Introduction to Gauge Theories

- **Basics of SU(*n*)**
- **Classical Fields**
- **U(1) Gauge Invariance**
- **SU(*n*) Gauge Invariance**
- **The Standard Model**

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Basics of SU(n)

- **Spin and Colour**
- **Notes on SU(n)**
- **Notes on SU(2)**
- **Notes on SU(3)**
- **Colour Factors**

Spin and Colour

From deep inelastic scattering, we found evidence for

- the fractional charge assignment of quarks;
- the spin $j = s = 1/2$ nature of quarks;
- gluons;
- asymptotic freedom (Bjorken scaling);
- confinement (no quarks observed in isolation).

Using the static quark model, all baryons are described as bound states of 3 quarks, but baryon wave functions violates Pauli exclusion principle (spin-statistics theorem), e.g. the $m_j = 3/2$ states

$$\Delta^{++}\left(\text{uuu}\right) \qquad \Delta^{-}\left(\text{ddd}\right) \qquad \Omega^{-}\left(\text{sss}\right)$$

All 3 quarks have same spin projection and same wave function, which implies a symmetric global wave function. This is not allowed for a fermion. We can solve the problem by introducing colour as an extra degree of freedom

$$\Psi_{\Omega^-}\left(J_z = \tfrac{3}{2}\hbar\right) = \Phi\left(\vec{r}_1, \vec{r}_2, \vec{r}_3, m_1 = m_2 = m_3 = \tfrac{1}{2}\right)\chi\left(c_1, c_2, c_3\right)$$

where Φ is a totally symmetric space-spin function of the 3 quarks, and χ is a totally antisymmetric colour function of the three quarks.

At least 3 distinct colours are needed here.

Spin and Colour

From $e^+ e^- \to$ hadrons, there is evidence for 3 different colours of quarks (Factor of 3 in *R*).

The quark model without colour predicts an infinity of multiquark states not observed in nature, e.g. states with 1,2,4,5,7,… quarks. But adding an extra degree of freedom (colour) increases the number of multiquark linearly independent states… This must be controlled.

QCD does it with colour confinement.

First, let's look at spin.

Consider a spin s = 1/2 particle A

$$|\mathrm{A}\rangle = \sum_{\alpha=1}^{2} \eta_\alpha |m_\alpha\rangle, \quad m_1 = \tfrac{1}{2}, \quad m_2 = -\tfrac{1}{2}$$

The norm $\sum_{\alpha=1}^{2} |\eta_\alpha|^2 = 1$ is left invariant by the transformation

$$|\mathrm{A}\rangle \xrightarrow{\hat{U}} \hat{U}|\mathrm{A}\rangle$$

Therefore $\hat{U}^\dagger \hat{U} = I$

Spin and Colour

Setting $\langle m_\alpha | \hat{U} | m_\beta \rangle = U_{\alpha\beta}$ we get $U^\dagger U = I \Rightarrow (\det U)^2 = 1$

We choose $\det U = 1$ so U is a 2 × 2 (special) unitary matrix. Then

$$\langle m_\alpha | \hat{U} | \mathrm{A} \rangle = U_{\alpha\beta}\eta_\beta \qquad \eta_\alpha \xrightarrow{\hat{U}} U_{\alpha\beta}\eta_\beta$$

Let's now consider the possible states $|J, J_z\rangle$ resulting from the combination of 2 spin 1/2 particles. Using CG coefficients,

$$|0,0\rangle = \tfrac{1}{\sqrt{2}}\left[\eta_1^{(1)}\eta_2^{(2)} \left|\tfrac{1}{2}\right\rangle_1 \left|\tfrac{-1}{2}\right\rangle_2 - \eta_2^{(1)}\eta_1^{(2)} \left|\tfrac{-1}{2}\right\rangle_1 \left|\tfrac{1}{2}\right\rangle_2\right]$$

$$|1,1\rangle = \eta_1^{(1)}\eta_1^{(2)} \left|\tfrac{1}{2}\right\rangle_1 \left|\tfrac{1}{2}\right\rangle_2$$

$$|1,0\rangle = \tfrac{1}{\sqrt{2}}\left[\eta_1^{(1)}\eta_2^{(2)} \left|\tfrac{1}{2}\right\rangle_1 \left|\tfrac{-1}{2}\right\rangle_2 + \eta_2^{(1)}\eta_1^{(2)} \left|\tfrac{-1}{2}\right\rangle_1 \left|\tfrac{1}{2}\right\rangle_2\right]$$

$$|1,-1\rangle = \eta_2^{(1)}\eta_2^{(2)} \left|\tfrac{-1}{2}\right\rangle_1 \left|\tfrac{-1}{2}\right\rangle_2$$

