial_\mu \eta)$$
$$- \frac{1}{2}(v + \sigma + i\eta)(\partial_\mu \sigma - i\partial_\mu \eta) \Big] A^\mu$$
$$+ \frac{1}{2} q^2 A^\mu A_\mu (v^2 + 2v\sigma + \sigma^2 + \eta^2)$$

But $(v+\sigma - i\eta)(\partial_\mu \sigma + i\partial_\mu \eta) - (v + \sigma + i\eta)(\partial_\mu \sigma - i\partial_\mu \eta)$

$$= 2i\,\mathrm{Im}\left[ (v + \sigma - i\eta)(\partial_\mu \sigma + i\partial_\mu \eta) \right]$$
$$= 2i \left[ v\partial_\mu \eta + \sigma \partial_\mu \eta - \eta \partial_\mu \sigma \right]$$

So
$$\mathcal{L}_{INT} = q\left[ v\partial_\mu \eta + \sigma\partial_\mu \eta - \eta \partial_\mu \sigma \right] A^\mu$$
$$+ \frac{1}{2} q^2 A^\mu A_\mu (v^2 + 2v\sigma + \sigma^2 + \eta^2) \Big]$$

So finally

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) - \mu^2 \sigma^2 \qquad \longrightarrow \frac{m_\sigma^2}{2} = \mu^2$$
$$+ \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) \qquad \longrightarrow m_\eta = 0$$
$$- \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$
$$+ \frac{1}{2} q^2 v^2 A^\mu A_\mu \qquad \longrightarrow m_A = qv$$
$$+ qv\, \partial_\mu \eta\, A^\mu \qquad \text{quadratic coupled}$$
$$+ \mathcal{L}'_{INT}$$

where
$$\mathcal{L}'_{INT} = -\lambda v \sigma (\sigma^2 + \eta^2) + q\sigma (\partial_\mu \eta) A^\mu - q\eta (\partial_\mu \sigma) A^\mu$$
$$+ q^2 v \sigma A_\mu A^\mu$$
$$- \frac{\lambda}{4}(\sigma^2 + \eta^2)^2 + \frac{1}{2} q^2 A^\mu A_\mu (\sigma^2 + \kappa^2)$$

In the unitary gauge,

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v+\sigma(x)\right)$$

Then

$$\begin{aligned}\mathcal{L} = &\ \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \mu^2\sigma^2 \\ &- \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}g^2v^2A_\mu A^\mu \\ &+ \mathcal{L}_{INT}\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{INT} &= -\lambda v\sigma^3 + g^2 v\sigma A_\mu A^\mu - \frac{\lambda}{4}\sigma^4 + \frac{1}{2}g^2 A_\mu A^\mu\sigma^2 \\ &= -\lambda v\sigma^3 - \frac{\lambda}{4}\sigma^4 + \frac{1}{2}g^2 A^\mu A_\mu\left(2v\sigma + \sigma^2\right)\end{aligned}$$

Question 6

$$D'_\mu \varphi' = (\partial_\mu + igk A'_\mu)\, e^{-i\varepsilon(x)k} \varphi$$

$$= -ik(\partial_\mu \varepsilon)\, e^{-i\varepsilon k} \varphi + e^{-i\varepsilon k}(\partial_\mu + igk A'_\mu)\varphi$$

$$= -ik(\partial_\mu \varepsilon)\, e^{-i\varepsilon k} \varphi + e^{-i\varepsilon k}(\partial_\mu + igk A_\mu)\varphi$$
$$+ e^{-i\varepsilon k}(igk A'_\mu - igk A_\mu)\varphi$$

$$= e^{-i\varepsilon k} D_\mu \varphi + igk\, e^{-i\varepsilon k}\left(A'_\mu - A_\mu - \frac{1}{g}(\partial_\mu \varepsilon)\right)$$

$$= e^{-i\varepsilon k} D_\mu \varphi$$

So $A_\mu \longrightarrow A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \varepsilon(x)$

Question 7

We have $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ and $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$

a) The suitable Lagrangian density is

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - m_S \phi^\dagger \phi$$
$$+ \bar{\Psi} (i\gamma^\mu D_\mu - m_D) \psi$$
$$- \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

where $D_\mu = \partial_\mu - igT^a A^a_\mu$

$$F^a_{\mu\nu} = A^a_{\mu\nu} + g f^{abc} A^b_\mu A^c_\nu$$
$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$
$$T^a = \frac{\sigma^a}{2} \qquad \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\frac{\sigma^c}{2}\varepsilon^{abc}$$
$$f^{abc} = \varepsilon^{abc} \qquad \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right]_+ = \frac{1}{2}\delta^{ab}$$

