The Parton Model

- Naive Parton Model
- QCD Improved Parton Model

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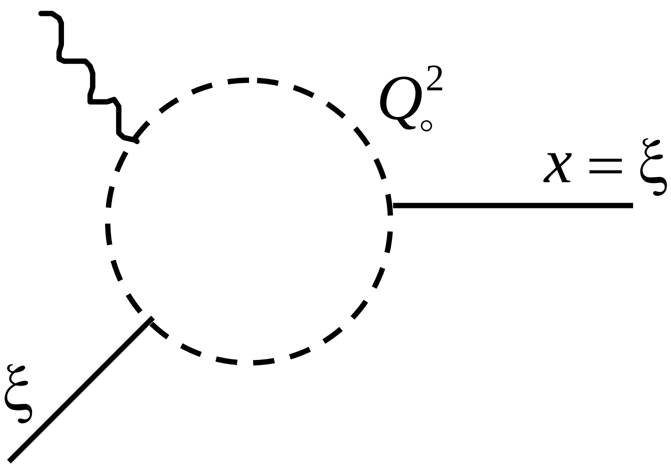
QCD Improved Parton Model

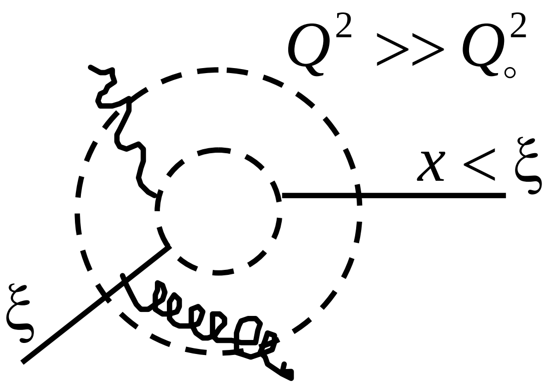
- **Scaling Violations**
- **The Altarelli-Parisi Equation**

Scaling Violations

In the naïve Parton Model, the structure functions were found to scale, which led to parton density functions independent of Q. This corresponds to probing partons in the nucleon assuming that the partons are not interacting.

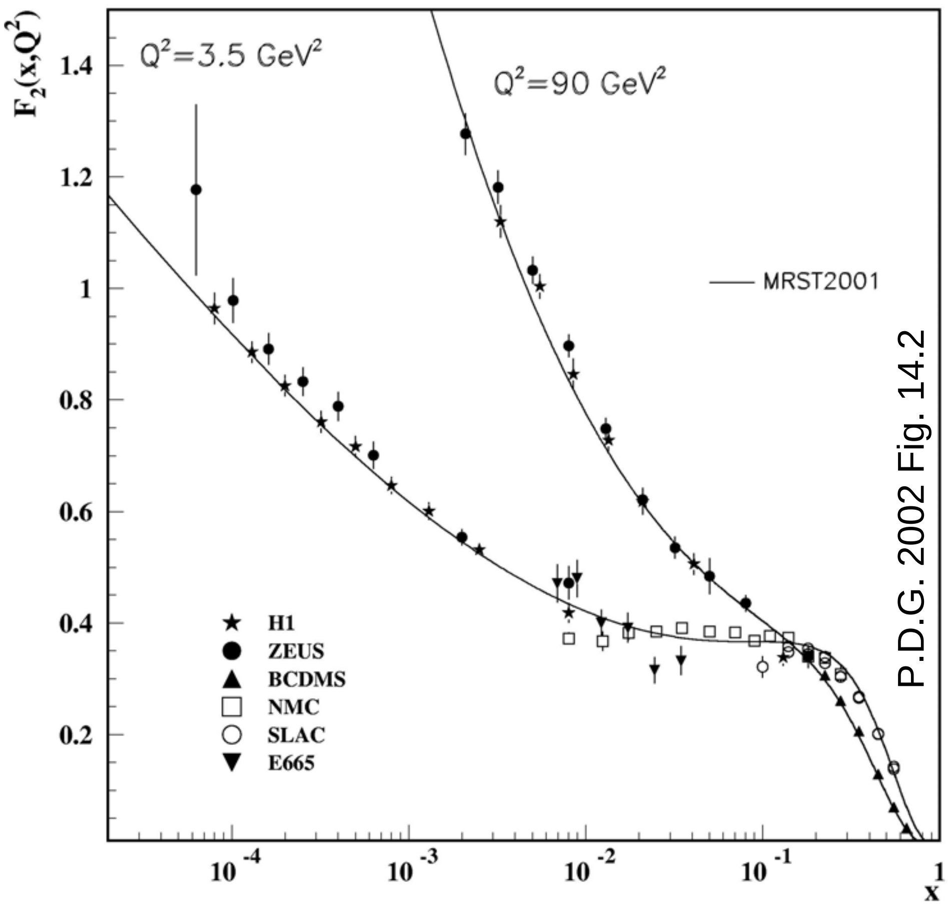
But if we turn on QCD, we expect that each parton will reveal itself surrounded by clouds of partons. More and more partons therefore share the nucleus momentum as Q increases. This can be depicted as follows in the case of a photon probe:



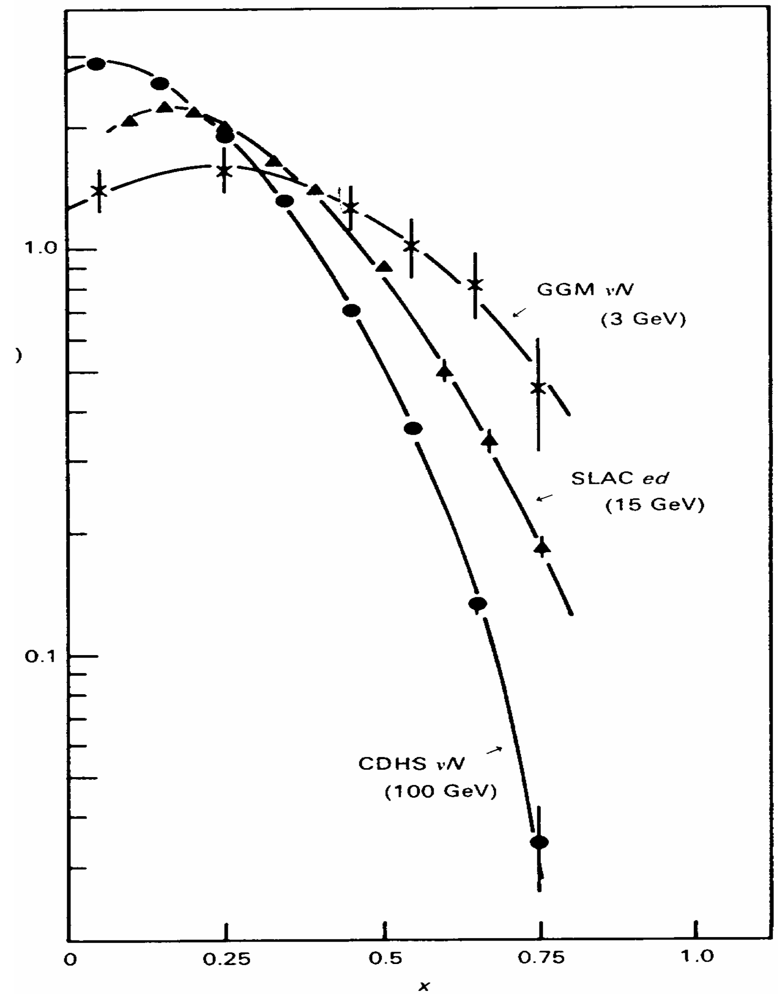


Scaling Violations

Therefore, as *Q* increases we expect the structure functions to shift towards low *x*. This is indeed found by experiment.



