# Introduction to Gauge Theories

- Basics of SU(*n*)
- Classical Fields
- U(1) Gauge Invariance
- SU(*n*) Gauge Invariance
- **The Standard Model**

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# The Standard Model

- **Chirality**
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- **Higgs Mechanism**
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## Chirality

The chirality of a fermionic matter field $\psi$ is defined as the eigenvalue of $\gamma^5$. There are two possible eigenvalues

$+1 \Rightarrow$ positive chirality

$-1 \Rightarrow$ negative chirality

The chirality of the corresponding adjoint spinor field $\overline{\Psi} \equiv \psi^\dagger \gamma^0$ is the same by definition.

Spinor space can be divided in the corresponding two chiral subspaces by the use of the following projection operators

$$P_R \equiv \tfrac{1}{2}\left(1+\gamma^5\right) \qquad \text{positive chirality}$$

$$P_L \equiv \tfrac{1}{2}\left(1-\gamma^5\right) \qquad \text{negative chirality}$$

Indeed, using $\left(\gamma^5\right)^2 = I$

we obtain

$$P_R + P_L = I \qquad P_R P_L = P_L P_R = 0$$

$$P_R^2 = P_R \qquad P_L^2 = P_L$$

## Chirality

Also using $\gamma^0\gamma^5\gamma^0 = -\left(\gamma^5\right)^\dagger = -\gamma^5$ and $\left(\gamma^0\right)^2 = I$

we obtain

$$P_R^\dagger = P_R \qquad P_L^\dagger = P_L$$
$$P_R\gamma^0 = \gamma^0 P_L \qquad P_L\gamma^0 = \gamma^0 P_R$$

Hence we write

$$\gamma^5\psi_R = \psi_R \qquad \bar\psi_R\gamma^5 = -\bar\psi_R$$
$$\gamma^5\psi_L = -\psi_L \qquad \bar\psi_L\gamma^5 = \bar\psi_L$$
$$\psi_R + \psi_L = \psi \qquad \bar\psi_R + \bar\psi_L = \bar\psi$$

where

$$\psi_R \equiv P_R\psi \quad \bar\psi_R = \bar\psi P_L = \overline{\left(\psi_R\right)} \neq \left(\bar\psi\right)_R \qquad + \text{ chirality}$$
$$\psi_L \equiv P_L\psi \quad \bar\psi_L = \bar\psi P_R = \overline{\left(\psi_L\right)} \neq \left(\bar\psi\right)_L \qquad - \text{ chirality}$$

we note the important result, using $\left[\gamma^\mu, \gamma^5\right]_+ = 0$

$$\bar\psi\psi = \bar\psi_L\psi_R + \bar\psi_R\psi_L$$
$$\bar\psi\gamma^\mu\psi = \bar\psi_L\gamma^\mu\psi_L + \bar\psi_R\gamma^\mu\psi_R$$

## Chirality

The subscripts R and L are used because, in the limit of massless Dirac fields, we have

helicity $=$ chirality for massless Dirac spinors $\psi$

$+1$ chirality $\Rightarrow$ Right handed helicity

$-1$ chirality $\Rightarrow$ Left handed helicity

helicity $=$ $-$chirality for massless Dirac adjoint spinors $\bar{\psi}$

$+1$ chirality $\Rightarrow$ Left handed helicity

$-1$ chirality $\Rightarrow$ Right handed helicity

## U(1) × SU(2)$_L$

We now construct a Lagrangian density that is invariant under a U(1) × SU(2)$_L$ gauge transformation, which features only one massless neutral gauge field. We only treat one family of leptons, namely the neutrino and the electron. We adopt the following notations:

$$\nu(x) \equiv \nu_e \quad \text{spinor}$$

$$e(x) \equiv e^- \quad \text{spinor}$$

$$l_L(x) \equiv \begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}$$

$$l_R(x) \in \{\nu_R(x), e_R(x)\}$$

$$\psi(x) = \begin{pmatrix} \nu(x) \\ e(x) \end{pmatrix}$$

## ■ U(1) × SU(2)$_L$

Consider the Lagrangian density, assuming that neutrinos are Dirac particles,

$$\mathscr{L} = \bar{\nu}\left(i\gamma^{\mu}\partial_{\mu} - m_{\nu}\right)\nu + \bar{e}\left(i\gamma^{\mu}\partial_{\mu} - m_{e}\right)e$$

$$= \bar{l}_{L} i\gamma^{\mu}\partial_{\mu} l_{L} + \sum_{l} \bar{l}_{R} i\gamma^{\mu}\partial_{\mu} l_{R} - m_{\nu}\left(\bar{\nu}_{L}\nu_{R} + \bar{\nu}_{R}\nu_{L}\right) - m_{e}\left(\bar{e}_{L}e_{R} + \bar{e}_{R}e_{L}\right)$$

In order to obtain the electroweak Lagrangian density, we wish to impose to the lepton fields a gauge invariance for each one of the two forces. We therefore try

$$l_{L} \xrightarrow{U} l'_{L} = U l_{L} \qquad l_{R} \xrightarrow{U} l'_{R} = U l_{R}$$

where

$$U = U_1 \otimes U_2$$

$$U_1 \equiv \exp\left[-i t^0 \varepsilon^0\left(x\right)\right]$$ $t^0$ is the weak hypercharge of doublets or singlets

$$U_2 \equiv \begin{cases} \exp\left[-i T^a \varepsilon^a\left(x\right)\right] & \text{on SU}\left(2\right)_L \text{ doublets} \quad T^a = \tfrac{1}{2}\sigma^a \\ I & \text{on SU}\left(2\right)_L \text{ singlets} \end{cases}$$

We immediately see that the mass terms are not gauge invariant. Therefore the lepton masses must be generated by a spontaneous symmetry hiding of SU(2)$_L$. Furthermore, the SU(2)$_L$ gauge fields must be massive, given the short range of the weak force.

## U(1) × SU(2)$_L$

We introduce a doublet of complex scalar fields, and its conjugate

$$\varphi \equiv \begin{pmatrix} \varphi^{+} \\ \varphi^{0} \end{pmatrix} \qquad \widehat{\varphi} \equiv i\sigma_2 \varphi^{*} = \begin{pmatrix} \varphi^{0*} \\ -\varphi^{-} \end{pmatrix} \qquad \varphi^{-} = \left(\varphi^{+}\right)^{*}$$

and the potential
$$\mathscr{V}(\varphi) = -\mu^2 \varphi^\dagger \varphi + \lambda \left(\varphi^\dagger \varphi\right)^2 \qquad \lambda > 0$$

In order to generate the lepton masses after symmetry hiding, we introduce the following Yukawa coupling terms

$$\mathscr{L}_{\text{Yukawa}} = c_e \left[ \bar{l}_L \varphi e_R + \bar{e}_R \varphi^\dagger l_L \right] + c_\nu \left[ \bar{l}_L \widehat{\varphi} \nu_R + \bar{\nu}_R \widehat{\varphi}^\dagger l_L \right]$$

where $c_e$ and $c_\nu$ are real constants. Note that these terms are Lorentz scalars, and are invariant under the U(1) × SU(2)$_L$ gauge transformations

$$l_L \xrightarrow{U} l_L' = U l_L \qquad l_R \xrightarrow{U} l_R' = U l_R$$
$$\varphi \xrightarrow{U} \varphi' = U \varphi \qquad \widehat{\varphi} \xrightarrow{U} \widehat{\varphi}' = U \widehat{\varphi}$$

if
$$t^0(l_L) = t^0(\varphi) + t^0(e_R) = t^0(\widehat{\varphi}) + t^0(\nu_R)$$

## $U(1) \times SU(2)_L$

Members of a given weak isospin doublet have the same hypercharge. Given that $q(\nu) = 0$ and $q(e) = -1$, and noting that

$$q(\nu) - q(e) = t^3(\nu_L) - t^3(e_L) = 1$$

then we must have $Q = T^3 + \alpha t^0$, where $Q$ is the electric charge matrix (for $SU(2)_L$ doublets) or number (for $SU(2)_L$ singlets) in units of e. We find

$$t^0(\nu_R) = 0$$

$$\left.\begin{aligned} &q(\varphi^+) = 1 &&\Rightarrow q(\varphi^-) = -1 \\ &q(\varphi^0) = 0 &&\Rightarrow q(\varphi^{0*} = \varphi^0) = 0 \end{aligned}\right\} \text{ hence the notation}$$

The value of $\alpha$ is conventional. We choose it to be 1. This now fixes all the $t^0$'s.

## U(1) × SU(2)$_L$

We can therefore attempt the following Lagrangian density

$$\mathscr{L} = \bar{l}_L i\gamma^\mu D_\mu l_L + \sum_l \bar{l}_R i\gamma^\mu D_\mu l_R + \left(D_\mu\varphi\right)^\dagger \left(D^\mu\varphi\right) - \mathscr{V}\left(\varphi\right)$$

$$-\tfrac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \tfrac{1}{4} H_{\mu\nu} H^{\mu\nu} + \mathscr{L}_{\text{Yukawa}}$$

where

$$D_\mu \equiv \begin{cases} \partial_\mu + ig' t^0 W^0_\mu + igT^a W^a_\mu & \text{for SU(2)}_L \text{ doublets} \\ \partial_\mu + ig' t^0 W^0_\mu & \text{for SU(2)}_L \text{ singlets} \end{cases}$$

$g'$ and $g$ are real coupling constants

$t^0$ is the weak hypercharge

$W^0_\mu(x)$ is the gauge field associated to the U(1) gauge

$W^a_\mu(x)$ are the gauge fields associated to the SU(2)$_L$ gauge

$$H_{\mu\nu} \equiv \partial_\mu W^0_\nu - \partial_\nu W^0_\mu$$

$$G^a_{\mu\nu} \equiv W^a_{\mu\nu} - g\varepsilon^{abc} W^b_\mu W^c_\nu$$

$$W^a_{\mu\nu} \equiv \partial_\mu W^a_\nu - \partial_\nu W^a_\mu$$

## U(1) × SU(2)$_L$

and where $\mathscr{L}$ is invariant under the U(1) × SU(2)$_L$ gauge transformation

$$l_L \xrightarrow{U} l'_L = U l_L \qquad l_R \xrightarrow{U} l'_R = U l_R$$
$$\varphi \xrightarrow{U} \varphi' = U\varphi \qquad \widehat{\varphi} \xrightarrow{U} \widehat{\varphi}' = U\widehat{\varphi}$$
$$W_\mu^0 \xrightarrow{U} W_\mu'^0 = W_\mu^0 + \tfrac{1}{g'}\partial_\mu \varepsilon^0(x)$$
$$T^a W_\mu^a \xrightarrow{U} T^a W_\mu'^a = U T^a W_\mu^a U^{-1} + \tfrac{1}{g}\partial_\mu T^a \varepsilon^a(x)$$

