Introduction to Gauge Theories

- Basics of SU(*n*)
- **Classical Fields**
- U(1) Gauge Invariance
- SU(*n*) Gauge Invariance
- The Standard Model

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Classical Fields

- **Lagrangian Formulation for Discrete Systems**
- **Lagrangian Formulation for Fields**
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Lagrangian Formulation for Discrete Systems

Let us review the Lagrangian formulation of classical mechanics.
Consider a system described by a set of *n* generalized coordinates

$$q_i \qquad i = 1, \ldots, n \qquad \text{configuration space}$$

and the Lagrangian $\quad L(q_i, \dot{q}_i, t) = T - V$

Hamilton's principle states that the action $\quad S = \int_{t_1}^{t_2} \mathrm{d}t\, L$

has a stationary value for the correct path of the motion of the system in configuration space with respect to the parameter *t*:

with $\delta q_i = 0$ at t_1 and t_2

$\delta S = 0 \Rightarrow$ equations of motion

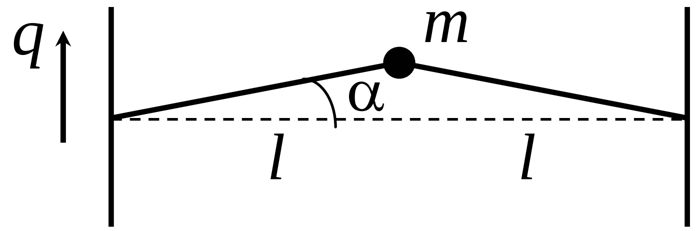
This yields the Euler-Lagrange equations of the coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \ldots, n$$

These *n* differential equations provide, when solved, the value of each coordinates as a function of *t*.
The generalized momenta can be defined as $\quad p_i \equiv \dfrac{\partial L}{\partial \dot{q}_i}$

Lagrangian Formulation for Discrete Systems

For example, consider the harmonic oscillator



Let Γ be the tension in the string.
The restoring force on the mass is

$$k \equiv \frac{\Gamma}{l}$$

$$F = -2\Gamma \sin\alpha \approx -2kq = -m\omega^2 q \qquad \text{where } \omega^2 = \frac{2k}{m}$$

Therefore

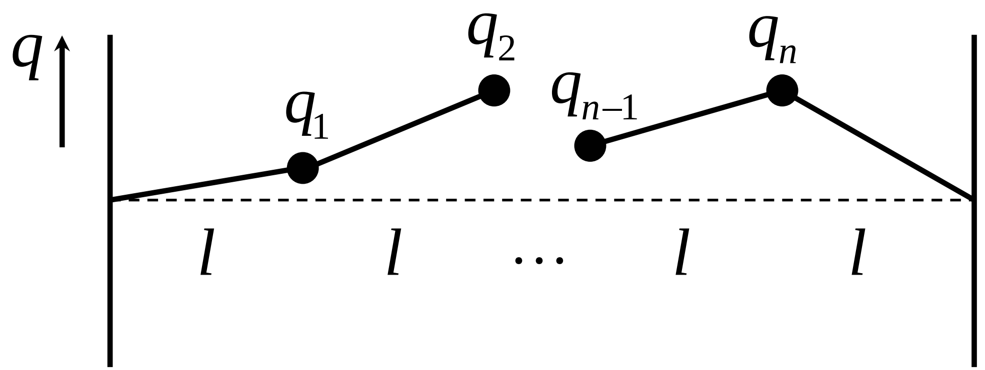
$$V(q,\dot{q}) = \tfrac{1}{2} m\omega^2 q^2$$
$$T(q,\dot{q}) = \tfrac{1}{2} m\dot{q}^2$$
$$L(q,\dot{q}) = T - V = \tfrac{1}{2} m\dot{q}^2 - \tfrac{1}{2} m\omega^2 q^2$$

Using the Euler-Lagrange equation we obtain the harmonic oscillator equation

$$\ddot{q} + \omega^2 q = 0$$

■ Lagrangian Formulation for Discrete Systems

Now consider an *n*-mass harmonic oscillator



$k \equiv \frac{\Gamma}{l}$

Setting $q_0 = q_{n+1} = 0$, we see that the force on the i^{th} mass is given by

$$F_i = -k\left(q_i - q_{i-1}\right) - k\left(q_i - q_{i+1}\right) = k\left(q_{i-1} - 2q_i + q_{i+1}\right)$$

Therefore

$$V\left(q_i, \dot{q}_i\right) = \tfrac{1}{2}k\sum_{i=0}^{n}\left(\Delta q_i\right)^2 \quad \text{where} \quad \Delta q_i \equiv q_{i+1} - q_i$$

$$T\left(q_i, \dot{q}_i\right) = \sum_{i=1}^{n}\tfrac{1}{2}m\dot{q}_i^2$$

and we have the Lagrangian

$$L\left(q_i, \dot{q}_i\right) = T - V = \sum_{i=1}^{n}\tfrac{1}{2}m\dot{q}_i^2 - \tfrac{1}{2}k\sum_{i=0}^{n}\left(\Delta q_i\right)^2$$

Lagrangian Formulation for Discrete Systems

The Euler-Lagrange equation yields

$$m\ddot{q}_i = k\left(q_{i-1} - 2q_i + q_{i+1}\right) \qquad i = 1, 2, \ldots, n$$

These are *n* coupled differential equations. They can be decoupled by suitable combinations of the q_i that form modes.

Lagrangian Formulation for Fields

Consider a system described by a set of *n* fields

$$\varphi_i = \varphi_i\left(t, x_j\right) \begin{cases} i = 1, 2, \ldots, n & \text{field configuration space} \\ j = 1, 2, 3 \end{cases}$$

and the Lagrangian density $\mathscr{L}\left(\varphi_i, \dot{\varphi}_i, \frac{\partial \varphi_i}{\partial x_j}, t, x_j\right)$

Hamilton's Principle states that the action $S = \int_\Omega \mathrm{d}t\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\, \mathscr{L}$

has a stationary value for the correct path of the motion of the system in field configuration space with respect to the parameters *t* and x_j:

with $\delta\varphi_i\left(t, x_j\right) = 0$ on the hypersurface bounding Ω

$$\delta S = 0 \quad \Rightarrow \quad \text{equations of motion}$$

This yields the Euler-Lagrange equations of the fields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\varphi}_i}\right) + \sum_{j=1}^{3} \frac{\mathrm{d}}{\mathrm{d}x_j}\left(\frac{\partial \mathscr{L}}{\partial\left(\frac{\partial \varphi_i}{\partial x_j}\right)}\right) - \frac{\partial \mathscr{L}}{\partial \varphi_i} = 0$$

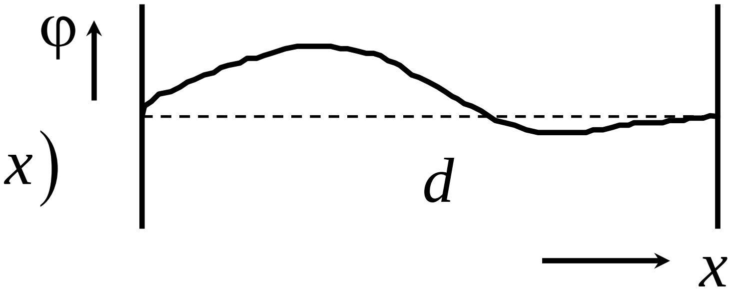
These are *n* differential equations which provide, when solved, the value of each field as a function of the parameters *t* and x_j.