Projecting onto the $|m_\alpha\rangle_1 \, |m_\beta\rangle_2$ space we can write

$$\Psi_{00} = \tfrac{1}{\sqrt{2}}\left[\eta_1^{(1)}\eta_2^{(2)} - \eta_2^{(1)}\eta_1^{(2)}\right]$$

Spin and Colour

If we apply *U* to both particles, we obtain

$$\Psi_{00} \xrightarrow{\hat{U}} \tfrac{1}{\sqrt{2}}\left[U_{1\alpha}\eta_{\alpha}^{(1)}U_{2\beta}\eta_{\beta}^{(2)} - U_{2\alpha}\eta_{\alpha}^{(1)}U_{1\beta}\eta_{\beta}^{(2)}\right]$$
$$= \tfrac{1}{\sqrt{2}}\left[U_{1\alpha}U_{2\beta} - U_{2\alpha}U_{1\beta}\right]\eta_{\alpha}^{(1)}\eta_{\beta}^{(2)}$$
$$= \tfrac{1}{\sqrt{2}}\left[U_{11}U_{22} - U_{21}U_{12}\right]\left[\eta_{1}^{(1)}\eta_{2}^{(2)} - \eta_{2}^{(1)}\eta_{1}^{(2)}\right]$$
$$= \left[U_{11}U_{22} - U_{21}U_{12}\right]\Psi_{00} = \left(\det U\right)\Psi_{00}$$
$$= \Psi_{00}$$

We see that the spin singlet Ψ_{00} is invariant under *U* as expected.
Members of the spin triplet mix amongst themselves under *U*.

Note that

$$\text{spin singlet } \left(J=0\right) \quad \Rightarrow \quad J_z = 0$$
$$\text{but} \qquad J_z = 0 \quad \not\Rightarrow \quad \text{spin singlet } \left(J=0\right)$$

Spin and Colour

Now let's look at colour.

We assume there are 3 possible colour states $\left|c_j\right\rangle$, j=1,2,3

often called red, green, blue. Therefore the general colour states of a quark is

$$\left|\mathrm{q}\right\rangle = \sum_{j=1}^{3} c_j \left|c_j\right\rangle$$

The norm $\sum_{j=1}^{3} \left|c_j\right|^2 = 1$ is left invariant by the transformation

$$\left|\mathrm{q}\right\rangle \xrightarrow{\hat{U}} \hat{U}\left|\mathrm{q}\right\rangle$$

Therefore $\hat{U}^\dagger \hat{U} = I$

Setting $\left\langle c_j \middle| \hat{U} \middle| c_k \right\rangle = U_{jk}$ we get $U^\dagger U = I \Rightarrow \left(\det U\right)^2 = 1$

We choose $\det U = 1$ so U is a 3 × 3 (special) unitary matrix. Then

$$\left\langle c_j \middle| \hat{U} \middle| \mathrm{q} \right\rangle = U_{jk} c_k \qquad c_j \xrightarrow{\hat{U}} U_{jk} c_k$$

Spin and Colour

We must seek multiquark states that are invariant under *U*, that is colour singlets. For 3 quark states, the state with this property is

$$\left|\mathrm{B}\right\rangle = \tfrac{1}{\sqrt{6}}\varepsilon_{ijk}c_i^{(1)}c_j^{(2)}c_k^{(3)}\left|c_i\right\rangle_1\left|c_j\right\rangle_2\left|c_k\right\rangle_3$$

Projecting onto the $|c_i\rangle_1|c_j\rangle_2|c_k\rangle_3$ space we can write

$$\Psi_{\mathrm{B}} = \tfrac{1}{\sqrt{6}}\varepsilon_{ijk}c_i^{(1)}c_j^{(2)}c_k^{(3)}$$