P.D.G. 2002 Fig. 14.2



Perkins, p. 304

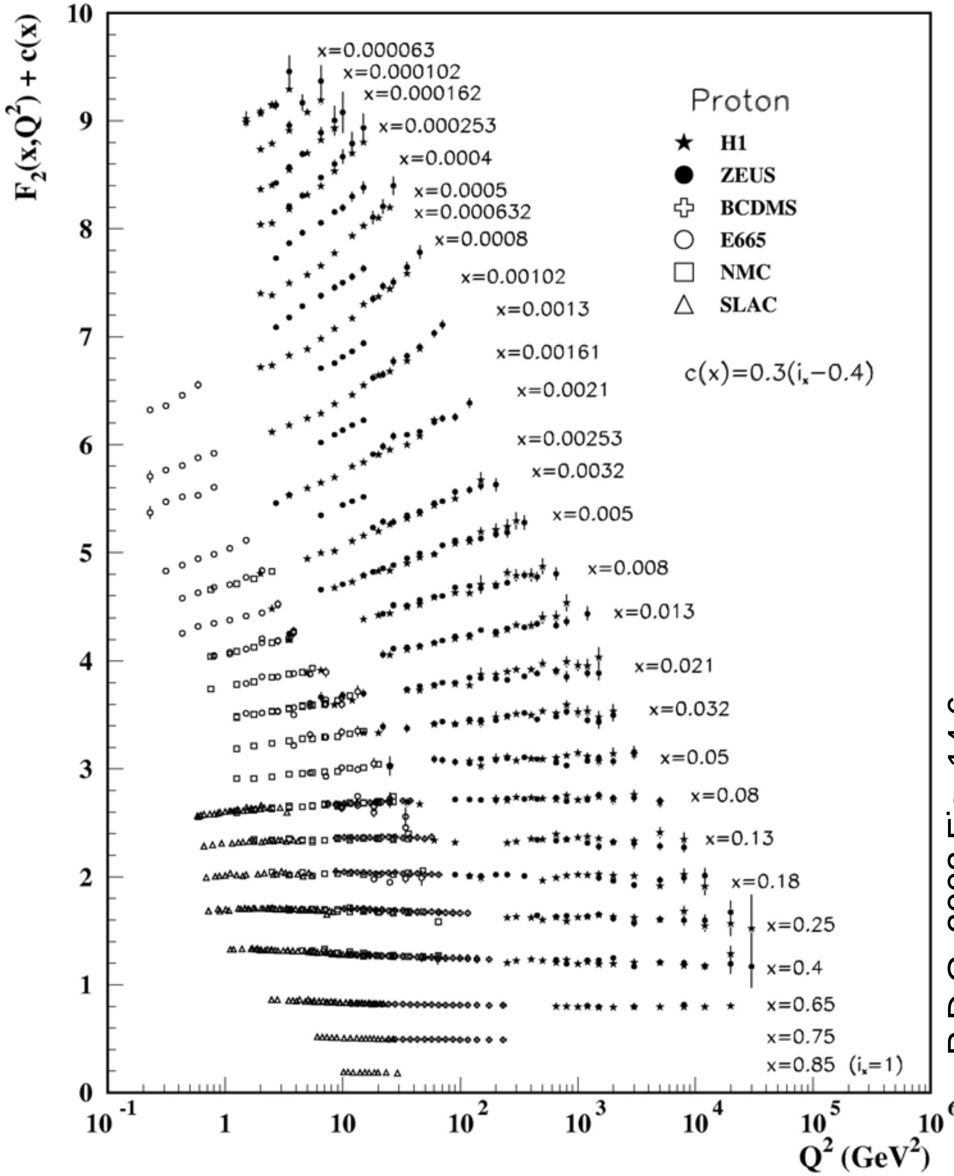
Scaling Violations

Proton structure function measured in electromagnetic scattering of positrons (H1, ZEUS), electrons (SLAC) and muons (BCDMS, E665, NMC). For plotting we have

$$c(x) = 0.3(i - 0.4)$$

$$i = 1 \quad (x = 0.85) \quad \text{to}$$

$$i = 28 \quad (x = 0.000063)$$



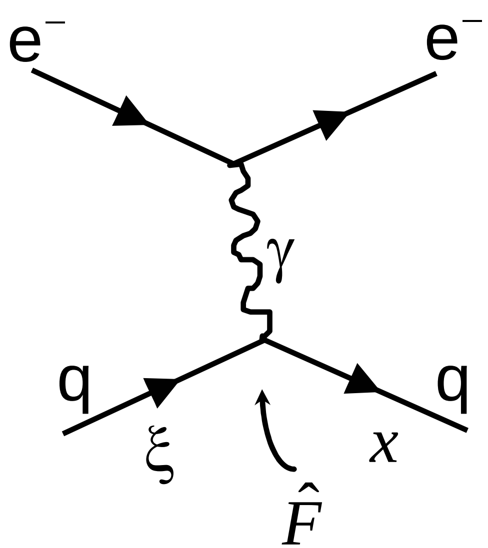
P.D.G. 2002 Fig. 14.6

The Altarelli-Parisi Equation

In the naïve parton model, the structure function scale in the limit $Q^2 \to \infty$, x fixed

$$\lim_{Q^2 \to \infty} F\left(x, Q^2\right) = F\left(x\right)$$

which led to parton density functions independent of Q. In QCD this scaling is broken by logarithms of Q which lead to a scale dependence of the p.d.f. In the naïve parton model, we considered the process