We can also define the canonical field conjugate

$$\pi_i\left(x\right) \equiv \frac{\partial \mathscr{L}}{\partial \dot{\varphi}_i}$$

Lagrangian Formulation for Fields

Now let us consider the n-mass harmonic oscillator in the limit $n \to \infty$. We then have a vibrating string



$$\lim_{n \to \infty} \left\{ q_i(t) \; ; \; i = 1, 2, \ldots, n \right\} = \varphi(t, x)$$

Now both t and x play the role of continuous parameters. Using

$\Delta x = \frac{d}{n+1} = l$ we get

$$V\left(\varphi, \dot{\varphi}, {}^{\partial\varphi}\!/\!_{\partial x}\right) = \lim_{n \to \infty} \tfrac{1}{2} k \sum_{i=0}^{n} \left(\Delta q_i\right)^2$$

$$= \lim_{n \to \infty} \frac{\Gamma}{2\Delta x} \sum_{i=0}^{n} \left(\frac{\Delta q_i}{\Delta x} \right)^2 \left(\Delta x\right)^2 = \tfrac{1}{2} \Gamma \int_0^d \mathrm{d}x \left(\frac{\partial \varphi}{\partial x} \right)^2$$

Let λ be the linear density of the string. Then $\Gamma = \lambda \mathrm{v}^2$

Lagrangian Formulation for Fields

and we can write

$$V\left(\varphi,\dot{\varphi},{}^{\partial\varphi}\!/\!_{\partial x}\right)=\tfrac{1}{2}\lambda \mathrm{v}^2\int_0^d \mathrm{d}x\,\left(\frac{\partial\varphi}{\partial x}\right)^2$$

$$T\left(\varphi,\dot{\varphi},{}^{\partial\varphi}\!/\!_{\partial x}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\tfrac{1}{2}m\dot{q}_i^2$$

$$=\lim_{n\to\infty}\frac{m}{2\Delta x}\sum_{i=1}^{n}\dot{q}_i^2\Delta x=\tfrac{1}{2}\lambda\int_0^d \mathrm{d}x\,\left(\frac{\partial\varphi}{\partial t}\right)^2$$

Finally we can write the Lagrangian of the string

$$L\left(\varphi,\dot{\varphi},{}^{\partial\varphi}\!/\!_{\partial x}\right)=T-V=\int_0^d \mathrm{d}x\,\left[\tfrac{1}{2}\lambda\left(\frac{\partial\varphi}{\partial t}\right)^2-\tfrac{1}{2}\lambda \mathrm{v}^2\left(\frac{\partial\varphi}{\partial x}\right)^2\right]$$

Lagrangian Formulation for Fields

The action can be written

$$S = \int \mathrm{d}t \; L = \int \mathrm{d}t\mathrm{d}x \; \mathscr{L}$$

where the Lagrangian density is given by

$$\mathscr{L}\left(\varphi, \dot{\varphi}, {}^{\partial\varphi}\!/_{\partial x}\right) = \tfrac{1}{2}\lambda\left(\frac{\partial\varphi}{\partial t}\right)^2 - \tfrac{1}{2}\lambda \mathrm{v}^2\left(\frac{\partial\varphi}{\partial x}\right)^2$$

Using the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{\varphi}}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\varphi}{\partial x}\right)}\right) - \frac{\partial\mathscr{L}}{\partial\varphi} = 0$$

we get the field equation

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\varphi}{\partial t^2} = 0$$

which is the wave equation of the string.

Covariant Formulation

The norm in Minkowski space is given by

$$(\mathrm{d}s)^2 = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

where the metric tensor is diagonal with

$$g_{00} = 1 \qquad g_{11} = g_{22} = g_{33} = -1 \qquad (g_{\mu\nu}) = (g^{\mu\nu})$$

$g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^\nu$, the unit tensor or rank 2

This norm is kept invariant under inhomogenous Lorentz (or Poincaré) transformations, which contain 10 parameters

Poincaré
- homogeneous Lorentz
  - $3 \rightarrow$ special Lorentz
  - $3 \rightarrow$ spatial rotations
- $4 \rightarrow$ space-time translations

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \text{where} \quad g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$$

Covariant Formulation

Infinitesimal (proper) Poincaré transformations can be expressed as

$$x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu{}_\nu x^\nu + \delta^\mu$$

where $\varepsilon^{\mu\nu}$ and $\varepsilon_{\mu\nu}$ are antisymmetric (6 parameters).

$$\left\{\varepsilon^\mu{}_\nu\right\} = \begin{pmatrix} 0 & -d\varsigma_1 & -d\varsigma_2 & -d\varsigma_3 \\ -d\varsigma_1 & 0 & d\omega_3 & -d\omega_2 \\ -d\varsigma_2 & -d\omega_3 & 0 & d\omega_1 \\ -d\varsigma_3 & d\omega_2 & -d\omega_1 & 0 \end{pmatrix}$$

$\vec{\omega}$ represents a spatial rotation of angle ω about its axis

$$\left\{\varepsilon_{\mu\nu}\right\} = \begin{pmatrix} 0 & -d\varsigma_1 & -d\varsigma_2 & -d\varsigma_3 \\ d\varsigma_1 & 0 & -d\omega_3 & d\omega_2 \\ d\varsigma_2 & d\omega_3 & 0 & -d\omega_1 \\ d\varsigma_3 & -d\omega_2 & d\omega_1 & 0 \end{pmatrix}$$

$$\vec{\varsigma} = \frac{\vec{\beta}}{\beta} \operatorname{argtanh}(\beta)$$

represents a boost of rapidity ζ about its axis

$$\left\{\varepsilon^{\mu\nu}\right\} = \begin{pmatrix} 0 & d\varsigma_1 & d\varsigma_2 & d\varsigma_3 \\ -d\varsigma_1 & 0 & -d\omega_3 & d\omega_2 \\ -d\varsigma_2 & d\omega_3 & 0 & -d\omega_1 \\ -d\varsigma_3 & -d\omega_2 & d\omega_1 & 0 \end{pmatrix}$$

Covariant Formulation

We have the following transformation properties under Poincaré transformations:

$$\varphi(x) \xrightarrow{\Lambda} \varphi'(x') = \varphi(x)$$

scalar field

$$A^{\mu}(x) \xrightarrow{\Lambda} A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} A^{\nu}(x)$$

$$A_{\mu}(x) \xrightarrow{\Lambda} A'_{\mu}(x') = \Lambda_{\mu}{}^{\nu} A_{\nu}(x)$$

vector field

$$T^{\mu\nu}(x) \xrightarrow{\Lambda} T'^{\mu\nu}(x') = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} T^{\alpha\beta}(x)$$

tensor field of rank 2

Note that since $A^{\mu}A_{\mu}$ is a scalar, we obtain

$$\Lambda_{\mu}{}^{\nu} = \left(\Lambda^{-1}\right)^{\nu}{}_{\mu}$$

We wish to consider the Lorentz transformations for a covariant formulation of physical laws. Then physical laws are formulated as covariant equations between 4-tensor fields.

