PHYS506B Asignment #3 SOLUTIONS

given 15/03/2005
due 05/04/2005

1
15pts

Consider the Dirac spinor field ψ. Let

$$\psi_{L} = P_{L}\psi \qquad \bar{\psi}_{L} \equiv \overline{(\psi_{L})} \neq (\bar{\psi})_{L}$$
$$\psi_{R} = P_{R}\psi \qquad \bar{\psi}_{R} \equiv \overline{(\psi_{R})} \neq (\bar{\psi})_{R}$$

where

$$P_{L} \equiv \tfrac{1}{2}(1-\gamma^{5})$$
$$P_{R} \equiv \tfrac{1}{2}(1+\gamma^{5})$$

Show that

a) $\bar{\psi}\psi = \bar{\psi}_{L}\psi_{R} + \bar{\psi}_{R}\psi_{L}$

b) $(\bar{\psi}_{L}\psi_{R})^{\dagger} = \bar{\psi}_{R}\psi_{L}$

c) $\bar{\psi}\gamma^{\mu}\psi = \bar{\psi}_{L}\gamma^{\mu}\psi_{L} + \bar{\psi}_{R}\gamma^{\mu}\psi_{R}$

2
30pts

From the lecture notes (Running Coupling Constant), obtain the equations

a) $\left[-\dfrac{\partial}{\partial t} + \beta(\alpha)\dfrac{\partial}{\partial \alpha}\right] R\left(\tfrac{Q^{2}}{\mu^{2}}, \alpha\right) = 0$

b) $\dfrac{\partial \alpha(Q)}{\partial t} = \beta(\alpha(Q))$ and $\dfrac{\partial \alpha(Q)}{\partial \alpha} = \dfrac{\beta(\alpha(Q))}{\beta(\alpha)}$

c) $\left[-\dfrac{\partial}{\partial t} + \beta(\alpha)\dfrac{\partial}{\partial \alpha}\right] R(1, \alpha(Q)) = 0$

3
25pts

Estimate the energy scale Q at which QED and QCD coupling constants meet. (Compare your result with the Planck mass scale given by $G^{-1/2}$).

Hint: use the renormalization point $\mu = M_{Z}$ with

$$\alpha_{s}(M_{Z}) = 0.118 \qquad \alpha(M_{Z}) = 128^{-1} \qquad M_{Z} = 91.187 \text{ GeV}$$

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4 30pts

Consider the definition of $\alpha(Q)$: $\ln\frac{Q^2}{\mu^2} = \int_{\alpha(\mu)}^{\alpha(Q)} \frac{\mathrm{d}x}{\beta(x)}$

a) Show that to leading order in $\beta(x)$, that is with $\beta(x)=-bx^2$, we obtain the following result for $\alpha(Q)$:

$$\alpha(Q)^{-1} - \alpha(\mu)^{-1} = b\ln\frac{Q^2}{\mu^2}$$

b) Using the next to leading order in $\beta(x)$, that is with $\beta(x)=-bx^2(1+b'x)$, obtain an expression for $\alpha(Q)$.

5 40pts

Consider the experimental results $\alpha_s(M_Z) = 0.118$ and $M_Z = 91.187$ GeV. Evolve this result for α_s down to the scales $Q = 35$ GeV and $Q = 2$ GeV

a) using the leading order running of α_s;

b) using the next to leading order for the running of α_s.

Hint: assume only one sharp mass threshold at the bottom quark mass, that is for each one of questions a) and b)

1) starting from the given $\alpha_s(M_Z)$, using $\mu = M_Z$, find α_s at $Q = 35$ GeV and at $Q = m_b = 4.3$ GeV;

2) starting from $\alpha_s(m_b)$ found in 1), using $\mu = m_b$, find α_s at $Q = 2$ GeV.

Photocopy the figure on page 200 and plot your points on it.

N.B.: This treatment neglects a very small α_s shift at m_b when going from n_f=5 to n_f=4.