and obtained the parton structure function

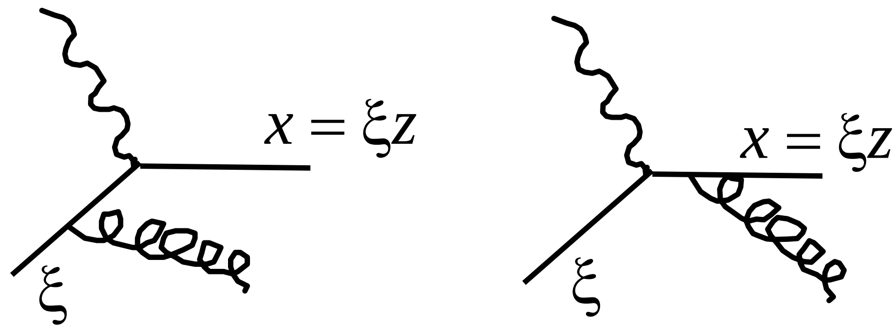
$$\hat{F}_2\left(\xi, x\right) = x e_{\mathrm{q}}^2 \delta\left(\xi - x\right)$$

We defined the parton density function $\mathrm{q}(\xi)\mathrm{d}\xi$, the probability that a parton carries a momentum fraction between ξ and $\xi + \mathrm{d}\xi$, and obtained the nucleon structure function by considering all ξ and all partons

$$F_2\left(x\right) = \sum_{\mathrm{q}} \int_0^1 \mathrm{d}\xi\ \mathrm{q}\left(\xi\right) \hat{F}_2\left(\xi, x\right) = \sum_{\mathrm{q}} e_{\mathrm{q}}^2 x \mathrm{q}\left(x\right)$$

The Altarelli-Parisi Equation

This exact scaling shows that the virtual photon sees a pointlike quark. QCD predicts that the photon should see each quark surrounded by a cloud of partons. For example, diagrams like the following must now be included



where z is the fraction of the parent parton momentum carried by the outgoing quark. The parton structure function then takes the form

$$\hat{F}_2\left(\xi, x, Q^2\right) = x\left[e_{\mathrm{q}}^2 \delta\left(\xi - x\right) + \theta\left(\xi - x\right)\int_0^1 \mathrm{d}z\ \hat{\sigma}\left(\gamma^* \mathrm{q} \to \mathrm{qg}\right)\ \delta\left(\xi z - x\right)\right]$$

where (Reya p269, Stirling p38)

$$\hat{\sigma}\left(\gamma^* \mathrm{q} \to \mathrm{qg}\right) = e_{\mathrm{q}}^2 \frac{\alpha_{\mathrm{s}}}{2\pi}\left[P_{\mathrm{qq}}\left(z\right)\ \ln\frac{Q^2}{\mu^2} + f\left(z\right) + O\left(Q^{-2}\right)\right]$$

That is, the cross section for eq $\to$ eqg is of order $\alpha\alpha_{\mathrm{s}}$.

The Altarelli-Parisi Equation

The function $P_{qq}(z)$ is uniquely defined as the coefficient of $\ln(Q^2/\mu^2)$ and can be computed by perturbative QCD. The function $f(z)$ depends on the scale μ, but does not depend on Q. It will not affect the results below.

It can be shown that (Halzen and Martin, p213)

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}p_{\mathrm{T}}^2}\left(\gamma^* \mathrm{q} \to \mathrm{qg}\right) = e_{\mathrm{q}}^2 \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{P_{\mathrm{qq}}(z)}{p_{\mathrm{T}}^2}$$

This is a measure of the probability that the quark-gluon system has p_{T}, and that the quark has a fraction z of the longitudinal momentum of the parent parton. We see that

$$\int_0^{Q^2} \frac{\mathrm{d}P_{\mathrm{T}}^2}{P_{\mathrm{T}}^2} \to \infty$$

We regularize the divergence by introducing a normalization scale μ

$$\int_{\mu^2}^{Q^2} \frac{\mathrm{d}P_{\mathrm{T}}^2}{P_{\mathrm{T}}^2} = \ln \frac{Q^2}{\mu^2}$$

This is where the logarithm comes from.

The Altarelli-Parisi Equation

Then

$$\frac{\hat{F}_2\left(\xi, x, Q^2\right)}{x e_{\mathrm{q}}^2} = \delta\left(\xi - x\right) + \theta\left(\xi - x\right)\frac{\alpha_{\mathrm{s}}}{2\pi}\ln\frac{Q^2}{\mu^2}\int_0^1 \mathrm{d}z\; P_{\mathrm{qq}}\left(z\right)\delta\left(\xi z - x\right)$$
$$+\theta\left(\xi - x\right)\frac{\alpha_{\mathrm{s}}}{2\pi}\int_0^1 \mathrm{d}z\; f\left(z\right)\delta\left(\xi z - x\right)$$