Covariant Formulation

Let A^μ and B^μ be 4-vectors. Then

$$A^2 = A^\mu A_\mu = A_\mu A^\mu = g_{\mu\nu} A^\mu A^\nu = A^0 A^0 - \vec{A}\cdot\vec{A}$$

$$A^\mu = \left(A^0, \vec{A}\right) \qquad A_\mu = \left(A^0, -\vec{A}\right) \qquad \vec{A} \equiv \left(A^1, A^2, A^3\right)$$

$$\left(A^0\right) = \left(A_0\right) \qquad \left(A^i\right) = -\left(A_i\right) \quad , \; i = 1, 2, 3$$

$$A\cdot B = A^\mu B_\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A}\cdot\vec{B}$$

The differential operators are given by

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(\frac{\partial}{\partial x^0}, -\vec{\nabla}\right)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$$

The 4-divergence of a 4-vector is an invariant scalar:

$$\partial^\mu A_\mu = \partial_\mu A^\mu = \frac{\partial A^0}{\partial x^0} + \vec{\nabla}\cdot\vec{A}$$

The 4-Laplacian operator is defined to be the invariant contraction

$$\square \equiv \partial^\mu \partial_\mu = \frac{\partial^2}{\partial x^{0\,2}} - \vec{\nabla}^2$$

Covariant Formulation

It is straightforward to cast the Lagrangian field formalism in a covariant form. Consider a system described by a set of *n* fields

$$\varphi_i = \varphi_i\left(x^\mu\right) \qquad i = 1, 2, \ldots, n \quad \text{field configuration space}$$

and the Lagrangian density $\mathscr{L}\left(\varphi_i, \partial_\mu \varphi_i, x^\mu\right)$

Consider the action $\quad S = \int_\Omega \mathrm{d}^4 x \ \mathscr{L}$

Hamilton's principle states that the action has a stationary value for the correct path of the motion of the system in field configuration space with respect to the parameters x^μ:

$$\text{with} \quad \delta\varphi_i = 0 \quad \text{on the hypersurface bounding } \Omega$$

$$\delta S = 0 \ \Rightarrow \ \text{equations of motion}$$

This yields the Euler-Lagrange equations of the fields

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \varphi_i\right)} \right) - \frac{\partial \mathscr{L}}{\partial \varphi_\mathrm{i}} = 0$$

Covariant Formulation

Note that the Lagrangian density is uncertain to any four-divergence of a function of the fields. Consider

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} + \partial_\mu B^\mu \qquad \forall B^\mu\left(\varphi_j\right)$$

then
$$S \to S' = S + \int_\Omega \mathrm{d}^4x \; \partial_\mu B^\mu$$

We can use the generalized Gauss theorem $\int_\Omega \mathrm{d}^4x \; \partial_\mu B^\mu = \oint_\sigma \mathrm{d}\sigma \; n_\mu B^\mu$

where $\mathrm{d}\sigma$ is an element of hypersurface σ bounding Ω

n_μ is a unit 4-vector normal outward to $\mathrm{d}\sigma$

Therefore
$$\delta S \to \delta S' = \delta S + \delta \oint_\sigma \mathrm{d}\sigma \; n_\mu B^\mu$$

but we have
$$\delta\varphi_j = 0 \;\text{ on }\; \sigma \quad \Rightarrow \quad \delta B^\mu = 0 \;\text{ on }\; \sigma$$

Therefore
$$\delta S' = \delta S$$

which means that $\mathscr{L}$ and $\mathscr{L}'$ will yield the same equations of motion.

Noether's Theorem

From classical mechanics, we have seen that symmetries lead to conservation laws:

space translation symmetry $\rightarrow \quad \dfrac{\mathrm{d}\vec{P}}{\mathrm{d}t} = 0$

time translation symmetry $\rightarrow \quad \dfrac{\mathrm{d}E}{\mathrm{d}t} = 0$

space rotation symmetry $\rightarrow \quad \dfrac{\mathrm{d}\vec{L}}{\mathrm{d}t} = 0$

Noether's theorem concerns continous transformations on fields:

symmetry of $\mathscr{L}$ $\rightarrow$ conserved currents and charges

Consider a system described by a set of *n* fields

$$\varphi_i = \varphi_i\left(x^\mu\right) \qquad i = 1, 2, \ldots, n$$

Consider the infinitesimal transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu\left(x\right)$$

$$\varphi_i\left(x\right) \rightarrow \varphi'_i\left(x'\right) = \varphi_i\left(x\right) + \delta\varphi_i\left(x\right)$$

where $\delta\varphi_i(x)$ includes changes in x and $\varphi_i(x)$.

Noether's Theorem

Under this transformation we have

$$\mathscr{L}\left(\varphi_i(x), \partial_\mu \varphi_i(x)\right) \to \mathscr{L}'\left(\varphi_i'(x'), \partial_\mu' \varphi_i'(x')\right)$$

We demand that under this infinitesimal transformation we have

<u>form invariance</u>

$$\mathscr{L}'\left(\varphi_i'(x'), \partial_\mu' \varphi_i'(x')\right) = \mathscr{L}\left(\varphi_i'(x'), \partial_\mu' \varphi_i'(x')\right)$$

<u>scale invariance</u>

$$S = \int_{\Omega'} \mathrm{d}\Omega' \; \mathscr{L}'\left(\varphi_i'(x'), \partial_\mu' \varphi_i'(x')\right) = \int_{\Omega} \mathrm{d}\Omega \; \mathscr{L}\left(\varphi_i(x), \partial_\mu \varphi_i(x)\right)$$

If this is satisfied, we obtain (see Goldstein)

$$\partial_\mu f^\mu(x) = 0 \quad \text{where} \quad f^\mu(x) = \mathscr{L} \cdot \delta x^\mu + \sum_{i=1}^{n} \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \varphi_i\right)} \left[\delta\varphi_i - \left(\partial_\nu \varphi_i\right)\delta x^\nu\right]$$

That is we have a continuity equation for the (conserved) current $f^\mu(x)$.
We also obtain the conserved charge

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \quad \text{where} \quad Q = \int \mathrm{d}V \; f^0(x)$$