We require that there be only one massless neutral gauge field, the electromagnetic field $A_\mu(x)$ . In general, it will be a linear combination of $W_\mu^0$ and $W_\mu^3$.

$$\begin{pmatrix} W_\mu^3 \\ W_\mu^0 \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

or

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_\mu^0 \\ W_\mu^3 \end{pmatrix}$$

## ■ U(1) × SU(2)$_L$

where $\theta_W$ is the Weinberg angle, to be determined by experiments, and $Z_\mu$ will be a massive neutral gauge field. Using

$$W_\mu^\pm \equiv \tfrac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \qquad T^\pm \equiv T^1 \pm iT^2$$

we can rewrite the covariant derivative in the form

$$\begin{aligned} D_\mu = \partial_\mu &+ \frac{ig}{\sqrt{2}}\left(T^+W_\mu^+ + T^-W_\mu^-\right) \\ &+ i\left(gT^3\sin\theta_W + g't^0\cos\theta_W\right)A_\mu \\ &+ i\left(gT^3\cos\theta_W - g't^0\sin\theta_W\right)Z_\mu \end{aligned}$$

Since we want $A_\mu$ to be the electromagnetic gauge field, we require

$$gT^3\sin\theta_W + g't^0\cos\theta_W = eQ$$

where, as before, $Q$ is the electric charge matrix (for SU(2)$_L$ doublets) or number (for SU(2)$_L$ singlets) in units of $e$.

## U(1) × SU(2)$_L$

Since $$Q = T^3 + t^0$$

then we get $$e = g \sin\theta_W = g' \cos\theta_W$$

which yields $$\tan\theta_W = \frac{g'}{g} \qquad e = \frac{gg'}{\sqrt{g^2 + g'^2}}$$

The covariant derivative takes the form

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}}\left(T^+ W_\mu^+ + T^- W_\mu^-\right) + ieQA_\mu + ieQ'Z_\mu$$

where the neutral charge matrix is given by

$$gT^3 \cos\theta_W - g't^0 \sin\theta_W = eQ'$$

or

$$Q' = T^3 \cot\theta_W - t^0 \tan\theta_W$$

## U(1) × SU(2)$_L$

We can summarize the quantum numbers as follows:

| | $t$ | $t^3$ | $t^0$ | $q$ | $q' \sin 2\theta_W$ |
|---|---|---|---|---|---|
| $\nu_L$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $1$ |
| $e_L$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-1+2\sin^2\theta_W$ |
| $\nu_R$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_R$ | $0$ | $0$ | $-1$ | $-1$ | $2\sin^2\theta_W$ |
| $\varphi^+$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $1-2\sin^2\theta_W$ |
| $\varphi^0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ |
| $\varphi^0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $1$ |
| $-\varphi^-$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-1+2\sin^2\theta_W$ |

(The doublets $\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}$, $\begin{pmatrix}\varphi^+\\ \varphi^0\end{pmatrix}$, $\begin{pmatrix}\varphi^0\\ -\varphi^-\end{pmatrix}$ are grouped in parentheses.)

$t$, $t^3$ is for weak isospin and $t^0$ is for weak hypercharge

## U(1) × SU(2)$_L$

Nothing that

$$T^+W_\mu^+ + T^-W_\mu^- = \begin{pmatrix} 0 & W_\mu^+ \\ W_\mu^- & 0 \end{pmatrix}$$

we can also write

$$D_\mu = \begin{pmatrix} \partial_\mu + ieq_1A_\mu + ieq_1'Z_\mu & \frac{1}{\sqrt{2}}igW_\mu^+ \\ \frac{1}{\sqrt{2}}igW_\mu^- & \partial_\mu + ieq_2A_\mu + ieq_2'Z_\mu \end{pmatrix}$$

where $q_j$ and $q_j'$ are for the corresponding multiplet member

## Higgs Mechanism

The mass terms are generated by the Higgs mechanism, which leads to a renormalizable quantum field theory. As before, we consider the case $\mu^2 > 0$ where we choose the equilibrium point

$$\varphi_0 = \begin{pmatrix} 0 \\ \frac{\mathrm{v}}{\sqrt{2}} \end{pmatrix} \qquad \mathrm{v}^2 \equiv \frac{\mu^2}{\lambda} > 0$$

We then expand the scalar field about this equilibrium

$$\varphi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_1(x) + i\eta_2(x) \\ \mathrm{v} + \sigma(x) + i\eta_3(x) \end{pmatrix}$$

The $\eta_j$ fields are the unphysical would-be Goldstone bosons, and can be eliminated by going to the unitary gauge

$$\varphi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathrm{v} + \sigma(x) \end{pmatrix}$$

Gauging away the charged $\varphi^+$ will insure $A_\mu$ to be massless.

## Higgs Mechanism

After symmetry hiding, the Lagrangian density can be written as

$$\mathscr{L} = \mathscr{L}_{\text{D}} + \mathscr{L}_{\text{V}} + \mathscr{L}_{\text{H}} + \mathscr{L}_{\text{DV}} + \mathscr{L}'_{\text{H}}$$

where

$$\begin{aligned} \mathscr{L}_{\text{D}} &= \text{free Dirac spinor matter fields} \\ \mathscr{L}_{\text{V}} &= \text{pure gauge fields} \\ \mathscr{L}_{\text{H}} &= \text{pure Higgs field} \\ \mathscr{L}_{\text{DV}} &= \text{interaction between matter fields and gauge fields} \\ \mathscr{L}'_{\text{H}} &= \text{interaction between Higgs fields and other fields} \end{aligned}$$

## Masses

We first concentrate on obtaining the mass terms. We first obtain

$$D_\mu \varphi = D_\mu \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathrm{v}+\sigma \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \tfrac{1}{\sqrt{2}} igW_\mu^+ \left(\mathrm{v}+\sigma\right) \\ \partial_\mu \sigma + \left(ieq_\sigma A_\mu + ieq'_\sigma Z_\mu\right)\left(\mathrm{v}+\sigma\right) \end{pmatrix}$$

Extracting the kinetic and mass terms from

$$\left(D_\mu \varphi\right)^\dagger \left(D^\mu \varphi\right) \qquad \text{and} \qquad -\mathscr{V}\left(\varphi\right)$$

yields $\tfrac{1}{2}\Big[\tfrac{1}{2} g^2 \mathrm{v}^2 W_\mu^+ W^{-\mu} + \left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right)$

$$+\mathrm{v}^2 e^2 \left(q_\sigma A_\mu + q'_\sigma Z_\mu\right)\left(q_\sigma A^\mu + q'_\sigma Z^\mu\right)\Big] - \mu^2 \sigma^2$$

Since $q_\sigma = 0$, we obtain no mass term for the photon field, as desired

$$M_A = 0$$

## Masses

We obtain the kinetic and mass terms

$$M_W^2 W_\mu^+ W^{-\mu} + \tfrac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \tfrac{1}{2} M_H^2 \sigma^2 + \tfrac{1}{2} M_Z^2 Z_\mu Z^\mu$$

where

$$M_W^2 = \tfrac{1}{4} g^2 v^2$$

$$M_Z^2 = e^2 v^2 q_\sigma'^2 = \frac{g^2 v^2}{4\cos^2\theta_W} \qquad \frac{M_Z}{M_W} = \frac{1}{\cos\theta_W} > 1$$

$$M_H^2 = 2\mu^2 = 2v^2\lambda$$

for later use, we note that then

$$D_\mu \varphi = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \tfrac{1}{\sqrt{2}} i g W_\mu^+ \left(v + \sigma\right) \\ \partial_\mu \sigma - \dfrac{ieZ_\mu}{\sin 2\theta_W}\left(v + \sigma\right) \end{pmatrix}$$

## Masses

The lepton mass terms are extracted from the Yukawa coupling terms, which now read

$$\tfrac{1}{\sqrt{2}}\left(\mathrm{v}+\sigma\right)\left[c_{\mathrm{e}}\left(\bar{e}_{\mathrm{L}}e_{\mathrm{R}}+\bar{e}_{\mathrm{R}}e_{\mathrm{L}}\right)+c_{\nu}\left(\bar{\nu}_{\mathrm{L}}\nu_{\mathrm{R}}+\bar{\nu}_{\mathrm{R}}\nu_{\mathrm{L}}\right)\right]=\tfrac{1}{\sqrt{2}}\left(\mathrm{v}+\sigma\right)\left[c_{\mathrm{e}}\bar{e}e+c_{\nu}\bar{\nu}\nu\right]$$

yielding the mass terms

$$-m_{\nu}\bar{\nu}\nu-m_{\mathrm{e}}\bar{e}e$$

where

$$m_{\nu}=-\tfrac{1}{\sqrt{2}}\mathrm{v}c_{\nu}$$

$$m_{\mathrm{e}}=-\tfrac{1}{\sqrt{2}}\mathrm{v}c_{\mathrm{e}}$$

The corresponding kinetic terms are readily extracted yielding the free spinor Lagrangian density

$$\begin{aligned}\mathscr{L}_{\mathrm{D}}&=\bar{l}_{\mathrm{L}}i\gamma^{\mu}\partial_{\mu}l_{\mathrm{L}}+\sum_{l}\bar{l}_{\mathrm{R}}i\gamma^{\mu}\partial_{\mu}l_{\mathrm{R}}-m_{\nu}\bar{\nu}\nu-m_{\mathrm{e}}\bar{e}e\\&=\bar{e}\left[i\gamma^{\mu}\partial_{\mu}-m_{\mathrm{e}}\right]e+\bar{\nu}\left[i\gamma^{\mu}\partial_{\mu}-m_{\nu}\right]\nu\\&=\bar{\psi}\left[i\gamma^{\mu}\partial_{\mu}-M_{l}\right]\psi\end{aligned}$$