From this parton structure function, we can obtain the nucleon structure function

$$F_2\left(x, Q^2\right) = \sum_{\mathrm{q}}\int_0^1 \mathrm{d}\xi\; \mathrm{q}\left(x\right)\hat{F}_2\left(\xi, x, Q^2\right) \equiv \sum_{\mathrm{q}} e_{\mathrm{q}}^2 x \mathrm{q}\left(x, \tfrac{Q^2}{\mu^2}\right)$$

where

$$\mathrm{q}\left(x, \tfrac{Q^2}{\mu^2}\right) = \mathrm{q}\left(x\right) + \frac{\alpha_{\mathrm{s}}}{2\pi}\ln\frac{Q^2}{\mu^2}\int_x^1 \mathrm{d}\xi\;\int_0^1 \mathrm{d}z\; \mathrm{q}\left(\xi\right)P_{\mathrm{qq}}\left(z\right)\delta\left(\xi z - x\right)$$
$$= \mathrm{q}\left(x\right) + \frac{\alpha_{\mathrm{s}}}{2\pi}\ln\frac{Q^2}{\mu^2}\int_x^1 \frac{\mathrm{d}\xi}{\xi}\mathrm{q}\left(\xi\right)P_{\mathrm{qq}}\left(x/\xi\right)$$

We have omitted an additional function of *x* that comes from *f*(*z*). We see that the naïve p.d.f. is replaced by an effective p.d.f. as seen by the photon which depends on *Q*.

The Altarelli-Parisi Equation

The limit $\mu \to 0$ corresponds to gluons and quarks emitted parallel to the parent parton. It is called a collinear divergence and it is absorbed in a redefinition of the p.d.f. This procedure is called factorization, and is performed at the scale μ, that plays a similar role as the renormalization scale.

Factorization has been shown to be possible to all orders in α_s and to be process independent. This is a remarkable result that allows us to keep the parton model ideas and the p.d.f. concept.

We see that QCD predicts the Q dependence of the p.d.f., which is due to the fact that the photon resolves the fine structure of the nucleon at smaller distances.

With the proposed $\mathrm{q}(x) \to \mathrm{q}\left(x, \frac{Q^2}{\mu^2}\right)$ we finally obtain

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} \mathrm{q}\left(x, \frac{Q^2}{\mu^2}\right) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{\mathrm{d}\xi}{\xi} \mathrm{q}\left(\xi, \frac{Q^2}{\mu^2}\right) P_{\mathrm{qq}}\left(x/\xi\right)$$

Note that $\alpha_s = \alpha_s(\mu)$.

The Altarelli-Parisi Equation

We can now resum the propagator corrections by changing $\alpha_s \to \alpha_s(Q)$. Also, it is customary to write

$$\mathrm{q}\left(x, Q^2\right) \equiv \mathrm{q}\left(x, \tfrac{Q^2}{\mu^2}\right)$$

We finally obtain

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} \mathrm{q}\left(x, Q^2\right) = \frac{\alpha_s(Q)}{2\pi} \int_x^1 \frac{\mathrm{d}\xi}{\xi} \mathrm{q}\left(\xi, Q^2\right) P_{\mathrm{qq}}\left(x/\xi\right)$$

This is the Altarelli-Parisi equation. It predicts the Q evolution of the p.d.f.
The Altarelli-Parisi equation is the analogue of the coupling constant equation

$$\frac{\partial \alpha(Q)}{\partial t} = Q^2 \frac{\partial \alpha(Q)}{\partial Q^2} = \beta\left(\alpha(Q)\right)$$

We see that $P_{\mathrm{qq}}(x)$ is the analogue of $\beta(x)$ for the coupling constant. Introducing the convolution integral notation

$$g \otimes f(x) = f \otimes g(x) \equiv \int_x^1 \frac{\mathrm{d}y}{y} f(y)\, g\left(\tfrac{x}{y}\right)$$

$$= \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z\, f(y)\, g(z)\, \delta(yz - x)$$

The Altarelli-Parisi Equation

we can write in a compact form

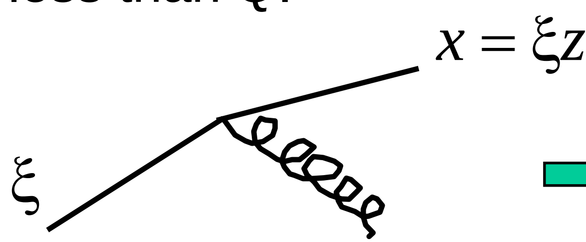
$$Q^2 \frac{\mathrm{dq}}{\mathrm{d}Q^2} = \frac{\alpha_s(Q)}{2\pi} P_{qq} \otimes \mathrm{q}$$

Strictly, this equation is only valid for differences of quark p.d.f., as we will see shortly. The P_{qq} functions can be obtained as a perturbative expansion in the running coupling constant

$$P_{qq}(z) \equiv P_{qq}(z, \alpha_s(Q)) = P_{qq}^{(0)}(z) + \frac{\alpha_s(Q)}{2\pi} P_{qq}^{(1)}(z) + \cdots$$

They are called Altarelli-Parisi splitting functions.