Question 1

From $P_L \equiv \frac{1}{2}(1-\gamma^5)$ $\qquad P_R \equiv \frac{1}{2}(1+\gamma^5)$

and recalling That

$$(\gamma^5)^2 = I$$

$$\gamma^0\gamma^5\gamma^0 = -(\gamma^5)^\dagger = -\gamma^5$$

$$[\gamma^5, \gamma^\mu]_+ = 0$$

We obtain

$$P_L P_R = \frac{1}{4}(1-\gamma^5)(1+\gamma^5) = \frac{1}{4}(1-(\gamma^5)^2) = 0$$

$$P_R P_L = \frac{1}{4}(1+\gamma^5)(1-\gamma^5) = \frac{1}{4}(1-(\gamma^5)^2) = 0$$

$$P_R^\dagger = \frac{1}{2}(1+(\gamma^5)^\dagger) = \frac{1}{2}(1+\gamma^5) = P_R$$

$$P_L^\dagger = \frac{1}{2}(1-(\gamma^5)^\dagger) = \frac{1}{2}(1-\gamma^5) = P_L$$

$$P_L\gamma^\mu = \frac{1}{2}(1-\gamma^5)\gamma^\mu = \frac{1}{2}(\gamma^\mu - \gamma^5\gamma^\mu)$$

$$= \frac{1}{2}(\gamma^\mu + \gamma^\mu\gamma^5) = \gamma^\mu\frac{1}{2}(1+\gamma^5) = \gamma^\mu P_R$$

$$P_R\gamma^\mu = \gamma^\mu P_L$$

We Then obtain

$$\psi_L \equiv P_L\psi \qquad \psi_R \equiv P_R\psi$$

$$\overline{\psi_L} \equiv (P_L\psi)^\dagger\gamma^0 = \psi^\dagger P_L\gamma^0 = \overline{\psi}P_R$$

$$\overline{\psi_R} \equiv (P_R\psi)^\dagger\gamma^0 = \psi^\dagger P_R\gamma^0 = \overline{\psi}P_L$$

$$\psi = \psi_R + \psi_L \qquad \overline{\psi} = \overline{\psi_R} + \overline{\psi_L}$$

a) $\bar{\psi}\psi = (\bar{\psi}_L + \bar{\psi}_R)(\psi_L + \psi_R) = \bar{\psi}_L\psi_L + \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L + \bar{\psi}_R\psi_R$

But $\bar{\psi}_L\psi_L = \bar{\psi} P_R P_L \psi = 0$

$\bar{\psi}_R\psi_R = \bar{\psi} P_L P_R \psi = 0$

$\bar{\psi}_L\psi_R = \bar{\psi} P_R P_R \psi \neq 0$

$\bar{\psi}_R\psi_L = \bar{\psi} P_L P_L \psi \neq 0$

So $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$

b) $(\bar{\psi}_L\psi_R)^\dagger = \psi_R^\dagger(\bar{\psi}_L)^\dagger = \psi_R^\dagger(\psi_L^\dagger\gamma^0)^\dagger = \psi_R^\dagger\gamma^0\psi_L$

$= \bar{\psi}_R\psi_L$

c) $\bar{\psi}\gamma^\mu\psi = (\bar{\psi}_L + \bar{\psi}_R)\gamma^\mu(\psi_L + \psi_R)$

$= \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_R + \bar{\psi}_R\gamma^\mu\psi_L$

but $\bar{\psi}_L\gamma^\mu\psi_R = \bar{\psi} P_R \gamma^\mu P_R \psi = \bar{\psi}\gamma^\mu P_L P_R \psi = 0$

$\bar{\psi}_R\gamma^\mu\psi_L = \bar{\psi} P_L \gamma^\mu P_L \psi = \bar{\psi}\gamma^\mu P_R P_L \psi = 0$

and $\bar{\psi}_L\gamma^\mu\psi_L = \bar{\psi} P_R \gamma^\mu P_L \psi = \bar{\psi}\gamma^\mu P_L P_L \psi \neq 0$

$\bar{\psi}_R\gamma^\mu\psi_R = \bar{\psi} P_L \gamma^\mu P_R \psi = \bar{\psi}\gamma^\mu P_R P_R \psi \neq 0$