where we have the lepton mass matrix

$$M_{l}=\begin{pmatrix}m_{\nu}&0\\0&m_{\mathrm{e}}\end{pmatrix}$$

## Interactions

Using
$$\begin{pmatrix} W_\mu^3 \\ W_\mu^0 \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_{\mathrm{W}} & \sin\theta_{\mathrm{W}} \\ -\sin\theta_{\mathrm{W}} & \cos\theta_{\mathrm{W}} \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

the pure gauge field Lagrangian density takes the form

$$\mathscr{L}_{\mathrm{V}} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \tfrac{1}{2} M_{\mathrm{Z}}^2 Z_\mu Z^\mu - \tfrac{1}{2} W_{\mu\nu}^{+} W^{-\mu\nu} + M_{\mathrm{W}}^2 W_\mu^{+} W^{-\mu} + \mathscr{L}_{\mathrm{VVV}} + \mathscr{L}_{\mathrm{VVVV}}$$

where
$$F_{\mu\nu} \equiv A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$$
$$W_{\mu\nu}^{\pm} \equiv \tfrac{1}{\sqrt{2}}\left(W_{\mu\nu}^1 \mp i W_{\mu\nu}^2\right) = \partial_\mu W_\nu^{\pm} - \partial_\nu W_\mu^{\pm}$$

and the vector boson gauge fields triple and quartic couplings

$$\mathscr{L}_{\mathrm{VVV}} = \tfrac{1}{2} g \varepsilon^{abc} W_{\mu\nu}^a W^{b\mu} W^{c\nu}$$
$$\mathscr{L}_{\mathrm{VVVV}} = -\tfrac{1}{4} g^2 \varepsilon^{abc} \varepsilon^{ars} W_\mu^b W_\nu^c W^{r\mu} W^{s\nu}$$

## Interactions

The interaction between the matter fields and the gauge fields is readily extracted to be

$$\mathscr{L}_{\mathrm{DV}} = \mathscr{L}_{\mathrm{em}} + \mathscr{L}_{\mathrm{neutral}} + \mathscr{L}_{\mathrm{charged}}$$

where

$$\mathscr{L}_{\mathrm{em}} = -eA_\mu \bar{l}_{\mathrm{L}} \gamma^\mu Q l_{\mathrm{L}} - \sum_l eA_\mu \bar{l}_{\mathrm{R}} \gamma^\mu Q l_{\mathrm{R}} = -eA_\mu \bar{\psi} \gamma^\mu Q \psi$$

$$\mathscr{L}_{\mathrm{neutral}} = -eZ_\mu \bar{l}_{\mathrm{L}} \gamma^\mu Q' l_{\mathrm{L}} - \sum_l eZ_\mu \bar{l}_{\mathrm{R}} \gamma^\mu Q' l_{\mathrm{R}} = -eZ_\mu \bar{\psi} \gamma^\mu Q' \psi$$

$$\begin{aligned}
\mathscr{L}_{\mathrm{charged}} &= -\tfrac{1}{\sqrt{2}} g W_\mu^+ \bar{\nu}_{\mathrm{L}} \gamma^\mu e_{\mathrm{L}} - \tfrac{1}{\sqrt{2}} g W_\mu^- \bar{e}_{\mathrm{L}} \gamma^\mu \nu_{\mathrm{L}} \\
&= -\frac{e}{\sqrt{2}\sin\theta_{\mathrm{W}}} \left[ W_\mu^+ \bar{\nu} \gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right) e + W_\mu^- \bar{e} \gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right) \nu \right] \\
&= -\frac{e}{\sqrt{2}\sin\theta_{\mathrm{W}}} \bar{\psi} \left( T^+ W_\mu^+ + T^- W_\mu^- \right) \gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right) \psi
\end{aligned}$$

## Interactions

Note that this can be also written as

$$\mathscr{L}_{\text{em}} = -A_\mu j^\mu_{\text{em}}$$

$$\mathscr{L}_{\text{neutral}} = -Z_\mu j^\mu_{\text{neutral}}$$

$$\mathscr{L}_{\text{charged}} = -W^+_\mu j^{+\mu}_{\text{charged}} - W^-_\mu j^{-\mu}_{\text{charged}}$$

where

$$j^\mu_{\text{em}} = e\bar{\psi}\gamma^\mu Q\psi$$

$$j^\mu_{\text{neutral}} = e\bar{\psi}\gamma^\mu Q'\psi$$

$$j^{\pm\mu}_{\text{charged}} = \frac{e}{\sqrt{2}\sin\theta_{\text{W}}}\bar{\psi}T^\pm\gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right)\psi$$

## Interactions

The interaction with the Higgs field is extracted

$$\mathscr{L}'_{\mathrm{H}} = \mathscr{L}_{\mathrm{HD}} + \mathscr{L}_{\mathrm{HV}}$$

where $\mathscr{L}_{\mathrm{HD}}$ is extracted from the Yukawa couplings

$$\mathscr{L}_{\mathrm{HD}} = -\tfrac{1}{\sqrt{2}} c_{\mathrm{e}} \sigma \bar{e} e - \tfrac{1}{\sqrt{2}} c_{\nu} \sigma \bar{\nu} \nu = -\tfrac{1}{\mathrm{v}} \sigma \bar{\psi} M_l \psi$$

and $\mathscr{L}_{\mathrm{HV}}$ is extracted from $\left(D_\mu \varphi\right)^\dagger \left(D^\mu \varphi\right)$

$$\mathscr{L}_{\mathrm{HV}} = \left(M_{\mathrm{W}}^2 W_\mu^+ W^{-\mu} + \tfrac{1}{2} M_{\mathrm{Z}}^2 Z_\mu Z^\mu\right) \tfrac{2}{\mathrm{v}} \sigma \left(1 + \tfrac{1}{2\mathrm{v}} \sigma\right)$$

Finally, the Higgs self interaction is obtained from $-\mathscr{V}(\varphi)$ and is included in the pure Higgs Lagrangian density

$$\mathscr{L}_{\mathrm{H}} = \tfrac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \tfrac{1}{2} M_{\mathrm{H}}^2 \sigma^2 - \tfrac{1}{2\mathrm{v}} M_{\mathrm{H}}^2 \sigma^3 \left(1 + \tfrac{1}{4\mathrm{v}} \sigma\right)$$

## Quarks and Families

We now wish to include quarks in our classical field electroweak theory, and allow for quarks and lepton family replication. Most of the development is straightforward, except for the emergence of a difference between gauge eigenstates and mass eigenstates.

First we need to extend our notation to family matrices, here shown for two families:

Leptons:

$$\nu(x) = \begin{pmatrix} \nu_{\mathrm{e}}(x) \\ \nu_{\mu}(x) \end{pmatrix} \qquad e(x) = \begin{pmatrix} \breve{e}(x) \\ \mu(x) \end{pmatrix}$$

$$l_{\mathrm{L}}(x) = \begin{pmatrix} \begin{pmatrix} \nu_{\mathrm{eL}}(x) \\ \breve{e}_{\mathrm{L}}(x) \end{pmatrix} \\ \begin{pmatrix} \nu_{\mu\mathrm{L}}(x) \\ \mu_{\mathrm{L}}(x) \end{pmatrix} \end{pmatrix}$$

$$l_{\mathrm{R}}(x) \in \left\{ \begin{pmatrix} \nu_{\mathrm{eR}}(x) \\ \nu_{\mu\mathrm{R}}(x) \end{pmatrix}, \begin{pmatrix} \breve{e}_{\mathrm{R}}(x) \\ \mu_{\mathrm{R}}(x) \end{pmatrix} \right\}$$

$$l(x) = \begin{pmatrix} \begin{pmatrix} \nu_{\mathrm{e}}(x) \\ \breve{e}(x) \end{pmatrix} \\ \begin{pmatrix} \nu_{\mu}(x) \\ \mu(x) \end{pmatrix} \end{pmatrix}$$

## Quarks and Families

Quarks:

Note that quarks come in 3 colours, that is each one of the quark symbols on the right is a matrix in colour space, for example

$$s(x) \equiv \begin{pmatrix} s^{red}(x) \\ s^{green}(x) \\ s^{blue}(x) \end{pmatrix}$$

$$u(x) = \begin{pmatrix} \breve{u}(x) \\ c(x) \end{pmatrix} \qquad d(x) = \begin{pmatrix} \breve{d}(x) \\ s(x) \end{pmatrix}$$

$$q_L(x) = \begin{pmatrix} \begin{pmatrix} \breve{u}_L(x) \\ \breve{d}_L(x) \end{pmatrix} \\ \begin{pmatrix} c_L(x) \\ s_L(x) \end{pmatrix} \end{pmatrix}$$

$$q_R(x) \in \left\{ \begin{pmatrix} \breve{u}_R(x) \\ c_R(x) \end{pmatrix}, \begin{pmatrix} \breve{d}_R(x) \\ s_R(x) \end{pmatrix} \right\}$$

$$q(x) = \begin{pmatrix} \begin{pmatrix} \breve{u}(x) \\ \breve{d}(x) \end{pmatrix} \\ \begin{pmatrix} c(x) \\ s(x) \end{pmatrix} \end{pmatrix}$$

## Quarks and Families

We can then expand the fermion quantum number table:

| | $t$ | $t^3$ | $t^0$ | $q$ | $q' \sin 2\theta_W$ |
|---|---|---|---|---|---|
| $\nu_L$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $1$ |
| $e_L$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-1+2\sin^2\theta_W$ |
| $\nu_R$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_R$ | $0$ | $0$ | $-1$ | $-1$ | $2\sin^2\theta_W$ |
| $u_L$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $1-\frac{4}{3}\sin^2\theta_W$ |
| $d_L$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{6}$ | $-\frac{1}{3}$ | $-1+\frac{2}{3}\sin^2\theta_W$ |
| $u_R$ | $0$ | $0$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $-\frac{4}{3}\sin^2\theta_W$ |
| $d_R$ | $0$ | $0$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{2}{3}\sin^2\theta_W$ |
| $\varphi^+$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $1-2\sin^2\theta_W$ |
| $\varphi^0$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ |
| $\varphi^0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $1$ |
| $-\varphi^-$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-1$ | $-1+2\sin^2\theta_W$ |

(Doublets grouped in parentheses: $\begin{pmatrix}\nu_L\\ e_L\end{pmatrix}$, $\begin{pmatrix}u_L\\ d_L\end{pmatrix}$, $\begin{pmatrix}\varphi^+\\ \varphi^0\end{pmatrix}$, $\begin{pmatrix}\varphi^0\\ -\varphi^-\end{pmatrix}$.)