From the notes we easily obtain

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_S + \mathcal{L}_A + \mathcal{L}_{D+A} + \mathcal{L}_{S+A}$$
$$\mathcal{L}_D = \bar{\Psi}[i\gamma^\mu \partial_\mu - m_D]\psi$$
$$\mathcal{L}_S = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m_S^2 \phi^\dagger \phi$$
$$\mathcal{L}_A = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$
$$\mathcal{L}_{D+A} = g\bar{\Psi}\gamma^\mu T^a \psi A^a_\mu$$
$$\mathcal{L}_{S+A} = ig\left[\phi^\dagger T^a (\partial^\mu \phi) - (\partial^\mu \phi)^\dagger T^a \varphi\right] A^a_\mu$$
$$+ \frac{1}{2} g^2 A^a_\mu A^{b\mu} \varphi^\dagger [T^a, T^b]_+ \varphi$$

The last term can also be written

$$\frac{1}{4} g^2 A^a_\mu A^{a\mu} \phi^+ \phi$$

b) The local SU(2) gauge transformations are

$$\varphi \to \varphi' = U\varphi$$
$$\psi \to \psi' = U\psi$$

$$A^a_\mu \sigma^a \to A'^a_\mu \sigma^a = A^a_\mu U \sigma^a U^{-1} - \frac{1}{g} \sigma^a \partial_\mu \varepsilon^a(x)$$

where

$$U = e^{-\frac{i}{2}\sigma^a \varepsilon^a(x)}$$

This leads to

$$F^a_{\mu\nu}\sigma^a \to F'^a_{\mu\nu}\sigma^a = F^a_{\mu\nu} U\sigma^a U^{-1}$$

c) $\overline{\Psi}\varphi_1\psi + \overline{\Psi}\varphi_2\psi$ is not invariant under local SU(2).

$\overline{\Psi}\varphi\psi^1 + \overline{\Psi}\varphi\psi^2$ is not invariant under local SU(2)
is not a Lorentz scalar.

$\overline{\Psi}\varphi + \varphi^+\psi$ is not a Lorentz scalar.

Therefore none of the 3 terms are valid for this theory.

Question 8

Consider $\mathcal{L}_A = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$

where $F^a_{\mu\nu} = A^a_{\mu\nu} - g f^{abc} A^b_\mu A^c_\nu$

$A^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$

The equations of motion of the gauge fields $A^a_\mu$ are given by the Euler-Lagrange equations

$$\partial_\mu \left[ \frac{\partial \mathcal{L}_A}{\partial(\partial_\mu A^a_\nu)} \right] - \frac{\partial \mathcal{L}_A}{\partial A^a_\nu} = 0$$

But
$$\frac{\partial \mathcal{L}_A}{\partial(\partial_\mu A^a_\nu)} = \frac{\partial}{\partial(\partial_\mu A^a_\nu)} \left[ -\frac{1}{4} F^b_{\alpha\beta} F^{b\alpha\beta} \right]$$

$$= -\frac{1}{4} \left[ \frac{\partial F^b_{\alpha\beta}}{\partial(\partial_\mu A^a_\nu)} F^{b\alpha\beta} + F^b_{\alpha\beta} \frac{\partial F^{b\alpha\beta}}{\partial(\partial_\mu A^a_\nu)} \right]$$

$$= -\frac{1}{2} \frac{\partial F^b_{\alpha\beta}}{\partial(\partial_\mu A^a_\nu)} F^{b\alpha\beta}$$

and
$$\frac{\partial F^b_{\alpha\beta}}{\partial(\partial_\mu A^a_\nu)} = \frac{\partial A^b_{\alpha\beta}}{\partial(\partial_\mu A^a_\nu)} = \frac{\partial}{\partial(\partial_\mu A^a_\nu)} \left[ \partial_\alpha A^b_\beta - \partial_\beta A^b_\alpha \right]$$

$$= \left[ \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right] \delta_{ab}$$