Noether's Theorem

Consider a Lagrangian density that is form and scale invariant under pure translations. We then have

$$\delta x^{\mu}\left(x\right) = \delta^{\mu} \qquad \text{constants}$$

$$\delta\varphi_i\left(x\right) = 0 \qquad \text{no mixing of fields}$$

Noether's theorem then yields

$$\partial_{\mu} T^{\mu\nu} = 0 \quad \text{where} \quad T^{\mu\nu} = -\mathscr{L} \cdot g^{\mu\nu} + \sum_{i=1}^{n} \frac{\partial \mathscr{L}}{\partial\left(\partial_{\mu}\varphi_i\right)} \partial^{\nu}\varphi_i$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the system. The following parts can be identified:

$T^{00} \rightarrow$ energy density

$T^{i0} \rightarrow$ i^{th}component of the energy current density

$T^{0j} \rightarrow$ j^{th}component of the momentum density

$T^{ij} \rightarrow$ i^{th}component of the current density

for the j^{th}component of the momentum density

$T^{ij} \rightarrow$ 3D stress tensor

Noether's Theorem

We also have the conserved charges

$$\frac{\mathrm{d}P^{\mu}}{\mathrm{d}t} = 0 \quad \text{where} \quad P^{\mu} = \int \mathrm{d}V \; T^{0\mu}$$

P^{μ} is the conserved 4-momentum of the system. Note that

$$P^{j} = \int \mathrm{d}V \; \sum_{k=1}^{n} \pi_k \partial^{j} \varphi_k$$

Noether's Theorem

Consider a Lagrangian density that is form and scale invariant under homogeneous Lorentz transformations. We then have

$$\delta x^{\mu}\left(x\right)=\varepsilon^{\mu\nu}x_{\nu} \qquad \text{Lorentz rotation}$$

$$\delta\varphi_{i}\left(x\right)=\varepsilon_{\mu\nu}\,\tfrac{1}{2}\sum_{j=1}^{n}Z_{ij}^{\mu\nu}\varphi_{j}\left(x\right) \quad \text{where} \quad Z_{ij}^{\mu\nu}=-Z_{ij}^{\nu\mu}$$

where Z depends on the transformation properties of the fields. For example,

<u>scalar field</u>

$$\varphi'\left(x'\right)=\varphi\left(x\right)\rightarrow\delta\varphi\left(x\right)=0\rightarrow Z=0$$

<u>vector field</u> (the index *i* becomes a Lorentz index)

$$\varphi'_{\alpha}\left(x'\right)=\varphi_{\alpha}\left(x\right)+\varepsilon_{\alpha}{}^{\beta}\varphi_{\beta}\left(x\right)\rightarrow\delta\varphi_{\alpha}\left(x\right)=\varepsilon_{\alpha}{}^{\beta}\varphi_{\beta}\left(x\right)=\varepsilon_{\alpha\beta}\varphi^{\beta}\left(x\right)$$

$$\rightarrow Z_{\alpha\beta}^{\mu\nu}=\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu}-\delta_{\alpha}^{\nu}\delta_{\beta}^{\mu}$$

Noether's Theorem

Noether's theorem yields

$$\partial_\mu \mathscr{M}^{\mu\alpha\beta} = 0 \qquad \mathscr{M}^{\mu\alpha\beta} = -\mathscr{M}^{\mu\beta\alpha}$$

$$\mathscr{M}^{\mu\alpha\beta} = \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} \right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \varphi_i \right)} Z_{ij}^{\alpha\beta} \varphi_j$$

We then have the conserved charges

$$\frac{\mathrm{d}M^{\alpha\beta}}{\mathrm{d}t} = 0 \quad \text{where} \quad M^{\alpha\beta} = -M^{\beta\alpha} = \int \mathrm{d}V \, \mathscr{M}^{0\alpha\beta}$$

This is the conserved angular-momentum 4-tensor of the system. Its space components form the angular momentum 3-vector of the system

$$\vec{M} = \left(M^{23}, M^{31}, M^{12} \right)$$

$$M^{ij} = \int \mathrm{d}V \left\{ \left[x^i T^{oj} - x^j T^{oi} \right] + \sum_{a=1}^{n} \sum_{b=1}^{n} \frac{\partial \mathscr{L}}{\partial \left(\partial_0 \varphi_a \right)} Z_{ab}^{ij} \varphi_b \right\}$$

We see that the conserved angular momentum is composed of an orbital part and an intrinsic or spin part that depends on the transformation properties of the field!

Noether's Theorem

In terms of the canonical field conjugates, we have

$$M^{ij} = \int \mathrm{d}V \left\{ \sum_{a=1}^{n} \left[x^i \pi_a \partial^j \varphi_a - x^j \pi_a \partial^i \varphi_a \right] + \sum_{a=1}^{n} \sum_{b=1}^{n} \pi_a Z_{ab}^{ij} \varphi_b \right\}$$

The orbital term does not mix field components, while the spin term does through *Z*.

Noether's Theorem

Consider a Lagrangian density that is form and scale invariant under a global phase transformation on the fields

$$\varphi_j\left(x\right) \to \varphi'_j\left(x\right) = e^{i\varepsilon_j}\varphi_j\left(x\right) \qquad \text{(no sum on } j\text{)}$$

where ε_j are real constants. We then have

$$\delta x^\mu = 0$$

$$\delta\varphi_j = i\varepsilon_j\varphi_j \quad \text{(no sum on } j\text{)}$$

Noether's theorem yields

$$\partial_\mu j^\mu = 0 \quad \text{where} \quad j^\mu = i\sum_{j=1}^{n}\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\varphi_j\right)}\varepsilon_j\varphi_j$$

and the conserved charge

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \quad \text{where} \quad Q = \int \mathrm{d}V\, j^0$$

Klein-Gordon Field

In non relativistic quantum mechanics, the Schrödinger equation can be obtained from

$$E = \frac{\vec{P}^2}{2m}$$

with the substitution

$$\left.\begin{array}{l} E \to i\dfrac{\partial}{\partial t} \\ \vec{P} \to -i\vec{\nabla} \end{array}\right\} P^\mu \to i\partial^\mu$$

An early attempt at building a one particle relativistic wave equation considered the same substitution on

$$E^2 = \vec{P}^2 + m^2 \quad \text{or} \quad P^2 = P^\mu P_\mu = m^2$$

This yields the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + m^2\right)\varphi = 0$$

For $m = 0$ this is the same equation as a vibrating string with $v = c = 1$. Assuming φ complex, we can write

$$\left(\Box + m^2\right)\varphi = 0 \qquad \left(\Box + m^2\right)\varphi^* = 0$$

Klein-Gordon Field

As a one particle quantum mechanic wave equation, the Klein-Gordon equation leads to negative probabilities and negative energies, and was abandoned. Of course these problems do not occur in quantum field theory, where we have only particles and antiparticles with positive energy. Let's therefore treat the Klein-Gordon field as a classical field. We therefore consider the Lagrangian density

$$\mathscr{L} = \left(\partial_\mu \varphi\right)^* \left(\partial^\mu \varphi\right) - m^2 \varphi^* \varphi$$

Since $\varphi(x)$ is complex, we must consider

$$\mathscr{L} = \mathscr{L}\left(\varphi, \partial_\mu \varphi, \varphi^*, \partial_\mu \varphi^*\right)$$

The Euler-Lagrange equation then yields the

<u>Klein-Gordon equation</u> $\left(\Box + m^2\right)\varphi = 0$ $\left(\Box + m^2\right)\varphi^* = 0$

Only one of them is independent.