## Quarks and Families

In the case of one lepton family, we had

$$-\mathscr{L}_{\text{Yukawa}} = \left[ \bar{l}_{\text{L}} \frac{\sqrt{2}}{\text{v}} \hat{\varphi} m_\nu \nu_{\text{R}} + \text{c.c.} \right] + \left[ \bar{l}_{\text{L}} \frac{\sqrt{2}}{\text{v}} \varphi m_e e_{\text{R}} + \text{c.c.} \right]$$

which became, in the unitary gauge,

$$-\mathscr{L}_{\text{Yukawa}} = \left(1 + \tfrac{1}{\text{v}} \sigma\right) \left[ \bar{\nu} m_\nu \nu + \bar{e} m_e e \right]$$

In the case of $n_f$ families of leptons and quarks, we have

$$-\mathscr{L}_{\text{Yukawa}} = \left[ \bar{l}_{\text{L}} \frac{\sqrt{2}}{\text{v}} \hat{\varphi} M'_\nu \nu_{\text{R}} + \text{c.c.} \right] + \left[ \bar{l}_{\text{L}} \frac{\sqrt{2}}{\text{v}} \varphi M'_e e_{\text{R}} + \text{c.c.} \right]$$
$$+ \left[ \bar{\text{q}}_{\text{L}} \frac{\sqrt{2}}{\text{v}} \hat{\varphi} M'_{\text{u}} \text{u}_{\text{R}} + \text{c.c.} \right] + \left[ \bar{\text{q}}_{\text{L}} \frac{\sqrt{2}}{\text{v}} \varphi M'_{\text{d}} \text{d}_{\text{R}} + \text{c.c.} \right]$$

where the *M*' are $n_f \times n_f$ complex matrices. In the unitary gauge, this becomes

$$-\mathscr{L}_{\text{Yukawa}} = \left(1 + \tfrac{1}{\text{v}} \sigma\right) \left[ \bar{\nu} M'_\nu \nu + \bar{e} M'_e e + \bar{\text{u}} M'_{\text{u}} \text{u} + \bar{\text{d}} M'_{\text{d}} \text{d} \right]$$

It turns out that an arbitrary complex matrix *M* ' can be diagonalized with real positive elements using a biunitary transformation

$$A_1^{-1} M' A_2 = M \qquad \text{where} \qquad A_1^\dagger A_1 = A_2^\dagger A_2 = I$$

and *M* is diagonal with real positive elements.

## Quarks and Families

Concentrating on the quark sector, we can write

$$\bar{u}M'_u u = \bar{u}_L M'_u u_R + \bar{u}_R M'^{\dagger}_u u_L = \bar{u}_L M'_u u_R + \text{c.c.}$$
$$\bar{d}M'_d d = \bar{d}_L M'_d d_R + \bar{d}_R M'^{\dagger}_d d_L = \bar{d}_L M'_d d_R + \text{c.c.}$$

since $M'$ does not act in Dirac space. There exists 4 unitary matrices such that

$$A_L^{-1} M'_u A_R = M_u \qquad B_L^{-1} M'_d B_R = M_d$$
$$A_L^{\dagger} A_L = A_R^{\dagger} A_R = B_L^{\dagger} B_L = B_R^{\dagger} B_R = I$$

where $M_u$ and $M_d$ are diagonal with real positive elements. We then set

$$u_L \to u'_L = A_L^{-1} u_L \qquad u_R \to u'_R = A_R^{-1} u_R$$
$$d_L \to d'_L = B_L^{-1} d_L \qquad d_R \to d'_R = B_R^{-1} d_R$$

such that we obtain

$$\bar{u}M'_u u = \bar{u}_L A_L A_L^{-1} M'_u A_R A_R^{-1} u_R + \text{c.c.} = \bar{u}'_L M_u u'_R + \text{c.c.} = \bar{u}' M_u u'$$
$$\bar{d}M'_d d = \bar{d}' M_d d'$$

That is the u' and d' fields are mass eigenstates.

## Quarks and Families

The u and fields are gauge eigenstates, that is they are the ones mixing in the weak charged current, eg

$$j^{+\mu}_{\substack{\text{charged}\\ \text{quarks}}} = \frac{e}{\sqrt{2}\sin\theta_W}\bar{\mathrm{q}}T^{+}\gamma^{\mu}\tfrac{1}{2}\left(1-\gamma^{5}\right)\mathrm{q} = \frac{e}{\sqrt{2}\sin\theta_W}\bar{\mathrm{u}}\gamma^{\mu}\tfrac{1}{2}\left(1-\gamma^{5}\right)\mathrm{d}$$

$$= \frac{e}{\sqrt{2}\sin\theta_W}\bar{\mathrm{u}}_L\gamma^{\mu}\mathrm{d}_L$$

Since $\bar{\mathrm{u}}'\mathrm{u}' = \bar{\mathrm{u}}\mathrm{u}$ and $\bar{\mathrm{d}}'\mathrm{d}' = \bar{\mathrm{d}}\mathrm{d}$

the electromagnetic and neutral currents are not affected by the change of basis for the fermion fields. We can now *redefine* u and d to be mass eigenstates fields, and u' and d' to be the gauge eigenstates fields. The Lagrangian density after symmetry hiding does not change, except the charged current, eg

$$j^{+\mu}_{\substack{\text{charged}\\ \text{quarks}}} = \frac{e}{\sqrt{2}\sin\theta_W}\bar{\mathrm{u}}'_L\gamma^{\mu}\mathrm{d}'_L = \frac{e}{\sqrt{2}\sin\theta_W}\bar{\mathrm{u}}_L\gamma^{\mu}A_L^{-1}B_L\mathrm{d}_L$$

$$= \frac{e}{\sqrt{2}\sin\theta_W}\bar{\mathrm{u}}_L\gamma^{\mu}\underline{\mathrm{d}}_L$$

## Quarks and Families

where $$\underline{\mathrm{d}}_\mathrm{L} \equiv A_\mathrm{L}^{-1} B_\mathrm{L} \mathrm{d}_\mathrm{L} = V \mathrm{d}_\mathrm{L} \qquad V^\dagger V = I$$

*V* is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix. It can be shown that the $n_\mathrm{f}$ x $n_\mathrm{f}$ unitary matrix *V* has

$$\left.\begin{array}{ll} \tfrac{1}{2} n_\mathrm{f}\left(n_\mathrm{f}-1\right) & \text{angles} \\ \tfrac{1}{2}\left(n_\mathrm{f}-1\right)\left(n_\mathrm{f}-2\right) & \text{ind. phases} \end{array}\right\} \left(n_\mathrm{f}-1\right)^2 \ \text{free parameters}$$

Similarly we have

$$j^{-\mu}_{\substack{\text{charged}\\ \text{quarks}}} = \left( j^{+\mu}_{\substack{\text{charged}\\ \text{quarks}}} \right)^\dagger = \frac{e}{\sqrt{2}\sin\theta_\mathrm{W}} \underline{\mathrm{d}}_\mathrm{L} \gamma^\mu \mathrm{u}_\mathrm{L}$$

After symmetry hiding, we then have the Lagrangian density

$$\mathscr{L} = \mathscr{L}_\mathrm{D} + \mathscr{L}_\mathrm{V} + \mathscr{L}_\mathrm{H} + \mathscr{L}_\mathrm{DV} + \mathscr{L}'_\mathrm{H}$$

where the modified terms are

$$\mathscr{L}_\mathrm{D} = \overline{e}\left[ i\gamma^\mu \partial_\mu - M_\mathrm{e} \right] e + \overline{\nu}\left[ i\gamma^\mu \partial_\mu - M_\nu \right]\nu + \overline{\mathrm{u}}\left[ i\gamma^\mu \partial_\mu - M_\mathrm{u} \right]\mathrm{u} + \overline{\mathrm{d}}\left[ i\gamma^\mu \partial_\mu - M_\mathrm{d} \right]\mathrm{d}$$

$$\mathscr{L}_\mathrm{DV} = \mathscr{L}_\mathrm{em} + \mathscr{L}_\mathrm{neutral} + \mathscr{L}_\mathrm{charged}$$