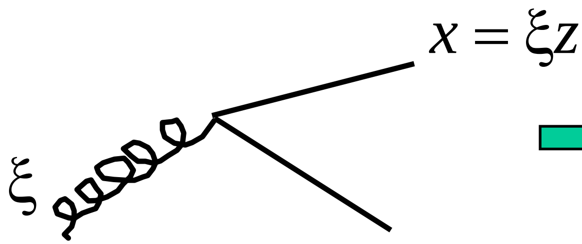
The leading-log part $P_{qq}^{(0)}(z)$ has an attractive physical interpretation. It is the probability density of finding a quark in a parent quark with a fraction z of the parent quark longitudinal momentum, and a transverse momentum much less than Q.



$$\Rightarrow P_{qq}^{(0)}(z)$$

The Altarelli-Parisi Equation

But a quark can also arise from a quark pair produced by a parent gluon

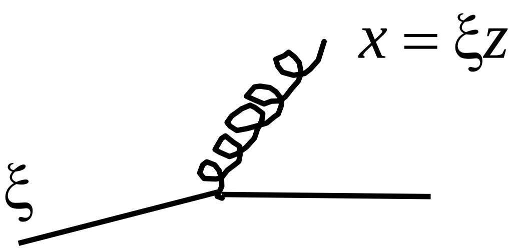


$$P_{qg}^{(0)}(z)$$

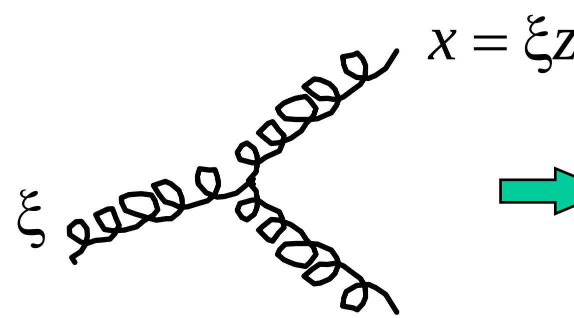
Therefore we obtain, for any flavour q of quarks, the complete evolution equation

$$Q^2 \frac{\mathrm{d}q}{\mathrm{d}Q^2} = \frac{\alpha_s(Q)}{2\pi}\left[P_{qq}^{(0)} \otimes q + P_{qg}^{(0)} \otimes g\right] + O\left(\alpha_s^2(Q)\right)$$

Similarly, we can obtain the evolution equation for the gluon p.d.f.
Obviously, the gluons cannot be probed by a photon!



$$P_{gq}^{(0)}(z)$$



$$P_{gg}^{(0)}(z)$$

The Altarelli-Parisi Equation

We obtain

$$Q^2 \frac{\mathrm{dg}}{\mathrm{d}Q^2} = \frac{\alpha_s(Q)}{2\pi}\left[P_{\mathrm{gq}}^{(0)} \otimes \sum_{i=1}^{n_f}\left(\mathrm{q}_i + \bar{\mathrm{q}}_i\right) + P_{\mathrm{gg}}^{(0)} \otimes \mathrm{g}\right] + O\left(\alpha_s^2(Q)\right)$$

where n_f is the number of quark flavours such that $m_q \ll Q \ll$ all other masses. We note that $P_{ab}^{(0)}(x)$ is the probability density to find a parton of type *a* in a parent parton of type *b* with a fraction *x* of the parent parton's longitudinal momentum and a transverse momentum much less than *Q*.