We also have the following canonical conjugate fields

$$\varphi\left(x\right) \to \pi\left(x\right) = \frac{\partial \mathscr{L}}{\partial \dot{\varphi}} = \dot{\varphi}^*\left(x\right) \qquad \varphi^*\left(x\right) \to \pi^*\left(x\right) = \frac{\partial \mathscr{L}}{\partial \dot{\varphi}^*} = \dot{\varphi}\left(x\right)$$

Klein-Gordon Field

The Lagrangian density is clearly form and scale invariant under Poincaré transformations. Noether's theorem then yields the conserved 4-vector momentum and 4-tensor angular momentum of the field. Their spatial parts are

$$\vec{P} = \int \mathrm{d}V \left[\left(-i\pi\right)\left(-i\vec{\nabla}\right)\left(\varphi\right) + \left(-i\pi^*\right)\left(-i\vec{\nabla}\right)\left(\varphi^*\right) \right]$$

$$\vec{M} = \int \mathrm{d}V \left[\left(-i\pi\right)\left(\vec{r}\times -i\vec{\nabla}\right)\left(\varphi\right) + \left(-i\pi^*\right)\left(\vec{r}\times -i\vec{\nabla}\right)\left(\varphi^*\right) \right]$$

Note that only orbital angular momentum contributes. The Klein-Gordon field has intrinsic spin 0. Upon quantization, it will represent spin 0 particles and antiparticles.

We also note that the Lagrangian density is form and scale invariant under a global phase transformation

$$\varphi\left(x\right) \to \varphi'\left(x\right) = e^{i\varepsilon}\varphi\left(x\right)$$

where ε is a real constant. Noether's theorem then yields the conserved current

$$\partial_\mu j^\mu = 0 \quad \text{where} \quad j^\mu\left(x\right) = i\left[\varphi^*\left(\partial^\mu\varphi\right) - \left(\partial^\mu\varphi^*\right)\varphi\right]$$

Klein-Gordon Field

and the conserved charge Q where $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$ and

$$Q = \int \mathrm{d}V\ j^0 = \int \mathrm{d}V\ i\left(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi\right) = \int \mathrm{d}V\ \left[\left(-i\pi\right)\varphi - \left(-i\pi^*\right)\varphi^*\right]$$

If we couple the Klein-Gordon field to the Maxwell field, this charge becomes proportional to the electric charge.

It is interesting to compare with the Schrödinger probability density and current density

$$\left.\begin{aligned} \rho(x) &= j^0(x) = \varphi^*\varphi \\ s^k(x) &= j^k(x) = \tfrac{i}{2m}\left[\varphi^*\left(\partial^k\varphi\right) - \left(\partial^k\varphi^*\right)\varphi\right] \end{aligned}\right\} \partial_\mu j^\mu = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{s} = 0$$

Note that the spatial components are identical in form. But in the Klein-Gordon case, $\rho(x)$ contains time derivatives which lead to a non positive definite $\rho(x)$.

We can also consider a real Klein-Gordon field $\varphi(x)$. In this case we have

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right) - \tfrac{1}{2}m^2\varphi^2 \quad \Rightarrow \quad \left(\Box + m^2\right)\varphi = 0$$

Note the 1/2 factor. Clearly, this Lagrangian density is not invariant under a global U(1) phase transformation.

Dirac Field

In an attempt to obtain a one particle relativistic quantum mechanical wave equation that yields a positive definite probability density, Dirac postulated the equation

$$i\frac{\partial\psi}{\partial t} = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\psi$$

and required $\psi(x)$ to also satisfy the Klein-Gordon equation

$$\left(\Box + m^2\right)\psi = 0$$

The following requirements on α_j and β are obtained

$$\left[\alpha_j, \alpha_k\right]_+ \equiv \alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}$$

$$\left[\alpha_j, \beta\right]_+ \equiv \alpha_j\beta + \beta\alpha_j = 0$$

$$\beta^2 = 1$$

Clearly α_j and β are no ordinary numbers. They can be represented by matrices. Simple arguments show that they must be of even dimension of at least 4.

ψ is then a column matrix called a Dirac spinor.

Dirac Field

One possibility is the Dirac representation $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

with the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

We must ensure that the Dirac equation is covariant, that is invariant under homogeneous Lorentz transformations. We introduce the notation

$$\gamma^0 \equiv \beta \qquad \gamma^j \equiv \beta\alpha_j$$

The γ^μ are 4 4×4 matrices and they satisfy

$$\left[\gamma^\mu, \gamma^\nu\right]_+ = 2g^{\mu\nu} I \qquad \left.\begin{aligned} \gamma^{j\dagger} &= -\gamma^j \\ \gamma^{0\dagger} &= \gamma^0 \end{aligned}\right\} \; \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

In the Dirac representation we have

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}$$

We can also introduce the useful slash notation: $\not{A} \equiv \gamma^\mu A_\mu$

Dirac Field

The Dirac equation can now be written in the compact form

$$\left(i\partial\!\!\!/ - m\right)\psi = 0$$

which stands for

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0$$

or, with the Dirac spinor indices explicit,

$$\sum_{k=1}^{4}\left[i\left(\gamma^{\mu}\right)_{jk}\frac{\partial}{\partial x^{\mu}} - m\delta_{jk}\right]\psi_k\left(x\right) = 0$$

which are 4 equations for $j = 1, 2, 3, 4$.