## Quarks and Families

where

$$\mathscr{L}_{\text{em}} = -A_\mu j^\mu_{\text{em}}$$

$$\mathscr{L}_{\text{neutral}} = -Z_\mu j^\mu_{\text{neutral}}$$

$$\mathscr{L}_{\text{charged}} = -W^+_\mu j^{+\mu}_{\text{charged}} - W^-_\mu j^{-\mu}_{\text{charged}}$$

$$j^\mu_{\text{em}} = e\bar{l}\gamma^\mu Q_l l + e\bar{\mathrm{q}}\gamma^\mu Q_{\mathrm{q}}\mathrm{q}$$

$$j^\mu_{\text{neutral}} = e\bar{l}\gamma^\mu Q'_l l + e\bar{\mathrm{q}}\gamma^\mu Q'_{\mathrm{q}}\mathrm{q}$$

$$j^{\pm\mu}_{\text{charged}} = \frac{e}{\sqrt{2}\sin\theta_{\mathrm{W}}}\left[\bar{l}T^\pm\gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right)\underline{l} + \bar{\mathrm{q}}T^\pm\gamma^\mu \tfrac{1}{2}\left(1-\gamma^5\right)\underline{\mathrm{q}}\right]$$

$$j^{+\mu}_{\text{charged}} = \frac{e}{\sqrt{2}\sin\theta_{\mathrm{W}}}\left[\bar{\nu}_{\mathrm{L}}\gamma^\mu \underline{e}_{\mathrm{L}} + \bar{\mathrm{u}}_{\mathrm{L}}\gamma^\mu \underline{\mathrm{d}}_{\mathrm{L}}\right]$$

$$j^{-\mu}_{\text{charged}} = \frac{e}{\sqrt{2}\sin\theta_{\mathrm{W}}}\left[\bar{\underline{e}}_{\mathrm{L}}\gamma^\mu \nu_{\mathrm{L}} + \bar{\underline{\mathrm{d}}}_{\mathrm{L}}\gamma^\mu \mathrm{u}_{\mathrm{L}}\right]$$

$$\underline{\mathrm{d}}_{\mathrm{L}} \equiv V\mathrm{d}_{\mathrm{L}}$$

$$\underline{e}_{\mathrm{L}} \equiv V' e_{\mathrm{L}}$$

## Quarks and Families

and $$\mathscr{L}'_{\mathrm{H}} = \mathscr{L}_{\mathrm{HD}} + \mathscr{L}_{\mathrm{HV}}$$

where $$\mathscr{L}_{\mathrm{HD}} = -\tfrac{1}{\mathrm{v}}\sigma\left[\bar{e}M_{\mathrm{e}}e + \bar{\nu}M_{\nu}\nu + \bar{\mathrm{u}}M_{\mathrm{u}}\mathrm{u} + \bar{\mathrm{d}}M_{\mathrm{d}}\mathrm{d}\right]$$

Note that if $M_{\nu} \equiv 0$, then $$\underline{e}_{\mathrm{L}} = e_{\mathrm{L}}$$

and there are no differences between neutrino mass and gauge eigenstates.

## Free Parameters

We have first considered a classical field theory with one lepton field family, but with massive neutrino. We identify the following free parameters, to be determined by experiments:

Before symmetry hiding:

$$g', g, \mu^2, \lambda, c_\mathrm{e}, c_\nu$$

After symmetry hiding:

We can consider the set

$$e, \theta_\mathrm{W}, \mathrm{v}, \lambda, m_e, m_\nu$$

where

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} \qquad \tan\theta_\mathrm{W} = \frac{g'}{g} \qquad \mathrm{v}^2 = \frac{\mu^2}{\lambda}$$

$$m_\mathrm{e} = -\frac{\mu}{\sqrt{2\lambda}} c_\mathrm{e} \qquad m_\nu = -\frac{\mu}{\sqrt{2\lambda}} c_\nu$$

## Free Parameters

Or we can also consider the set

$$\alpha, \theta_{\mathrm{W}}, M_{\mathrm{Z}}, M_{\mathrm{H}}, m_{\mathrm{e}}, m_{\nu}$$

where

$$\alpha = \tfrac{1}{4\pi} e^2 = \tfrac{1}{4\pi} \frac{g^2 g'^2}{g^2 + g'^2}$$

$$M_{\mathrm{Z}}^2 = \frac{g^2 \mathrm{v}^2}{4\cos^2\theta_{\mathrm{W}}} = \frac{g^2\mu^2}{4\lambda\cos^2\theta_{\mathrm{W}}} \qquad M_{\mathrm{H}}^2 = 2\mathrm{v}^2\lambda = 2\mu^2$$

Note that we also have

$$M_{\mathrm{W}}^2 = \tfrac{1}{4} g^2 \mathrm{v}^2 = \frac{g^2\mu^2}{4\lambda} = \frac{\sqrt{2}g^2}{8G_{\mathrm{F}}} = \frac{\sqrt{2}e^2}{8G_{\mathrm{F}}\sin^2\theta_{\mathrm{W}}}$$

$$\frac{M_{\mathrm{W}}^2}{M_{\mathrm{Z}}^2} = \frac{1}{\cos\theta_{\mathrm{W}}} \qquad G_{\mathrm{F}} = \frac{\sqrt{2}g^2}{8M_{\mathrm{W}}^2}$$

## Free Parameters

In the case of $n_f$ families, we have

After symmetry hiding:

| | | $M_\nu \neq 0$ | $M_\nu = 0$ |
|---|---|---|---|
| | $\alpha, \theta_W, M_Z, M_H$ | 4 | 4 |
| Leptons | | | |
| | $M_e$ | $n_f$ | $n_f$ |
| | $M_\nu$ | $n_f$ | 0 |
| | $V'$ | $(n_f - 1)^2$ | 0 |
| Quarks | | | |
| | $M_u, M_d$ | $2n_f$ | $2n_f$ |
| | $V$ | $(n_f - 1)^2$ | $(n_f - 1)^2$ |
| TOTAL | | $2(n_f^2 + 3)$ | $n_f(n_f + 1) + 5$ |

## Free Parameters

Therefore

$$
\begin{aligned}
n_{\mathrm{f}} = 2 \quad & M_{\nu} \neq 0 \rightarrow 14 \\
& M_{\nu} = 0 \rightarrow 11 \\
n_{\mathrm{f}} = 3 \quad & M_{\nu} \neq 0 \rightarrow 24 \\
& M_{\nu} = 0 \rightarrow 17
\end{aligned}
$$

Note that other parameters appear in the corresponding quantized theory!

## Running Coupling Constant

Consider a dimensionless observable *R* that depends only on one physical scale *Q*. This means that we assume *Q* >> all masses. From dimensional scaling, one would expect that *R* is then a constant independent of *Q*. This is not the case in a renormalizable quantum field theory, where we obtain

$$R = R\left(\frac{Q^2}{\mu^2}, \alpha\right)$$

where $\mu$ is the renormalization scale at which the subtraction of divergences are performed

$\alpha$ is the renormalized coupling constant used as a basis for a perturbation expansion of *R*. It depends on $\mu$.

Therefore in general *R* will depend on *Q*.

But $\mu$ is arbitrary, and is not part of the Lagrangian of the theory. Any observable cannot depend on the choice of $\mu$. We can therefore write the renormalization group equation

$$\mu^2 \frac{d}{d\mu^2} R\left(\tfrac{Q^2}{\mu^2}, \alpha\right) = \left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha}{\partial \mu^2} \frac{\partial}{\partial \alpha}\right] R\left(\tfrac{Q^2}{\mu^2}, \alpha\right) = 0$$

## Running Coupling Constant

Let $t \equiv \ln\frac{Q^2}{\mu^2}$ , $\beta(\alpha) \equiv \mu^2 \frac{\partial\alpha}{\partial\mu^2}$ then $\left[-\frac{\partial}{\partial t} + \beta(\alpha)\frac{\partial}{\partial\alpha}\right] R\left(\frac{Q^2}{\mu^2}, \alpha\right) = 0$

We can solve this equation by introducing a new function $\alpha(Q)$, the running coupling constant, defined by

$$t = \int_{\alpha}^{\alpha(Q)} \frac{dx}{\beta(x)} \qquad\qquad \alpha(\mu) \equiv \alpha$$

The $\beta(\alpha)$ functions can be obtained from perturbation theory. They govern the running of $\alpha(Q)$. From this definition we can obtain

$$\frac{\partial\alpha(Q)}{\partial t} = \beta(\alpha(Q)) \qquad\qquad \frac{\partial\alpha(Q)}{\partial\alpha} = \frac{\beta(\alpha(Q))}{\beta(\alpha)}$$

From these we can show that $R(1, \alpha(Q))$ is also a solution of the renormalization group equation:

$$\left[-\frac{\partial}{\partial t} + \beta(\alpha)\frac{\partial}{\partial\alpha}\right] R(1, \alpha(Q)) = 0$$

This is an important result. It shows that ALL the physical scale *Q* dependence of *R* enters through the running of the coupling constant.

## Running Coupling Constant

That is, one computes $R\left(\frac{Q^2}{\mu^2},\alpha\right)$ to fixed order in perturbation theory.

Then setting $\mu = Q$ and $\alpha \to \alpha(Q)$ allows a prediction of the variation of $R$ with $Q$; the residual dependence of $R$ on $\mu$ is at the next order.

Note that only the variation of $\alpha$ with the scale $Q$ is predicted, not the absolute value of $\alpha$. The value of $\alpha$ (at a given scale where the theory is in the perturbation domain) has to be obtained from experiment.

Another approach (useful in QCD) is to introduce a parameter $\Lambda$ (dimension of energy) that represents the scale at which the coupling becomes very large

$$\ln\frac{Q^2}{\Lambda^2} = -\int_{\alpha(Q)}^{\infty}\frac{\mathrm{d}x}{\beta(x)}$$

Consider the perturbative development of $\beta(x)$

$$\beta(x) = -bx^2\left(1 + b'x + O\left(x^2\right)\right)$$

## Running Coupling Constant

If we keep only the leading order we obtain

$$\alpha(Q) = \frac{\alpha}{1+b\alpha t} = \frac{\alpha}{1+b\alpha \ln\frac{Q^2}{\mu^2}} = \frac{1}{b\ln\frac{Q^2}{\Lambda^2}}$$

We see that $\alpha(\mu) = \alpha$ and $\alpha(\Lambda) \to \infty$

In QED, not including any QCD effect but including the quarks, one obtains $\beta(x) = -bx^2 + O(x^3)$ where $b = -\frac{1}{3\pi}\sum_{\mathrm{f}} e_{\mathrm{f}}^2 N_{\mathrm{c}}$

and the sum runs over all fermions of charge $e_{\mathrm{f}}$ such that $m_{\mathrm{f}} << Q <<$ all other masses. The number of colours $N_{\mathrm{c}}$ is 3 for quarks and 1 for leptons.

Since $b < 0$, we see that $\alpha_{\mathrm{QED}}(Q)$ increases with $Q$.

Experimentally, therefore, we can obtain $\alpha(Q)$ at vanishing $Q$.