So $\bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R$

Question 2

a) We have

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha(\mu)\right) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha(\mu)}{\partial \mu^2} \frac{\partial}{\partial \alpha(\mu)}\right] R\left(\frac{Q^2}{\mu^2}, \alpha(\mu)\right) = 0$$

Let $\tau \equiv \ln \frac{Q^2}{\mu^2}$

$$\beta(\alpha(\mu)) \equiv \mu^2 \frac{\partial \alpha(\mu)}{\partial \mu^2}$$

Then

$$\frac{\partial \tau}{\partial \mu^2} = -\frac{\mu^2}{Q^2} \frac{Q^2}{\mu^4} = -\frac{1}{\mu^2}$$

$$\frac{\partial R}{\partial \mu^2} = \frac{\partial R}{\partial \tau} \frac{\partial \tau}{\partial \mu^2} = -\frac{1}{\mu^2} \frac{\partial R}{\partial \tau}$$

This gives

$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha(\mu)) \frac{\partial}{\partial \alpha(\mu)}\right] R\left(\frac{Q^2}{\mu^2}, \alpha(\mu)\right) = 0$$

$$\left[-\frac{\partial}{\partial \tau} + \beta(\alpha(\mu)) \frac{\partial}{\partial \alpha(\mu)}\right] R\left(e^{\tau}, \alpha(\mu)\right) = 0$$

b) Consider the definition of $\alpha(Q)$

$$\tau(Q) = \int_{\alpha(\mu)}^{\alpha(Q)} \frac{dx}{\beta(x)}$$

We then have

$$\frac{\partial \tau(Q)}{\partial \tau} = 1 = \frac{1}{\beta(\alpha(Q))} \frac{\partial \alpha(Q)}{\partial \tau} - \frac{1}{\beta(\alpha(\mu))} \frac{\partial \alpha(\mu)}{\partial \tau}$$

But $\frac{\partial \alpha(\mu)}{\partial \tau} = 0$. Therefore

$$\frac{\partial \alpha(Q)}{\partial \tau} = \beta(\alpha(Q))$$

Also, we have

$$\frac{\partial \tau(Q)}{\partial \alpha(\mu)} = 0 = \frac{1}{\beta(\alpha(Q))} \frac{\partial \alpha(Q)}{\partial \alpha(\mu)} - \frac{1}{\beta(\alpha(\mu))} \frac{\partial \alpha(\mu)}{\partial \alpha(\mu)}$$

But $\frac{\partial \alpha(\mu)}{\partial \alpha(\mu)} = 1$. Therefore

$$\frac{\partial \alpha(Q)}{\partial \alpha(\mu)} = \frac{\beta(\alpha(Q))}{\beta(\alpha(\mu))}$$

c) With $R = R(1, \alpha(Q))$, we have

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha(\mu))\frac{\partial}{\partial \alpha(\mu)}\right] R$$

$$= -\frac{\partial R}{\partial t} + \beta(\alpha(\mu))\frac{\partial R}{\partial(\alpha(\mu))}$$

$$= -\frac{\partial R}{\partial \alpha(Q)}\,\frac{\partial \alpha(Q)}{\partial t} + \beta(\alpha(\mu))\frac{\partial R}{\partial \alpha(Q)}\,\frac{\partial \alpha(Q)}{\partial \alpha(\mu)}$$

$$= \frac{\partial R}{\partial \alpha(Q)}\left[-\frac{\partial \alpha(Q)}{\partial t} + \beta(\alpha(\mu))\frac{\partial \alpha(Q)}{\partial \alpha(\mu)}\right]$$

$$= \frac{\partial R}{\partial \alpha(Q)}\left[-\beta(\alpha(Q)) + \beta(\alpha(\mu)) \cdot \frac{\beta(\alpha(Q))}{\beta(\alpha(\mu))}\right]$$

$$= 0$$

Question 3

QED and QCD.

To do a proper job, we should.

1° evolve α from $Q = m_Z$ to $Q = m_{top}$ using $n_f = 5$
evolve α_s from $Q = m_Z$ to $Q = m_{top}$ using $n_f = 5$

2° Let $\alpha(Q^2) = \alpha_s(Q^2)$ using $n_f = 6$
starting from $\alpha(m_{top})$ and $\alpha_s(m_{top})$.