So
$$\frac{\partial \mathcal{L}_A}{\partial(\partial_\mu A^a_\nu)} = -\frac{1}{2} \left[ F^{a\mu\nu} - F^{a\nu\mu} \right] = -F^{a\mu\nu}$$

Also
$$\frac{\partial \mathcal{L}_A}{\partial A^a_\nu} = -\frac{1}{2} \frac{\partial F^b_{\alpha\beta}}{\partial A^a_\nu} F^{b\alpha\beta}$$

But

$$\frac{\partial F^b_{\alpha\beta}}{\partial A^a_\nu} = -g f^{brs} \frac{\partial}{\partial A^a_\nu}\left(A^r_\alpha A^s_\beta\right)$$

$$= -g f^{brs}\left[A^s_\beta \delta^\nu_\alpha \delta_{ar} + A^r_\alpha \delta^\nu_\beta \delta_{as}\right]$$

$$= -g \left[f^{abc} A^c_\alpha \delta^\nu_\beta - f^{abc} A^c_\beta \delta^\nu_\alpha\right]$$

$$= -g f^{abc}\left[A^c_\alpha \delta^\nu_\beta - A^c_\beta \delta^\nu_\alpha\right]$$

So

$$\frac{\partial \mathcal{L}_A}{\partial A^a_\nu} = \frac{g}{2} f^{abc}\left[A^c_\alpha F^{b\alpha\nu} - A^c_\beta F^{b\nu\beta}\right]$$

$$= +g f^{abc} A^c_\mu F^{b\mu\nu} = -g f^{abc} A^b_\mu F^{c\mu\nu}$$

Therefore we obtain

$$-\partial_\mu F^{a\mu\nu} + g f^{abc} A^b_\mu F^{c\mu\nu} = 0$$

The Lagrangian density

$$\mathcal{L}_A = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

is also invariant under a global SU(N) Transformation

$$A^a_\mu T^a \xrightarrow{\varepsilon^a_0} A'^a_\mu T^a = U A^a_\mu T^a U^{-1}$$

$$U = \exp\left(-i\varepsilon^a T^a\right) \quad , \quad \varepsilon^a \text{ real constants}$$

The corresponding infinitesimal Transformation is obtained from

$$U = 1 - i\varepsilon^a T^a + \mathcal{O}(\varepsilon^2)$$

$$A'^a_\mu T^a = A^a_\mu \left(1 - i\varepsilon^b T^b\right) T^a \left(1 + i\varepsilon^c T^c\right) + \mathcal{O}(\varepsilon^2)$$

$$= A^a_\mu \left[T^a - i\left(\varepsilon^b T^b T^a - \varepsilon^c T^a T^c\right)\right] + \mathcal{O}(\varepsilon^2)$$

$$= A^a_\mu T^a - i\left[\varepsilon^b T^b, A^a_\mu T^a\right]$$

So

$$\delta(A^a_\mu T^a) = -i\left[\varepsilon^b T^b, A^c_\mu T^c\right] = -i\varepsilon^b A^c_\mu \left[T^b, T^c\right]$$

$$= \varepsilon^b A^c_\mu f^{bca} T^a$$

So

$$\delta A^a_\mu = f^{abc} \varepsilon^b A^c_\mu$$

Noether's Theorem yields

$$\partial_\mu j^\mu = 0$$

$$j^\mu = \frac{\partial \mathcal{L}_A}{\partial(\partial_\mu A^a_\nu)} \delta A^a_\nu \qquad \text{(note the sum over } \nu \text{ \underline{and} } a\text{)}$$

$$= -F^{a\mu\nu} f^{abc} \varepsilon^b A^c_\nu = f^{abc} A^b_\nu \varepsilon^c F^{a\mu\nu}$$

$$= -f^{abc} A^b_\nu \varepsilon^c F^{a\nu\mu}$$

Hence

$$\partial_\nu f^\nu = 0$$

$$f^\nu = -f^{abc} A^b_\mu \varepsilon^c F^{a\mu\nu}$$

Let $f^\nu = +\frac{1}{g} J_A^{c\nu} \varepsilon^c$

Then since the $\varepsilon^c$ are arbitrary, we set

$$J_A^{c\nu} = -g f^{abc} A^b_\mu F^{a\mu\nu}$$

or $J_A^{a\nu} = g f^{abc} A^b_\mu F^{a\mu\nu}$

with $\partial_\nu J_A^{a\nu} = 0$