Setting $Q = \mu = m_{\mathrm{e}}$ we obtain

$$\alpha = \alpha_{\mathrm{QED}} = \alpha(m_{\mathrm{e}}) \approx \frac{1}{137.0}$$

The coupling becomes strong at $\Lambda_{\mathrm{QED}} = m_{\mathrm{e}} e^{-\frac{1}{2b\alpha}}$

## Running Coupling Constant

In this case, we assume $Q$ << all masses except $m_e$, and we have

$$b = -\tfrac{1}{3\pi} \quad \rightarrow \quad \frac{\Lambda^{1}_{\text{QED}}}{m_e} \sim e^{646} \sim 10^{280}$$

If we consider a very large physical scale $Q = M$, then

$$b = -\tfrac{1}{3\pi}\left[3 + 9\left(\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2\right)\right] = -\tfrac{8}{3\pi} \quad \rightarrow \quad \frac{\Lambda^{(9)}_{\text{QED}}}{M} \sim e^{80.7} \sim 10^{35}$$

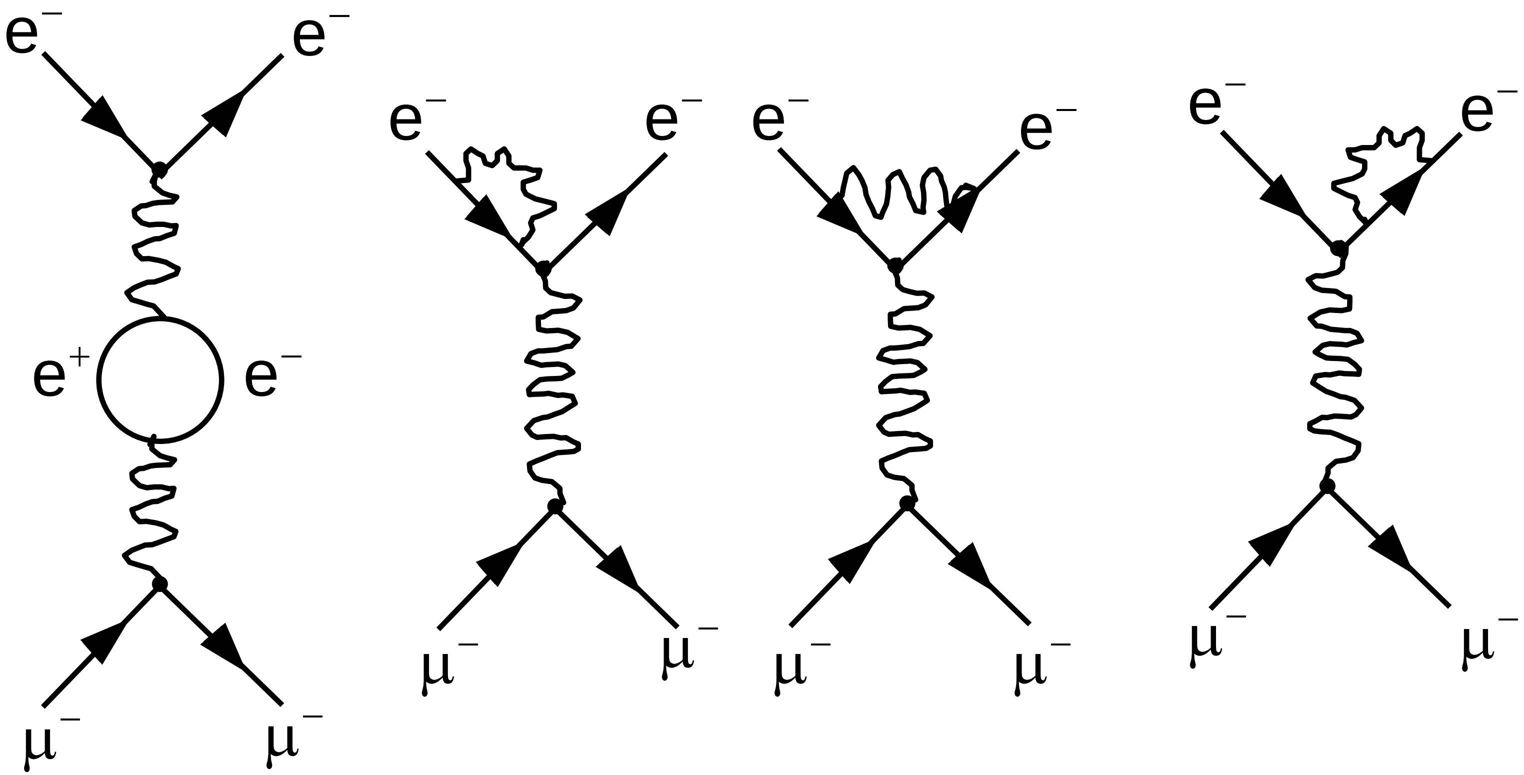
where we have assumed $\alpha(M) \approx \alpha$ since it runs very slowly in QED. We see that $\alpha(Q)$ runs faster with $Q$ if more fermions are included.

So QED is safely in the perturbative domain at all experimentally reachable energies.

We see that in QED higher order corrections and renormalization modify the coupling, and hence the electron charge.

## Running Coupling Constant

For example, in the case of $e^- \mu^-$ scattering, we have the following Feynman diagrams of order $\alpha^2$ that modify the electron charge

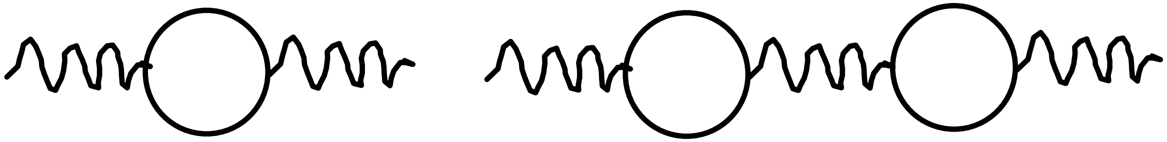



Due to a Ward identity, only the propagator loop diagram contributes to the modification of the charge. This is true to all order in perturbation.

Thus we see that using the running coupling constant is equivalent to summing all diagrams with loops in the photon propagator.

## Running Coupling Constant

Summing all diagrams with detached loops



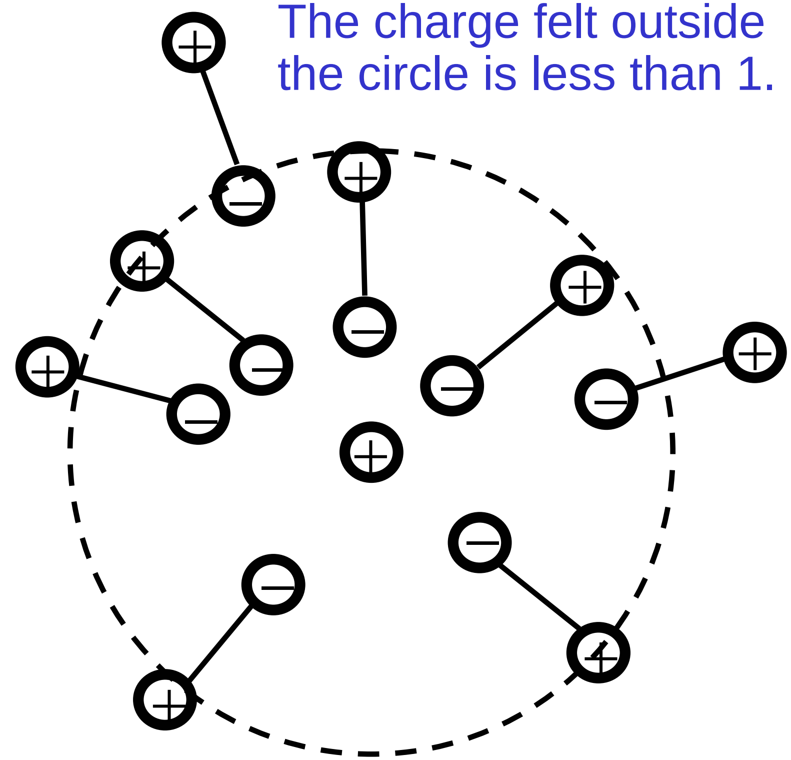
is equivalent to using the running coupling constant obtained with $\beta(x) = -bx^2$, which is also called the leading order in $\beta(x)$ or the leading log order in $\alpha(Q)$.

The increase of $\alpha(Q)$ with $Q$ means that the effective charge of the electron increases with decreasing distance $1/Q$. This is attributed to the cloud of $e^+ e^-$ around the electron that effectively screen its charge. At large distances

$$r \geq Q^{-1} \approx m_e^{-1}$$

the screening felt is maximum, and the charge measured is the conventional electron charge.

The charge felt outside the circle is less than 1.



## Running Coupling Constant

The inclusion of mass effects is a tricky business. For example, one obtains (hep-ph-9502298) the leading log result

$$\alpha\left(M_Z\right)=\frac{\alpha}{1-\frac{\alpha}{3\pi}\sum_{f} e_f^2 N_c\left[\ln\frac{M_Z^2}{m_f^2}-\frac{5}{3}\right]}$$

where the sum includes all fermions except the top quark. Using the PDG average for the mass of each quark, one obtains

$$\alpha\left(M_Z\right)=\frac{1}{128}>\alpha$$

So we see that in QED the coupling constant does not run very fast.