But to obtain an estimate, we proceed with step 2° using $n_f = 6$ starting from $\alpha(m_Z)$ and $\alpha_s(m_Z)$.

at LO we have

$$\alpha(Q) = \alpha_s(Q) \quad \longleftrightarrow \quad \frac{1}{\alpha(\mu)} + b \ln\frac{Q^2}{\mu^2} = \frac{1}{\alpha_s(\mu)} + b_s \frac{Q^2}{\mu^2}$$

or $$\ln\frac{Q^2}{\mu^2} = \frac{1}{b_s - b}\left[\frac{1}{\alpha(\mu)} - \frac{1}{\alpha_s(\mu)}\right]$$

where $$b = -\frac{1}{3\pi}\left[\sum_{i=1}^{n_f^l} e_i^2 + \sum_{i=1}^{n_f^q} 3\, e_i^2\right] \quad \text{where } n_f^l = n_f^q = 6$$

so $$b = -\frac{1}{3\pi}\left[3 + 3^2\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right)\right] = -\frac{8}{3\pi}$$

where $$b_s = \frac{1}{12\pi}(33 - 2n_f) = \frac{7}{4\pi}$$

Starting from
$$\alpha_s(M_Z) = 0.118 = 8.475^{-1}$$
$$\alpha(M_Z) = 0.00781 = 128^{-1}$$
$$M_Z = 91.187 \text{ GeV}.$$

We obtain $$\ln\frac{Q^2}{M_Z^2} = 85.02 \longrightarrow \frac{Q^2}{M_Z^2} = e^{85.0} = 8.2 \times 10^{36}$$

So $$\frac{Q}{M_Z} \sim 2.9 \times 10^{18} \longrightarrow Q \sim 2.6 \times 10^{20} \text{ GeV}$$

Note that $G^{-1/2} = 1.22 \times 10^{19}$ GeV

Also see Halzen-Martin p348.

Question 4

Let $\beta(x) = -bx^2$

Then

$$T = \int_{\alpha_0}^{\alpha(Q)} \frac{dx}{\beta(x)} = \int_{\alpha_0}^{\alpha(Q)} \frac{dx}{-bx^2} = \frac{1}{b}\left[\frac{1}{x}\right]_{\alpha_0}^{\alpha(Q)}$$

$$= \frac{1}{b}\left(\frac{1}{\alpha(Q)} - \frac{1}{\alpha_0}\right)$$

This yields

$$\alpha^{-1}(Q) - \alpha_0^{-1} = b \ln\left(\frac{Q^2}{\mu^2}\right)$$

OR

$$\alpha(Q) = \frac{\alpha_0}{1 + \alpha_0 b \ln\left(\frac{Q^2}{\mu^2}\right)}$$

This can also be written

$$\alpha^{-1}(Q) - \alpha^{-1}(\mu) = b \ln\left(\frac{Q^2}{\mu^2}\right)$$

which is invariant under $Q \longleftrightarrow \mu$

Though the <u>running</u> of α depends on the order at which $\beta(x)$ is expressed, it does not depend on the renormalization scale by definition !

We also verify that $\alpha(Q=\mu) = \alpha(\mu)$!

Indeed, we verify that

$$\frac{\alpha(Q_1)}{\alpha(Q_2)}(\mu) = \frac{\frac{1}{\alpha(\mu)} + b\ln\left(\frac{Q_2^2}{\mu^2}\right)}{\frac{1}{\alpha(\mu)} + b\ln\left(\frac{Q_1^2}{\mu^2}\right)}$$

Let $\mu = Q_3$; then $\frac{1}{\alpha(Q_3)} = \frac{1}{\alpha(Q_2)} + b\ln\frac{Q_3^2}{Q_2^2}$

$$= \frac{1}{\alpha(Q_1)} + b\ln\frac{Q_3^2}{Q_1^2}$$

Therefore

$$\frac{\alpha(Q_1)}{\alpha(Q_2)}(\mu = Q_3) = \frac{\frac{1}{\alpha(Q_2)} + b\ln\frac{Q_2^2}{Q_3^2}\cdot\frac{Q_3^2}{Q_2^2}}{\frac{1}{\alpha(Q_1)} + b\ln\frac{Q_1^2}{Q_3^2}\,\frac{Q_3^2}{Q_1^2}} = \frac{\alpha(Q_1)}{\alpha(Q_2)}$$

We see that indeed the running of α does not depend on μ.