From charge conjugation we have

$$P_{\mathrm{qq}} = P_{\bar{\mathrm{q}}\bar{\mathrm{q}}} \qquad P_{\mathrm{gq}} = P_{\mathrm{g}\bar{\mathrm{q}}} \qquad P_{\mathrm{qg}} = P_{\bar{\mathrm{q}}\mathrm{g}}$$

Neglecting mass effects, that is assuming n_f massless quark flavours, we have

$$P_{\mathrm{q}_i\mathrm{g}} = P_{\mathrm{q}_j\mathrm{g}} = P_{\bar{\mathrm{q}}_j\mathrm{g}} \qquad P_{\mathrm{gq}_i} = P_{\mathrm{gq}_j} = P_{\mathrm{g}\bar{\mathrm{q}}_j}$$

Furthermore, at the lowest order we have

$$P_{\mathrm{q}_i\bar{\mathrm{q}}_j}^{(0)} = 0 \quad \forall ij \qquad P_{\mathrm{q}_i\mathrm{q}_j}^{(0)} = 0 \quad i \neq j$$

Some of these results have already implicitly been used.

The Altarelli-Parisi Equation

We can define the flavour non-singlet and singlet combinations

$$\eta \equiv \mathrm{q}_i - \mathrm{q}_j \quad \text{or} \quad \mathrm{q}_i - \bar{\mathrm{q}}_j \qquad\qquad \Sigma \equiv \sum_{i=1}^{n_\mathrm{f}} \left(\mathrm{q}_i + \bar{\mathrm{q}}_i \right)$$

We can then write the complete Altarelli-Parisi equation in matrix form

$$Q^2 \frac{\mathrm{dq}}{\mathrm{d}Q^2} = \frac{\alpha_\mathrm{s}(Q)}{2\pi} \begin{pmatrix} P_{\mathrm{qq}}^{(0)} & P_{\mathrm{qg}}^{(0)} \end{pmatrix} \otimes \begin{pmatrix} \mathrm{q} \\ \mathrm{g} \end{pmatrix} + O\left(\alpha_\mathrm{s}^2(Q)\right)$$

$$Q^2 \frac{\mathrm{dg}}{\mathrm{d}Q^2} = \frac{\alpha_\mathrm{s}(Q)}{2\pi} \begin{pmatrix} P_{\mathrm{gq}}^{(0)} & P_{\mathrm{gg}}^{(0)} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ \mathrm{g} \end{pmatrix} + O\left(\alpha_\mathrm{s}^2(Q)\right)$$

or

$$Q^2 \frac{\mathrm{d}\eta}{\mathrm{d}Q^2} = \frac{\alpha_\mathrm{s}(Q)}{2\pi} P_{\mathrm{qq}}^{(0)} \otimes \eta + O\left(\alpha_\mathrm{s}^2(Q)\right)$$

$$Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2} \begin{pmatrix} \Sigma \\ g \end{pmatrix} = \frac{\alpha_\mathrm{s}(Q)}{2\pi} \begin{pmatrix} P_{\mathrm{qq}}^{(0)} & 2n_\mathrm{f} P_{\mathrm{qg}}^{(0)} \\ P_{\mathrm{gq}}^{(0)} & P_{\mathrm{gg}}^{(0)} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix} + \begin{pmatrix} O\left(\alpha_\mathrm{s}^2(Q)\right) \\ O\left(\alpha_\mathrm{s}^2(Q)\right) \end{pmatrix}$$

The Altarelli-Parisi Equation

The splitting functions satisfy the relations

$$\int_0^1 \mathrm{d}x \; P_{\mathrm{qq}}^{(0)}(x) = 0$$

charge conservation

$$\int_0^1 \mathrm{d}x \; x\left[P_{\mathrm{qq}}^{(0)}(x) + P_{\mathrm{gq}}^{(0)}(x)\right] = 0$$

$$\int_0^1 \mathrm{d}x \; x\left[2n_{\mathrm{f}} P_{\mathrm{qg}}^{(0)}(x) + P_{\mathrm{gg}}^{(0)}(x)\right] = 0$$

total momentum conservation

Momentum conservation also predicts, for $z < 1$,

$$P_{\mathrm{qq}}^{(0)}(z) = P_{\mathrm{gq}}^{(0)}(1-z)$$

$$P_{\mathrm{qg}}^{(0)}(z) = P_{\mathrm{qg}}^{(0)}(1-z)$$

$$P_{\mathrm{gg}}^{(0)}(z) = P_{\mathrm{gg}}^{(0)}(1-z)$$

The Altarelli-Parisi Equation

Explicit calculation gives

$$P_{\mathrm{qq}}^{(0)}(z) = \frac{4}{3}\frac{1+z^2}{(1-z)_+} + 2\delta(1-z)$$

$$P_{\mathrm{qg}}^{(0)}(z) = \frac{1}{2}\left[z^2 + (1-z)^2\right]$$

$$P_{\mathrm{gq}}^{(0)}(z) = \frac{4}{3}\left[\frac{1+(1-z)^2}{z_+}\right]$$

$$P_{\mathrm{gg}}^{(0)}(z) = 6\left[\frac{z}{(1-z)_+} + \frac{(1-z)}{z} + z(1-z)\right] + \left(\frac{33-2n_{\mathrm{f}}}{6}\right)\delta(1-z)$$