If we apply *U* to each three quarks, we obtain

$$\begin{aligned}\Psi_{\mathrm{B}} &\xrightarrow{\hat{U}} \tfrac{1}{\sqrt{6}}\varepsilon_{ijk}U_{ia}c_a^{(1)}U_{jb}c_b^{(2)}U_{kc}c_c^{(3)}\\ &= \varepsilon_{abc}\left(\det U\right)\tfrac{1}{\sqrt{6}}c_a^{(1)}c_b^{(2)}c_c^{(3)} = \left(\det U\right)\Psi_{\mathrm{B}}\\ &= \Psi_{\mathrm{B}}\end{aligned}$$

where we have used $\varepsilon_{abc}\det U = \varepsilon_{ijk}U_{ia}U_{jb}U_{kc}$

We see that Ψ_{B} is invariant under *U*. It is a colour singlet. None of the other 26 colour combinations are singlets.

Ψ_{B} is a suitable wavefunction for baryons.

Spin and Colour

For quark-antiquark states, we need the anticolour vector

$$\left|\mathrm{q}\right\rangle = \sum_{j=1}^{3} c_j \left|c_j\right\rangle \qquad \left|\bar{\mathrm{q}}\right\rangle = \sum_{j=1}^{3} c_j^* \left|\bar{c}_j\right\rangle$$

Consider

$$\left|\mathrm{M}\right\rangle = \tfrac{1}{\sqrt{3}}\left[c_1^{(1)}c_1^{(2)*}\left|c_1\right\rangle_1\left|\bar{c}_1\right\rangle_2 + c_2^{(1)}c_2^{(2)*}\left|c_2\right\rangle_1\left|\bar{c}_2\right\rangle_2 + c_3^{(1)}c_3^{(2)*}\left|c_3\right\rangle_1\left|\bar{c}_3\right\rangle_2\right]$$

or, projecting onto the $|c_j\rangle_1|c_k\rangle_2$ space

$$\Psi_{\mathrm{M}} = \tfrac{1}{\sqrt{3}}\left[c_1^{(1)}c_1^{(2)*} + c_2^{(1)}c_2^{(2)*} + c_3^{(1)}c_3^{(2)*}\right]$$

which is invariant under *U*. It is a colour singlet.

Ψ_{M} is a suitable wavefunction for mesons.

Spin and Colour

In fact,

- Ψ_B is the only invariant qqq state;
- Ψ_M is the only invariant quark-antiquark state;
- The only other invariant multiquark states contain 3,6,9,… quarks and an arbitrary number of quark-antiquark pairs.

All observed hadrons are colour singlets

There are no long range confinement forces between colour singlets, e.g. pn can be separated. The nuclear force is a "Van der Waals" colour (strong) force.

Leptons and photons are colour singlets, so they don't feel the strong force. They are not made of colour objects either, so they don't feel the nuclear force.

Notes on SU(n)

The commutation relation $\left[t_a, t_b\right] = i f_{abc} t_c \qquad a = 1, 2, \ldots, n^2 - 1$

forms the Lie algebra of SU(n), where f_{abc} are the structure constants of the group. The t_a are the generators of the group. An element u of the group is then

$$u = e^{i\omega_a t_a}$$

where the ω_a are real parameters that label the group element u. By convention, we will assume sum over repeated group generator indices.

Consider the representation $u \to U(u) = U$

If there exist a non-singular matrix M, independent of the group elements, such that

$$MU(u)M^{-1} = \begin{pmatrix} U_1(u) & 0 & 0 \\ 0 & U_2(u) & 0 \\ 0 & 0 & \ddots \end{pmatrix} \qquad \forall u \in \text{SU}(n)$$

then U is called a reducible representation.

$$U(u) = U_1(u) \oplus U_2(u) \oplus \ \cdots$$

Otherwise it is an irreducible representation. In this case $U = e^{i\omega_a T_a}$

Notes on SU(n)

From $U^\dagger U = I$ we get $T^\dagger = T$

From $\det U = 1$ we get $\mathrm{Tr}\, T = 0$

Therefore the generators representation T_a are n^2-1 hermitian traceless matrices. They follow the SU(n) algebra

$$\left[T_a, T_b\right] = i f_{abc} T_c$$

Since $U(u_1)U(u_2) = U(u_1 u_2)$ implies $U^*(u_1)U^*(u_2) = U^*(u_1 u_2)$ we see that U^* forms a complex conjugate representation. This implies that $-T_a^*$ forms a representation of the generators.