Let $\beta(x) = -bx^2 - bb'x^3 = -bx^2(1+b'x)$

Then

$$T = \int_{\alpha_0}^{\alpha(Q)} \frac{dx}{\beta(x)} = \int_{\alpha_0}^{\alpha(Q)} \frac{dx}{-bx^2(1+b'x)}$$

but $\displaystyle\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \ln\left(\frac{ax+b}{x}\right)$

Therefore

$$-bT = \left[-\frac{1}{x} + b' \ln\left(\frac{1+b'x}{x}\right)\right]_{\alpha_0}^{\alpha(Q)}$$

$$= -\frac{1}{\alpha(Q)} + \frac{1}{\alpha_0} + b' \ln\left(\frac{1+b'\alpha(Q)}{\alpha(Q)}\right) - b' \ln\left(\frac{1+b'\alpha_0}{\alpha_0}\right)$$

which can be written as

$$\alpha^{-1}(Q) - \alpha^{-1}(\mu) + b' \ln\left(\frac{\alpha(Q)}{1+b'\alpha(Q)}\right) - b' \ln\left(\frac{\alpha(\mu)}{1+b'\alpha(\mu)}\right) = b \ln \frac{Q^2}{\mu^2}$$

note that this is invariant under $Q \leftrightarrow \mu$.

We also verify that indeed $\alpha(Q=\mu) = \alpha(\mu)$!

More elegantly:

$$\alpha^{-1}(Q) - \alpha^{-1}(\mu) + b' \ln\left[\frac{\alpha(Q)}{\alpha(\mu)} \frac{1+b'\alpha(\mu)}{1+b'\alpha(Q)}\right] = b \ln \frac{Q^2}{\mu^2}$$

Question 5

L.O. $\quad \alpha^{-1}(Q) - \alpha^{-1}(\mu) - b\ln\frac{Q^2}{\mu^2} = 0$

N.L.O. $\quad \alpha^{-1}(Q) - \alpha^{-1}(\mu) + b'\ln\left[\frac{\alpha(Q)}{1+b'\alpha(Q)}\right] - b'\ln\left[\frac{\alpha(\mu)}{1+b'\alpha(\mu)}\right] - b\ln\frac{Q^2}{\mu^2} = 0$

$$b = \frac{1}{12\pi}(33 - 2n_f)$$

$$b' = \frac{153 - 19n_f}{2\pi(33 - 2n_f)}$$

So

	$n_f = 5$	$n_f = 4$
b	$\frac{23}{12\pi} = 0.610$	$\frac{25}{12\pi} = 0.663$
b'	$\frac{29}{23\pi} = 0.401$	$\frac{77}{50\pi} = 0.490$

a) L.O. α_s

$n_f = 5$: $\quad M_Z$ 0.118 $\longrightarrow \boxed{35\text{ GeV} \;\; 0.1369} \longrightarrow$ 4.3 GeV 0.2106

$n_f = 4$: $\quad$ 4.3 GeV 0.2106 $\longrightarrow \boxed{2\text{ GeV} \;\; 0.2679}$

b) N.L.O. α_s

$n_f = 5$: $\quad M_Z$ 0.118 $\longrightarrow \boxed{35\text{ GeV} \;\; 0.1380} \longrightarrow$ 4.3 GeV 0.2217

$n_f = 4$: $\quad$ 4.3 GeV 0.2217 $\longrightarrow \boxed{2\text{ GeV} \;\; 0.2969}$

Running Coupling Constant

Summary of the values of $\alpha_s(Q)$ at the values of Q where they are measured. The lines show the central values and the $\pm 1\sigma$ limits of the PDG average. The figure clearly shows the decrease in α_s with increasing Q.