Dirac Field

Consider the homogeneous Lorentz transformation

$$x^\mu \xrightarrow{\Lambda} x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

and the corresponding spinor transformation

$$\psi(x) \xrightarrow{\Lambda} \psi'(x') = S\psi(x)$$

If we require covariance of the Dirac equation

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi(x) = 0 \xrightarrow{\Lambda} \left(i\gamma^\mu \partial'_\mu - m\right)\psi'(x') = 0$$

we obtain the constraint defining *S*

$$\gamma^\nu = S^{-1}\gamma^\mu S \Lambda_\mu{}^\nu \quad \text{or} \quad \gamma^\mu \Lambda_\mu{}^\nu = S\gamma^\nu S^{-1} \quad \text{or} \quad \Lambda^\mu{}_\nu \gamma^\nu = S^{-1}\gamma^\mu S$$

The matrix *S* that satisfies this constraint is

$$S = \exp\left(+\tfrac{i}{2}\vec{\omega}\cdot\vec{\bar{\sigma}} + \tfrac{i}{2}\vec{\varsigma}\cdot\vec{\bar{\sigma}}\right) = \exp\left(-\tfrac{i}{4}\sigma^{\mu\nu}A_{\mu\nu}\right)$$

where $\vec{\omega}$ represents a spatial rotation of angle ω about the axis $\hat{\omega}$

$\vec{\varsigma} = \frac{\vec{\beta}}{\beta}\arg\tanh\beta$ represents a boost of rapidity ζ about the axis $\hat{\beta}$

$A_{\mu\nu}$ is obtained from $\Lambda^\mu{}_\nu = \exp(A^\mu{}_\nu)$

Dirac Field

and $\vec{\tilde{\sigma}} \equiv \left(\sigma^{23}, \sigma^{31}, \sigma^{12}\right)$ $\qquad$ $\vec{\breve{\sigma}} \equiv \left(\sigma^{01}, \sigma^{02}, \sigma^{03}\right)$

$$\sigma^{\mu\nu} \equiv \tfrac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] \qquad \sigma^{\mu\nu\dagger} = \gamma^{0}\sigma^{\mu\nu}\gamma^{0}$$

Therefore $S^{\dagger} = \gamma^{0} S^{-1} \gamma^{0}$

The Dirac field transforms under an infinitesimal Lorentz transformation as follows: $\delta\psi_{i}\left(x\right) = \varepsilon_{\mu\nu}\,\tfrac{1}{2} Z_{ij}^{\mu\nu}\psi_{j}\left(x\right)$ where $Z_{ij}^{\mu\nu} = -\tfrac{i}{2}\sigma_{ij}^{\mu\nu}$

Also of interest is the γ^5 matrix

$$\gamma^{5} = \gamma_{5} \equiv i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \tfrac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} \qquad \gamma^{5\dagger} = \gamma^{5} \qquad \left(\gamma^{5}\right)^{2} = I$$

$$\left[\gamma^{\mu}, \gamma^{5}\right]_{+} = 0 \qquad \gamma^{0}\gamma^{5}\gamma^{0} = -\gamma^{5\dagger} = -\gamma^{5}$$

$$\left[\gamma^{5}, \sigma^{\mu\nu}\right] = 0 \qquad \gamma^{5}\sigma^{\mu\nu} = \tfrac{i}{2}\varepsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$$

In the Dirac representation it becomes $\gamma^{5} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

Since $\varepsilon_{abcd}\left(\det\Lambda\right) = \varepsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}{}_{a}\Lambda^{\nu}{}_{b}\Lambda^{\rho}{}_{c}\Lambda^{\sigma}{}_{d}$

we obtain $S^{-1}\gamma^{5}S = \left(\det\Lambda\right)\gamma^{5}$

Dirac Field

Consider the adjoint spinor field $\bar{\psi} \equiv \psi^\dagger \gamma^0$

then $\psi(x) \xrightarrow{\Lambda} \psi'(x') = S\psi(x)$

$$\bar{\psi}(x) \xrightarrow{\Lambda} \bar{\psi}'(x') = \bar{\psi}(x) S^{-1}$$

In order to build a covariant theory with spinors, we are interested in classifying bilinear forms like

$$\bar{\psi}\Gamma\psi$$

where Γ is a 4x4 complex matrix according to their behaviour under homogeneous Lorentz transformations $\Lambda^\mu{}_\nu$ and parity transformations $\Lambda_P{}^\mu{}_\nu$. Clearly,

$$\Lambda_P{}^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \det\Lambda_P = -1$$

The equation $\Lambda_P{}^\mu{}_\nu \gamma^\nu = S_P^{-1}\gamma^\mu S_P$ is satisfied by $S_P = S_P^{-1} = S_P^\dagger = \gamma^0$

Then

$$S_P^{-1}\gamma^5 S_P = -\gamma^5$$

Dirac Field

With all these relations we obtain

$$\bar{\psi}\psi \xrightarrow{\Lambda} \bar{\psi}'\psi' = \bar{\psi}\psi$$

$$\bar{\psi}\psi \xrightarrow{\Lambda_P} \bar{\psi}'\psi' = \bar{\psi}\psi$$

$$\bar{\psi}\gamma^\mu\psi \xrightarrow{\Lambda} \bar{\psi}'\gamma^\mu\psi' = \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi$$

$$\bar{\psi}\gamma^\mu\psi \xrightarrow{\Lambda_P} \bar{\psi}'\gamma^\mu\psi' = \begin{cases} +\bar{\psi}\gamma^\mu\psi, & \mu = 0 \\ -\bar{\psi}\gamma^\mu\psi, & \mu = 1, 2, 3 \end{cases}$$

$$\bar{\psi}\sigma^{\mu\nu}\psi \xrightarrow{\Lambda} \bar{\psi}'\sigma^{\mu\nu}\psi' = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \bar{\psi}\sigma^{\rho\sigma}\psi$$

$$\bar{\psi}\gamma^5\gamma^\mu\psi \xrightarrow{\Lambda} \bar{\psi}'\gamma^5\gamma^\mu\psi' = \left(\det\Lambda\right)\Lambda^\mu{}_\nu \bar{\psi}\gamma^5\gamma^\nu\psi$$

$$\bar{\psi}\gamma^5\gamma^\mu\psi \xrightarrow{\Lambda_P} \bar{\psi}'\gamma^5\gamma^\mu\psi' = \begin{cases} -\bar{\psi}\gamma^5\gamma^\mu\psi, & \mu=0 \\ +\bar{\psi}\gamma^5\gamma^\mu\psi, & \mu=1,2,3 \end{cases}$$

$$\bar{\psi}\gamma^5\psi \xrightarrow{\Lambda} \bar{\psi}'\gamma^5\psi' = \left(\det\Lambda\right)\bar{\psi}\gamma^5\psi$$

$$\bar{\psi}\gamma^5\psi \xrightarrow{\Lambda_P} \bar{\psi}'\gamma^5\psi' = -\bar{\psi}\gamma^5\psi$$

Dirac Field

Therefore, under homogeneous Lorentz transformations, we have the following bilinear covariants

$\bar{\psi}\psi$	1	scalar	tensor rank 0
$\bar{\psi}\gamma^{\mu}\psi$	4	vector	tensor rank 1
$\bar{\psi}\sigma^{\mu\nu}\psi$	6	antisymmetric	tensor rank 2
$\bar{\psi}\gamma^{5}\gamma^{\mu}\psi$	4	pseudovector	tensor rank 1
$\bar{\psi}\gamma^{5}\psi$	1	pseudoscalar	tensor rank 0

We can then use these bilinear covariants to build a covariant theory.