If there exist a non-singular matrix S such that

$$S T_a S^{-1} = -T_a^* \qquad \forall a$$

then T_a and $-T_a^*$ are said to be equivalent. In this case the representation is said to be a real representation.

The dimension d of a representation is the dimension of the vector space on which it acts.

Notes on SU(n)

For $d = n$ we have the defining or fundamental representation denoted **n**. In this case the generators are $n \times n$ traceless hermitian matrices. There are $n^2 - 1$ of them, which sets the number of ω_a parameters needed to specify a group element.

In the fundamental representation, SU(n) is the group of all unitary $n \times n$ matrices of unit determinant.

Remember the properties of commutator algebra

$$\left[T_1, T_2\right] = -\left[T_2, T_1\right]$$
$$\left[aT_1 + bT_2, T_3\right] = a\left[T_1, T_3\right] + b\left[T_2, T_3\right]$$
$$\left[T_1, \left[T_2, T_3\right]\right] + \left[T_3, \left[T_1, T_2\right]\right] + \left[T_2, \left[T_3, T_1\right]\right] = 0$$

From the Jacobi identity we get

$$f_{abd}\, f_{cde} + f_{cad}\, f_{bde} + f_{bcd}\, f_{ade} = 0$$

Therefore the matrices T_a defined by

$$\left(T_a\right)_{bc} = -if_{abc}$$

also satisfy the algebra of the group. They generate the adjoint representation of dimension $d = n^2 - 1$.

Notes on SU(n)

Choosing the normalisation $\mathrm{Tr}\left(T_a T_b\right) = \kappa\delta_{ab}$

the structure constants f_{abc} are totally antisymmetric in all three indices.

Of the $n^2 - 1$ generators, only $n - 1$ are diagonal. They commute with one another, and their (real) eigenvalues are used to label group elements of an irreducible representation.

An irreducible representation (multiplet) of dimension d is denoted $\boldsymbol{d}$, though this does not generally uniquely label it.

Consider an infinitesimal SU(n) transformation on a generator

$$T_a \to U T_a U^\dagger \qquad U = 1 + i\omega_b T_b$$

We obtain $T_a \to T_a + \delta T_a$ where $\delta T_a = f_{abc}\omega_b T_c$

Any set of $n^2 - 1$ quantities that transform under SU(n) like T_a can be denoted $\vec{T} \equiv \left(T_1, T_2, \ldots, T_{n^2-1}\right)$

We can indeed define the following products

$$\left(\vec{S} \times \vec{T}\right)_a = f_{abc} S_b T_c \qquad \vec{S} \cdot \vec{T} = S_a T_a$$

Notes on SU(n)

Using the antisymmetry of f_{abc} we get the identity

$$\left(\vec{R}\times\vec{S}\right)\cdot\vec{T} = -\left(\vec{R}\times\vec{T}\right)\cdot\vec{S}$$

from which we can show that $\vec{S}\cdot\vec{T}$ is indeed a SU(n) scalar

$$\delta\left(\vec{S}\cdot\vec{T}\right) = \delta\vec{S}\cdot\vec{T} + \vec{S}\cdot\delta\vec{T} = \left(\vec{\omega}\times\vec{S}\right)\cdot\vec{T} + \vec{S}\cdot\left(\vec{\omega}\times\vec{T}\right) = 0$$

Therefore we can write

$$T^2 \equiv \vec{T}\cdot\vec{T} = \sum_{a=1}^{n^2-1} T_a^2 = kI$$

In general

$$\mathrm{Tr}T^2 = \sum_{a=1}^{n^2-1} \mathrm{Tr}T_a^2 = kd$$

where d is the dimension of the representation.

For U in SU(n), we can have

$$T_b = UT_aU^\dagger$$

Then

$$\mathrm{Tr}T_b^2 = \mathrm{Tr}\left[UT_aU^\dagger UT_aU^\dagger\right] = \mathrm{Tr}T_a^2$$

Notes on SU(n)

Let's choose $a = \alpha$ corresponding to one of the $n - 1$ diagonal generators. Then

$$\mathrm{Tr}T_a^2 = \mathrm{Tr}T_\alpha^2 = \sum_{i=1}^{d}\left(t_\alpha\right)_i^2 \quad \text{where} \quad \left(t_\alpha\right)_i \quad \text{are the eigenvalues of } T_\alpha$$