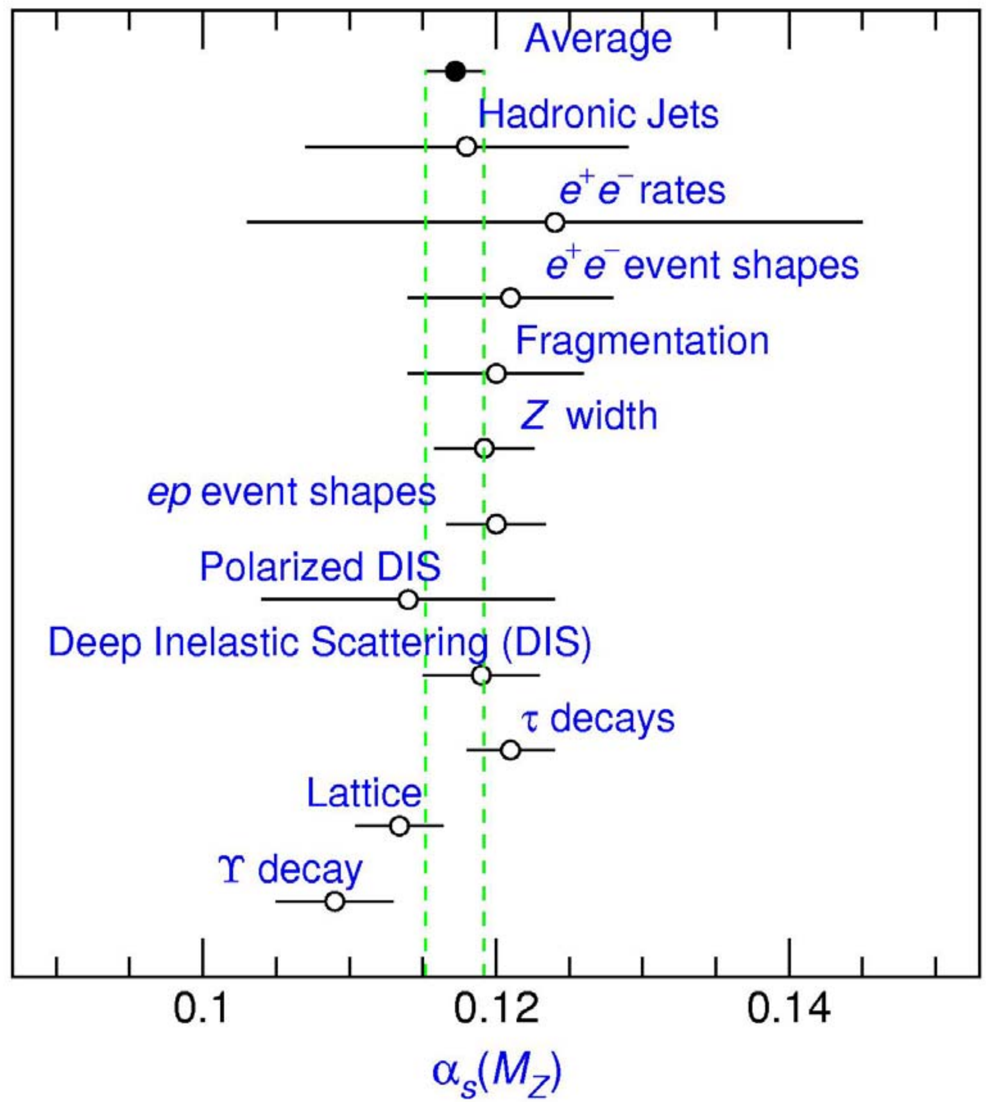
where the + prescription is defined by

$$\int_0^1 \mathrm{d}z\, f(z)\left[g(z)\right]_+ \equiv \int_0^1 \mathrm{d}z\, \left[f(z) - f(z_\circ)\right] g(z) \quad \text{where} \quad g(z_\circ) = 0$$

The Altarelli-Parisi Equation

Note that the splitting functions are positive definite for $z < 1$, which is required by their interpretation as probability densities.

Since the p.d.f. evolution is governed by $\alpha_s(Q)$, a fit to the structure function data at different x and Q yields a measurement of α_s.



P.D.G. 2002 Fig 9.1