In QCD, one obtains

$$\beta\left(x\right)=-bx^2\left(1+b'x+O\left(x^2\right)\right)$$

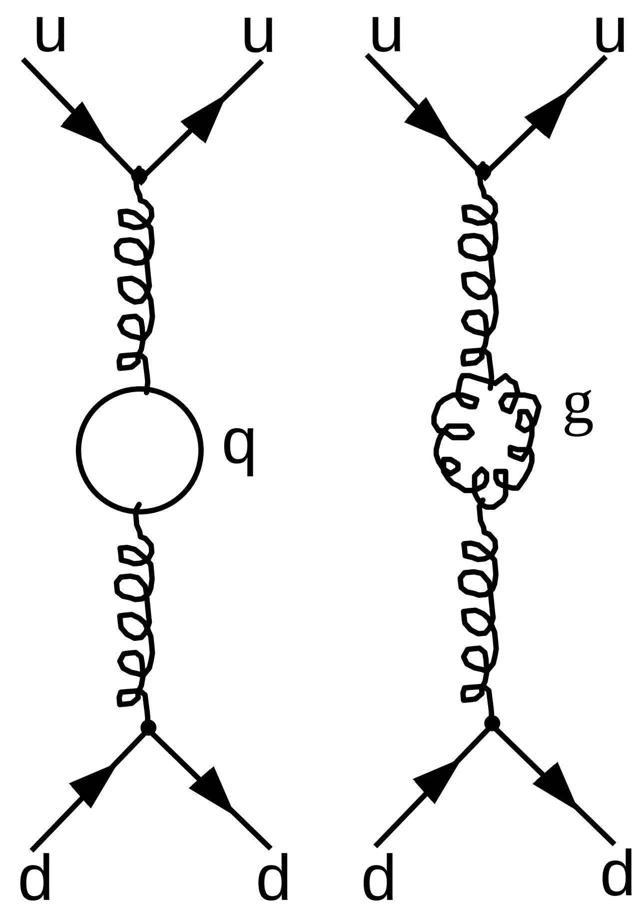
where

$$b=\frac{33-2n_f}{12\pi} \qquad b'=\frac{153-19n_f}{2\pi\left(33-2n_f\right)}$$

and $n_f$ is the number of flavours of quarks that satisfy $m_q << Q$. All other quark masses are assumed much heavier than $Q$.

## Running Coupling Constant

Since $b > 0$ we see that $\alpha_s(Q)$ decreases with $Q$. This leads to asymptotic freedom. The coupling becomes very small at large $Q$ (small distance). Experimentally, one measures $\alpha_s$ at a given scale $Q$. But since $\alpha_s(Q)$ diverges at small $Q$, it is customary to seek experimentally the scale $\Lambda = \Lambda_{QCD}$ at which $\alpha_s$ diverges. We expect $\Lambda$ to be of the order of meson and baryon masses.




The positive value of $b$ comes from the gluon loop contributions. It is a consequence of the non-abelian SU(3) nature of the colour group.

The gluons have an antiscreening effect on the colour charge which increases at large distances. This is due to the fact that gluons carry colour.

## Running Coupling Constant

Since $\beta(x)$ changes when the scale $Q$ crosses quark mass thresholds, $\Lambda$ must also change

$$\Lambda \rightarrow \Lambda^{(n_{\mathrm{f}})}$$

Furthermore, since the perturbation expansion is truncated at some order, the observable and the definition of $\Lambda$ depend on the renormalization scheme used.

Prescriptions exist on the correspondence between values of $\Lambda$ for $n_{\mathrm{f}}$ and $n_{\mathrm{f}} - 1$. The relation between $\Lambda$ for different renormalization schemes can be computed.

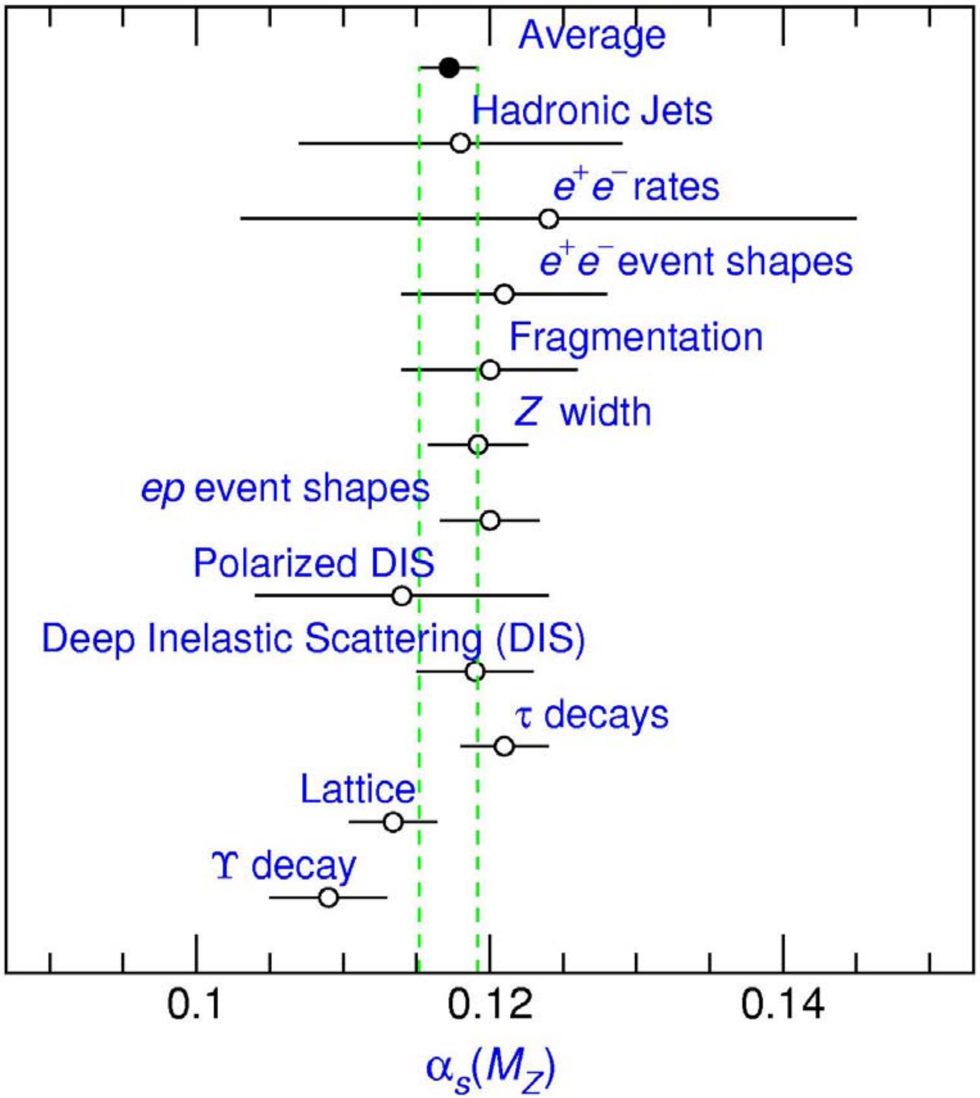
## Running Coupling Constant

Therefore a determination of $\alpha_s$ normally proceeds as follows:

- measure an observable with strong interaction effects at a certain energy scale $Q$;
- compute the expected observable to a given order in $\alpha_s(Q)$ (choose a renormalization scheme);
- extract $\alpha_s(Q)$ from data;
- evolve $\alpha_s(Q)$ to another scale (typically $M_Z$) to compare with other experiments;
- define $\alpha_s(Q)$ in terms of $\Lambda$;
- extract $\Lambda$ from $\alpha_s(Q)$ from data;
- convert $\Lambda$ to an appropriate renormalization scheme (typically the modified minimal subtraction scheme) and to an appropriate $n_f$ (typically 4 or 5) to compare with other experiments.

## Running Coupling Constant

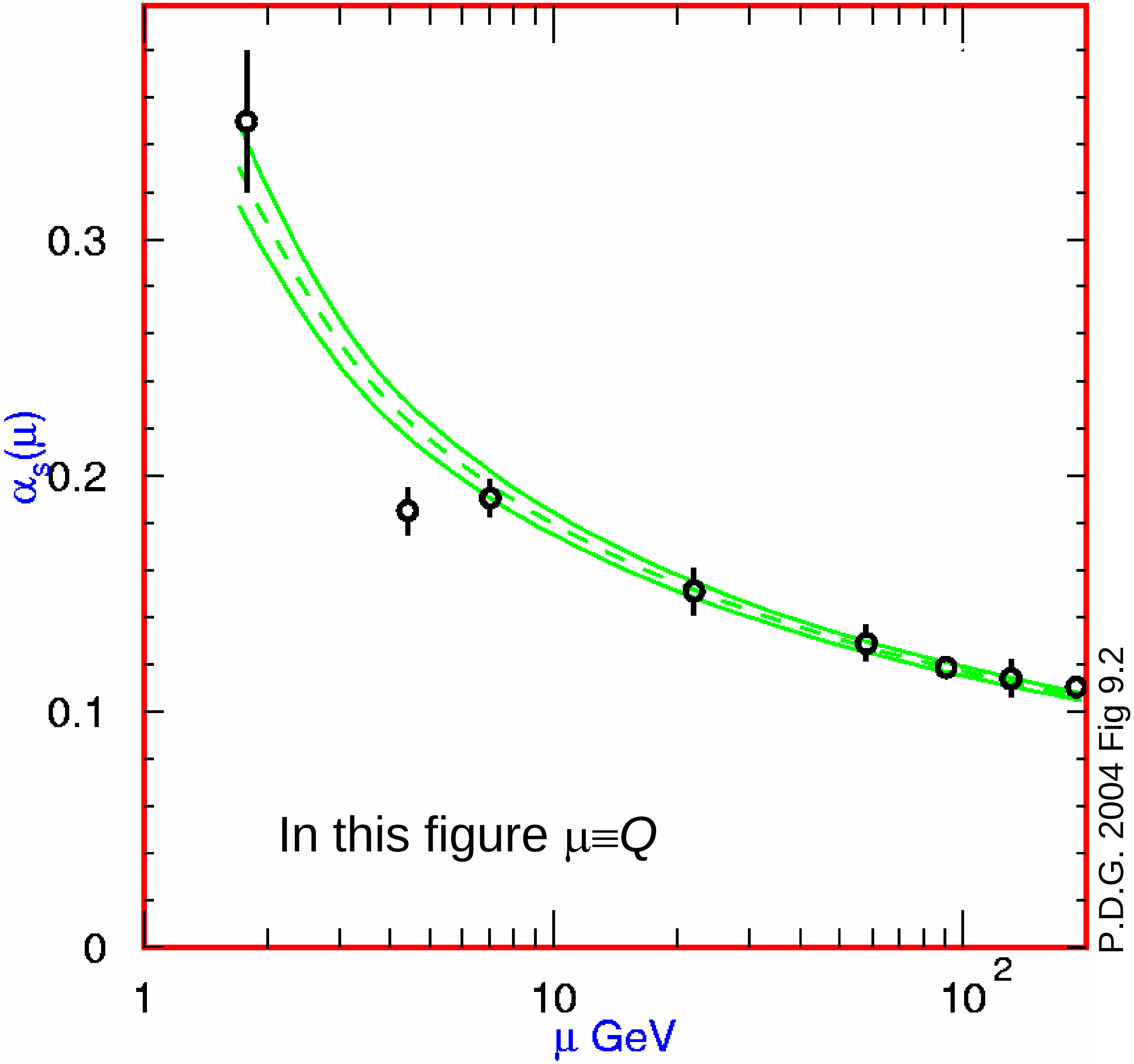
Summary of the values of $\alpha_s(M_Z)$ from various processes. The values shown indicate the process and the measured value of $\alpha_s$ extrapolated up to $\mu = M_Z$. The error shown is the total error including theoretical uncertainties.