Therefore

$$\mathrm{Tr}T^2 = \sum_{a=1}^{n^2-1}\mathrm{Tr}T_a^2 = \left(n^2-1\right)\sum_{i=1}^{d}\left(t_\alpha\right)_i^2 = kd$$

Finally we obtain the important result

$$T^2 \equiv \vec{T}\cdot\vec{T} = \frac{\left(n^2-1\right)}{d}\sum_{i=1}^{d}\left(t_\alpha\right)_i^2 I$$

We also note that the normalization condition can be written as

$$\mathrm{Tr}\left(T_a T_b\right) = \kappa\delta_{ab} = \delta_{ab}\sum_{i=1}^{d}\left(t_\alpha\right)_i^2$$

where $\left(t_\alpha\right)_i$ are the eigenvalues of T_α, any one of the $n - 1$ diagonal generators

Notes on SU(2)

In SU(2) there are $n^2 - 1 = 3$ generators. In the fundamental representation, we choose the Pauli matrices σ_a

$$T_a = \tfrac{1}{2}\sigma_a \quad , \; a = 1,2,3 \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They have the normalization

$$\mathrm{Tr}\left(T_a T_b\right) = \tfrac{1}{2}\delta_{ab}$$

and they verify the Lie algebra

$$\left[T_a, T_b\right] = i\varepsilon_{abc} T_c$$

where ε_{abc} is totally antisymmetric with $\varepsilon_{123} = 1$.

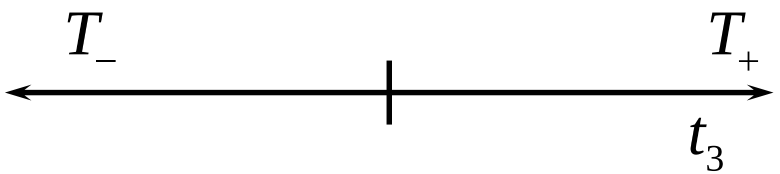
Note that T_3 is the only ($n - 1 = 1$) diagonal generator. Its eigenvalues t_3 label each state of a multiplet.

Setting $T_\pm \equiv T_1 \pm iT_2$ we obtain $\left[T_3, T_\pm\right] = \pm T_\pm$ therefore

T_+ raises t_3 by 1 $\qquad\qquad$ T_- lowers t_3 by 1

Notes on SU(2)

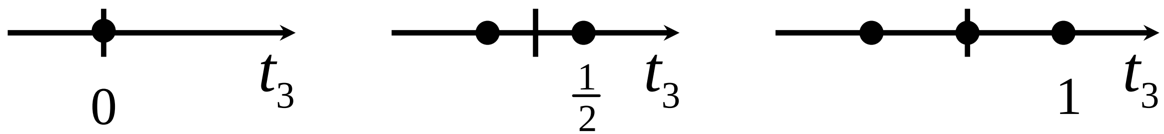
The action of $T_\pm$ can be represented by



Each irreducible representation of SU(2) is characterized by an integer *p* which corresponds to a multiplet of dimension *d*

$$p = 0, 1, 2, \ldots \qquad\qquad d = 1 + p$$

We can represent graphically the irreducible SU(2) multiplet by states on the t_3 axis



$p = 0$ $\qquad$ $p = 1$ $\qquad$ $p = 2$

1 $\qquad$ **2** $\qquad$ **3**

Note that $T_a \to -T_a^*$ does not change the spectrum of states. Indeed, there exist a *S* such that $S T_a S^{-1} = -T_a^* \qquad \forall a$

This can be verified in **2** with $S = T_2 = \frac{1}{2}\sigma_2$

In general this change implies $t_3 \to -t_3$. We see that

$\mathbf{d}$ and $\mathbf{d}^*$ are equivalent

Notes on SU(2)

Remember that in general $$T^2 \equiv \vec{T}\cdot\vec{T} = \frac{\left(n^2-1\right)}{d}\sum_{i=1}^{d}\left(t_\alpha\right)_i^2 I$$

where T_α is diagonal with eigenvalues $\left(t_\alpha\right)_i$

In the case of SU(2) we have

$$\mathbf{1}:\ T^2 = \tfrac{3}{1}0I = 0$$

$$\mathbf{2}:\ T^2 = \tfrac{3}{2}\left[\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2\right]I = \tfrac{3}{4}I$$

$$\mathbf{3}:\ T^2 = \tfrac{3}{3}\left[\left(-1\right)^2 + \left(0\right)^2 + \left(1\right)^2\right]I = 2I$$