Dirac Field

We therefore consider the Lagrangian density

$$\mathscr{L} = \bar{\psi}\left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi \qquad \bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$$

$$\mathscr{L} = \mathscr{L}\left(\psi_j, \partial_{\mu}\psi_j, \bar{\psi}_j, \partial_{\mu}\bar{\psi}_j\right) \qquad j = 1, 2, 3, 4$$

The Euler-Lagrange equations then yields the field equations:

Dirac equation $$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0$$

Adjoint Dirac equation $$i\partial_{\mu}\bar{\psi}\gamma^{\mu} + m\bar{\psi} = 0$$

Only one of them is independent. We also have the canonical field conjugates

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial \dot{\psi}} = i\bar{\psi}(x)\gamma^{0} = i\psi^{\dagger}(x)$$

$$\bar{\pi}(x) = \frac{\partial \mathscr{L}}{\partial \dot{\bar{\psi}}} \equiv 0$$

Dirac Field

We can verify that the Lagrangian density is a scalar:

$$\mathscr{L} \xrightarrow{\Lambda} \mathscr{L}' = \bar{\psi}'(x')\left[i\gamma^{\mu}\partial'_{\mu} - m\right]\psi'(x')$$
$$= \psi'^{\dagger}(x')\gamma^{0}\left[i\gamma^{\mu}\partial'_{\mu} - m\right]\psi'(x')$$
$$= \left[S\psi(x)\right]^{\dagger}\gamma^{0}\left[i\gamma^{\mu}\Lambda_{\mu}{}^{\nu}\partial_{\nu} - m\right]S\psi(x)$$
$$= \psi^{\dagger}S^{\dagger}\gamma^{0}\left[i\gamma^{\mu}\Lambda_{\mu}{}^{\nu}\partial_{\nu} - m\right]S\psi$$

But

$$S^{\dagger} = \gamma^{0}S^{-1}\gamma^{0} \qquad \left(\gamma^{0}\right)^{2} = I$$

so

$$\mathscr{L}' = \psi^{\dagger}\gamma^{0}S^{-1}\gamma^{0}\gamma^{0}\left[i\gamma^{\mu}\Lambda_{\mu}{}^{\nu}\partial_{\nu} - m\right]S\psi = \bar{\psi}\left[iS^{-1}\gamma^{\mu}S\Lambda_{\mu}{}^{\nu}\partial_{\nu} - m\right]\psi$$

But the covariance of the Dirac equation defined *S*

$$S^{-1}\gamma^{\mu}S\Lambda_{\mu}{}^{\nu} = \gamma^{\nu}$$

Therefore finally

$$\mathscr{L}' = \bar{\psi}\left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi = \mathscr{L}$$

Dirac Field

Clearly, the Lagrangian density is also scale and form invariant under Lorentz transformations. Noether's theorem then yields the conserved 4-momentum of the field

$$P^\mu = \int \mathrm{d}V \ \psi^\dagger i\partial^\mu \psi \qquad \text{where} \qquad \psi^\dagger(x) = -i\pi(x)$$

and the conserved 4-tensor angular momentum of the field whose spatial parts are

$$\vec{M} = \int \mathrm{d}V \ \psi^\dagger \left[\vec{r} \times -i\vec{\nabla}\right]\psi + \int \mathrm{d}V \ \psi^\dagger \tfrac{1}{2}\vec{\hat{\sigma}}\psi$$

Note that the orbital term does not mix field components, while the spin term does. The spin term can be written as

$$\int \mathrm{d}V \ \psi^\dagger \tfrac{1}{2}\vec{\hat{\sigma}}\psi = \int \mathrm{d}V \ \sum_{a=1}^{4}\sum_{b=1}^{4}(-i\pi)_a \left(\vec{S}\right)_{ab} \psi_b$$

where

$$\vec{S} = \left(S^1, S^2, S^3\right) \qquad \left[S^j, S^k\right] = i\sum_{l=1}^{3}\varepsilon^{jkl}S^l \qquad \vec{S}^2 = \tfrac{3}{4}$$

The S matrices are a representation of the SU(2) algebra with j = 1/2. Therefore the Dirac field has intrinsic spin 1/2. In the Dirac representation we have

$$\vec{S} = \tfrac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

Dirac Field

We also notice that the Lagrangian density is form and scale invariant under a global phase transformation

$$\psi(x) \rightarrow \psi'(x) = e^{i\varepsilon}\psi(x) \qquad \bar{\psi}(x) \rightarrow \bar{\psi}'(x) = e^{-i\varepsilon}\bar{\psi}(x)$$

where ε is a real constant.

Noether's theorem then yields the conserved current

$$\partial_\mu j^\mu = 0 \quad \text{where} \quad j^\mu(x) = \bar{\psi}\gamma^\mu\psi$$

and the conserved charge

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \quad \text{where} \quad Q = \int \mathrm{d}V\, j^0(x) = \int \mathrm{d}V\, \psi^\dagger\psi$$

If we couple the Dirac field to the Maxwell field, this charge becomes proportional to the electric charge. Note that

$$\rho(x) = j^0(x) = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$$

is positive definite, as desired by Dirac. But the single particle interpretation of the Dirac equation leads to negative energies, which where reinterpreted by Dirac. In the Dirac quantum field theory, there are only particles and antiparticles with positive energies.

Maxwell Field

Consider the electric $\vec{E}(t,\vec{r})$ and magnetic $\vec{B}(t,\vec{r})$ fields in the presence of a charge density $\rho(t,\vec{r})$ and a current density $\vec{j}(t,\vec{r})$. Maxwell's equations, in the Heaviside-Lorentz system, read

$$\vec{\nabla}\cdot\vec{E}=\rho \qquad \text{Gauss' Law}$$

$$\vec{\nabla}\times\vec{B}=\vec{j}+\frac{\partial\vec{E}}{\partial t} \qquad \text{Ampere's Law}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \text{Gauss' Law}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \text{Faraday's Law}$$

Local charge conservation is expressed as the continuity equation

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j}=0$$

From the second pair of Maxwell's equations follow the existence of scalar $\phi(t,\vec{r})$ and vector $\vec{A}(t,\vec{r})$ potentials defined by

$$\vec{B}=\vec{\nabla}\times\vec{A} \qquad \vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}$$

Maxwell Field

These equations do not determine the potential uniquely. For an arbitrary function $f\left(t,\vec{r}\right)$, the transformations

$$\phi \to \phi' = \phi + \frac{\partial f}{\partial t} \qquad \vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla} f$$

leave the electric and magnetic fields unaltered. These transformations are called gauge transformations, and since the electric and magnetic fields are the observables, any theory based on ϕ and the vector potential must be gauge invariant.