P.D.G. 2002 Fig 9.1

## Running Coupling Constant

Summary of the values of $\alpha_s(Q)$ at the values of $Q$ where they are measured. The lines show the central values and the $\pm 1\sigma$ limits of the PDG average. The figure clearly shows the decrease in $\alpha_s$ with increasing $Q$.



## Running Coupling Constant

Let us consider
$$R = \frac{\sigma\left(e^+e^- \to \text{ hadrons}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)}$$

An $O(\alpha_s^2)$ calculation in the modified minimal subtraction scheme gives

$$R^{(2)}\left(\tfrac{Q^2}{\mu^2}, \alpha_s\right) = R^{(0)}\left[1 + C_1 \frac{\alpha_s}{\pi} + C_2' \left(\frac{\alpha_s}{\pi}\right)^2\right]$$

where
$$R^{(0)} = 3\sum_f e_f^2$$

$$C_1 = 1$$

$$C_2' = C_2\left(\tfrac{Q^2}{\mu^2}\right) = \left(\frac{33-2n_f}{12}\right)\ln\tfrac{\mu^2}{Q^2} + \tfrac{365}{24} - 11\varsigma + \left(\tfrac{2}{3}\varsigma - \tfrac{11}{12}\right)n_f$$

$$= b\pi \ln\tfrac{\mu^2}{Q^2} + C_2(1), \qquad \varsigma = 1.2021\,, \qquad C_2(1) = 1.41$$

for $n_f = 5$. Here $\zeta$ is $\zeta(3)$, where $\varsigma(s) = \sum_{k=1}^{\infty} k^{-s}$

Remember that $\alpha_s = \alpha_s(\mu)$ is the renormalized strong coupling constant.

## Running Coupling Constant

Let us consider the renormalization scale dependence of $R$ at the leading order in $\alpha_s$

$$R^{(1)}\left(\frac{Q^2}{\mu^2},\alpha_s\right)=R^{(0)}\left(1+\frac{\alpha_s}{\pi}\right)$$

At this order in $\beta(x)$ we have

$$\alpha_s(Q)=\frac{\alpha_s(\mu)}{1+b\alpha_s(\mu)\ln\frac{Q^2}{\mu^2}}$$

Note that this expression is invariant under $Q\leftrightarrow\mu$:

$$\alpha_s(\mu)=\frac{\alpha_s(Q)}{1+b\alpha_s(Q)\ln\frac{\mu^2}{Q^2}}=\alpha_s(Q)\left[1+b\alpha_s(Q)\ln\frac{Q^2}{\mu^2}+O\left(b^2\alpha_s^2(Q)\ln^2\frac{Q^2}{\mu^2}\right)\right]$$

We can therefore write

$$R^{(1)}\left(\frac{Q^2}{\mu^2},\alpha_s\right)=R^{(0)}\left[1+\frac{\alpha_s(Q)}{\pi}+\frac{b}{\pi}\alpha_s^2(Q)\ln^2\frac{Q^2}{\mu^2}+\cdots\right]$$

where the higher terms are of order $b^2\alpha_s^3(Q)\ln^2\frac{Q^2}{\mu^2}$

Notice that the $\mu$ dependence occurs at order $\alpha_s^2(Q)$.

## Running Coupling Constant

We therefore verify the renormalization group formalism result

$$R^{(1)}\left(\tfrac{Q^2}{\mu^2},\alpha_s\right) = R^{(1)}\left(1,\alpha_s(Q)\right)$$

Looking at the next-to-leading order is instructive. We have

$$R^{(2)}\left(\tfrac{Q^2}{\mu^2},\alpha_s\right) = R^{(0)}\left(1+\frac{\alpha_s}{\pi}+C_2'\left(\frac{\alpha_s}{\pi}\right)^2\right)$$

$$= R^{(0)}\left[1+\frac{\alpha_s(Q)}{\pi}+\frac{b}{\pi}\alpha_s^2(Q)\ln\tfrac{Q^2}{\mu^2}+C_2'\left(\frac{\alpha_s}{\pi}\right)^2+\cdots\right]$$

But

$$C_2'\left(\frac{\alpha_s}{\pi}\right)^2 = \left[b\pi\ln\tfrac{\mu^2}{Q^2}+C_2(1)\right]\left(\frac{\alpha_s(Q)}{\pi}\right)^2\left[1+O\left(b\alpha_s(Q)\ln\tfrac{Q^2}{\mu^2}\right)\right]$$

$$= -\tfrac{b}{\pi}\alpha_s^2(Q)\ln\tfrac{Q^2}{\mu^2}+C_2(1)\left(\frac{\alpha_s(Q)}{\pi}\right)^2+O\left(b\alpha_s^3(Q)\ln\tfrac{Q^2}{\mu^2}\right)$$

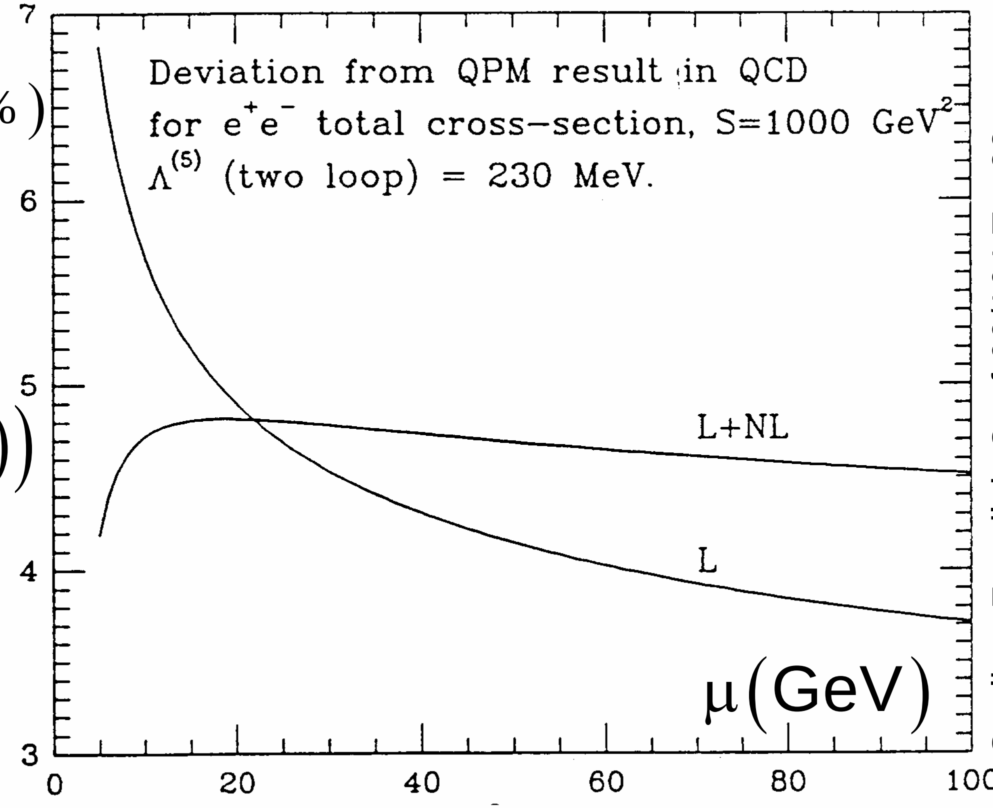
## Running Coupling Constant

Therefore
$$R^{(2)}\left(\frac{Q^2}{\mu^2},\alpha_s\right)=R^{(0)}\left(1+\frac{\alpha_s(Q)}{\pi}+C_2(1)\left(\frac{\alpha_s(Q)}{\pi}\right)^2+\cdots\right)$$

where the other terms are of order $b\alpha_s^3(Q)\ln\frac{Q^2}{\mu^2}$ and $b\alpha_s^3(Q)\ln^2\frac{Q^2}{\mu^2}$

Again we see that the $\mu$ dependence is at higher order in $\alpha_s(Q)$.

$$\frac{R^{(1,2)}\left(\frac{Q^2}{\mu^2},\alpha_s\right)}{R^0}-1\ (\%)$$




Sterling, Fermilab-Conf-90/164-T, p22.

We also verify that

$$R^{(2)}\left(\frac{Q^2}{\mu^2},\alpha_s\right)=R^{(2)}\left(1,\alpha_s(Q)\right)$$

As expected, the renormalization scale dependence is smaller at higher order in $\alpha_s$.

## Running Coupling Constant

A calculation to next-to-next-to-leading order gives (P.D.G.)

$$R^3\left(1,\alpha_s\left(Q\right)\right) = R^{(0)}\left[1 + C_1\frac{\alpha_s\left(Q\right)}{\pi} + C_2\left(\frac{\alpha_s\left(Q\right)}{\pi}\right)^2 + C_3\left(\frac{\alpha_s\left(Q\right)}{\pi}\right)^3\right]$$

where

$$C_1 = 1$$

$$C_2 = C_2(1) = 1.41$$

$$C_3 = C_3(1) = -12.8$$

With all available data in 20 < $Q$ < 65 GeV one obtains

$$\alpha_s\left(Q = 35\ \text{GeV}\right) = 0.146 \pm 0.03$$

If the third order is not included, the result is 0.142 $\pm$ 0.03 which indicates that the theoretical uncertainty is smaller than the experimental error.

Evolving this result to $Q = M_Z$ using the expression for $\alpha_s(Q)$ obtained to leading log order with 5 quark flavours (neglecting all other mass effects), we get 0.125.