We now express Maxwell's equations in terms of the potentials. The second pair of equations are satisfied automatically. The first pair becomes

$$\Box\phi - \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right) = \rho$$

$$\Box\vec{A} + \vec{\nabla}\left(\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A}\right) = \vec{j}$$

Maxwell Field

Maxwell's equations can be expressed in a covariant form with the 4-potential

$$A^{\mu}\left(x\right)\equiv\left(\phi\left(x\right),\vec{A}\left(x\right)\right)$$

and the antisymmetric tensor (and its dual)

$$F^{\mu\nu}\left(x\right)\equiv\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu} \qquad \tilde{F}^{\mu\nu}\left(x\right)\equiv\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\left(x\right)$$

Note that the other sign convention is also used. We then have

$$\nu\rightarrow \quad 0 \quad 1 \quad 2 \quad 3$$

$$F^{\mu\nu}\left(x\right)=\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\nu\rightarrow \quad 0 \quad 1 \quad 2 \quad 3$$

$$\tilde{F}^{\mu\nu}\left(x\right)=\begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

Maxwell Field

Using the current $j^{\mu}(x) \equiv \left(\rho(x), \vec{j}(x)\right)$

Maxwell's equations take the compact form

$$\partial_{\mu} F^{\mu\nu} = j^{\nu} \qquad \partial_{\mu} \tilde{F}^{\mu\nu} = 0$$

or equivalently

$$\Box A^{\mu} - \partial^{\mu}\left(\partial_{\nu} A^{\nu}\right) = j^{\mu}$$

$$\partial^{\rho} F^{\mu\nu} + \partial^{\mu} F^{\nu\rho} + \partial^{\nu} F^{\rho\mu} = 0$$

From the definition of $F^{\mu\nu}$, the second equation is a mathematical identity. The continuity equation for the current follows from the antisymmetry of $F^{\mu\nu}$

$$\partial_{\mu} j^{\mu} = 0$$

Maxwell's equations are invariant under the gauge transformation

$$A^{\mu} \to A'^{\mu} = A^{\mu} + \partial^{\mu} f \qquad \forall f(x)$$

Maxwell Field

Consider the Lagrangian density

$$\mathscr{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - j^{\mu} A_{\mu}$$

The Euler-Lagrange equation then yields the field equations:

Maxwell's equations $\quad \partial_{\mu} F^{\mu\nu} = j^{\nu}$

The canonical field conjugate is given by

$$\pi^{\mu}(x) = \frac{\partial \mathscr{L}}{\partial \dot{A}_{\mu}} = F^{\mu 0}$$

Since the Lagrangian density is scale and form invariant under Poincaré transformations, we obtain the conserved 4-vector momentum and 4 tensor angular momentum of the field. Their spatial parts are

$$\vec{P} = \int \mathrm{d}V \left[\left(-i\pi^{\mu}\right)\left(-i\vec{\nabla}\right)\left(A_{\mu}\right) \right]$$

$$\vec{M} = \int \mathrm{d}V \left[\left(-i\pi^{\mu}\right)\left(\vec{r} \times -i\vec{\nabla}\right)\left(A_{\mu}\right) \right] + \int \mathrm{d}V \; \vec{\pi} \times \vec{A}$$

Notice the orbital and the spin parts on the angular momentum.

Maxwell Field

The intrinsic spin can be written as

$$\int \mathrm{d}V \vec{\pi} \times \vec{A} = \int \mathrm{d}V \sum_{j=1}^{3} \sum_{k=1}^{3} \left(-i\pi^{j}\right) \left(\vec{S}\right)_{jk} A_k$$

where

$$\vec{S} = \left(S^1, S^2, S^3\right)$$

$$S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad S^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad S^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We obtain

$$\left[S^a, S^b\right] = i \sum_{c=1}^{3} \varepsilon^{abc} S^c \qquad \vec{S}^2 = 2I$$

We see that the *S* matrices are a representation of the SU(2) algebra with $j = 1$.

Therefore the Maxwell field has intrinsic spin 1. Upon quantization, it will represent spin 1 massless particles and antiparticles. Since the Maxwell field is real, particles (photons) are their own antiparticles.

Maxwell Field

Note that the Lagrangian density (with a current) is not gauge invariant. Under the gauge transformation

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu f \qquad \forall f(x)$$

it becomes

$$\mathscr{L} \to \mathscr{L}' = \mathscr{L} + j_\mu \partial^\mu f$$

but

$$j_\mu \partial^\mu f = \partial^\mu \left(j_\mu f \right) - f \partial^\mu j_\mu = \partial^\mu \left(j_\mu f \right)$$

is a 4-divergence. Therefore $\mathscr{L}$ and $\mathscr{L}'$ both yield Maxwell's equations.

Therefore the continuity equation for j^μ is a necessary and sufficient condition for the gauge invariance of the theory.

Proca Field

Consider the massive real vector field $Z^\mu(x)$ governed by the free field Lagrangian density

$$\mathscr{L} = -\tfrac{1}{4} G^{\mu\nu} G_{\mu\nu} + \tfrac{1}{2} M^2 Z^\mu Z_\mu \qquad G^{\mu\nu} \equiv \partial^\mu Z^\nu - \partial^\nu Z^\mu$$

The Euler-Lagrange equation then yields the field equations:

Proca equations

$$\partial_\mu G^{\mu\nu} + M^2 Z^\mu = 0 \qquad \text{or} \qquad \left(\Box + M^2\right) Z^\mu - \partial^\mu \left(\partial_\nu Z^\nu\right) = 0$$

Note that this can be obtained from the free Maxwell's equations with the substitution $\Box \to \Box + M^2$

Taking the divergence of the Proca equation yields

$$\partial_\mu Z^\mu = 0$$

which allows the Proca equation to be written as

$$\left(\Box + M^2\right) Z^\mu = 0$$

We see that each component of Z follows a Klein-Gordon equation. The canonical field conjugate is given by

$$\pi^\mu(x) = \frac{\partial \mathscr{L}}{\partial \dot{Z}_\mu} = G^{\mu 0}$$

Proca Field

Since this is the same as for the Maxwell field, the Proca field also has conserved 4-vector momentum and 4-tensor angular momentum of the field whose spatial parts are

$$\vec{P} = \int \mathrm{d}V \left[\left(-i\pi^{\mu}\right)\left(-i\vec{\nabla}\right)\left(Z_{\mu}\right)\right]$$

$$\vec{M} = \int \mathrm{d}V \left[\left(-i\pi^{\mu}\right)\left(\vec{r}\times -i\vec{\nabla}\right)\left(Z_{\mu}\right)\right] + \int \mathrm{d}V\ \vec{\pi}\times\vec{Z}$$

revealing the spin 1 nature of the Proca field. Upon quantization, it will represent massive spin 1 particles and antiparticles. Since the Proca field is real, particles (eg the Z^0) are their own antiparticles.

In the case of the free Maxwell field A^{μ}, both the Lagrangian density and the Maxwell's equations are invariant under the local gauge transformation

$$A^{\mu} \rightarrow A'^{\mu} = A^{\mu} + \partial^{\mu} f \qquad \forall f\left(x\right)$$

This is not the case for the free Proca field, where both the Lagrangian density and the Proca equation are not invariant under the local gauge transformation

$$Z^{\mu} \rightarrow Z'^{\mu} = Z^{\mu} + \partial^{\mu} f \qquad \forall f\left(x\right)$$

Note that the mass term is seen to break gauge invariance.