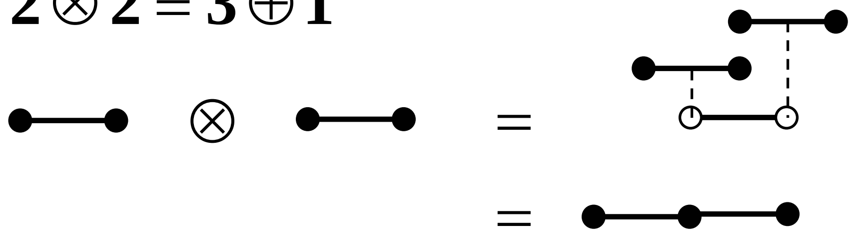
This is the familiar

$$J^2 = j\left(j+1\right) \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

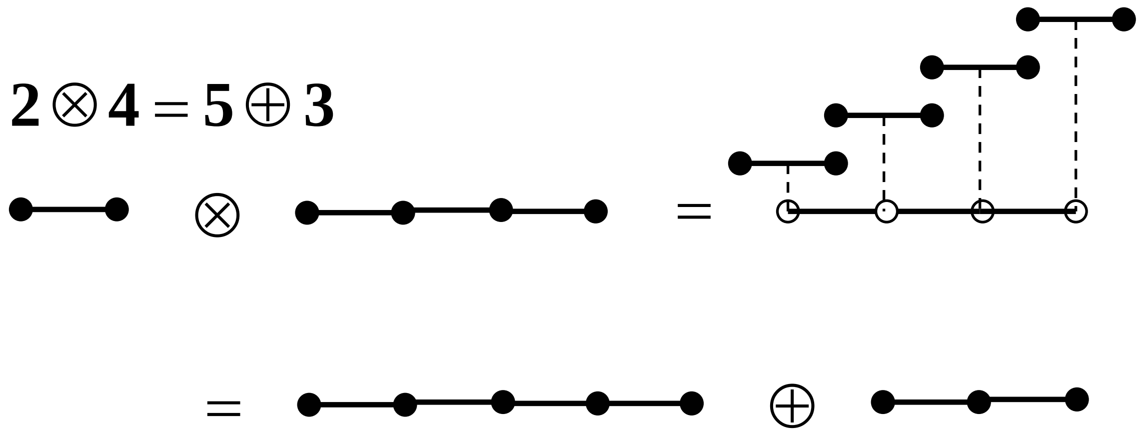
Notes on SU(2)

The product of irreducible representation can also be performed graphically

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$$



$$\mathbf{2} \otimes \mathbf{4} = \mathbf{5} \oplus \mathbf{3}$$



Notes on SU(3)

In SU(3) there are $n^2 - 1 = 8$ generators. In the fundamental representation, we choose the Gell-Mann matrices λ_a

$$T_a = \tfrac{1}{2}\lambda_a \qquad a = 1,2,\ldots,8$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

They have the normalization

$$\mathrm{Tr}\left(T_a T_b\right) = \tfrac{1}{2}\delta_{ab}$$

Notes on SU(3)

and they verify the Lie algebra $\left[T_a, T_b\right] = i f_{abc} T_c$

where f_{abc} is totally antisymmetric with nonvanishing elements

$$f_{123} = 1 \qquad f_{458} = f_{678} = \tfrac{\sqrt{3}}{2}$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \tfrac{1}{2}$$

There are $n - 1 = 2$ diagonal generators. They commute, so $\left[T_3, T_8\right] = 0$

Their eigenvalues t_3 and t_8 are used to label each states of a multiplet. We can define

$$T_\pm \equiv T_1 \pm i T_2 \qquad U_\pm \equiv T_6 \pm i T_7 \qquad V_\pm \equiv T_4 \pm i T_5$$

Note that when considering flavour SU(3), it is useful to define

$$Y \equiv \tfrac{2}{\sqrt{3}} T_8$$

Also note that T_1, T_2 and T_3 form an SU(2) sub-algebra. Therefore

$$f_{abc} = \varepsilon_{abc} \quad \text{for} \quad a, b, c \in 1, 2, 3$$

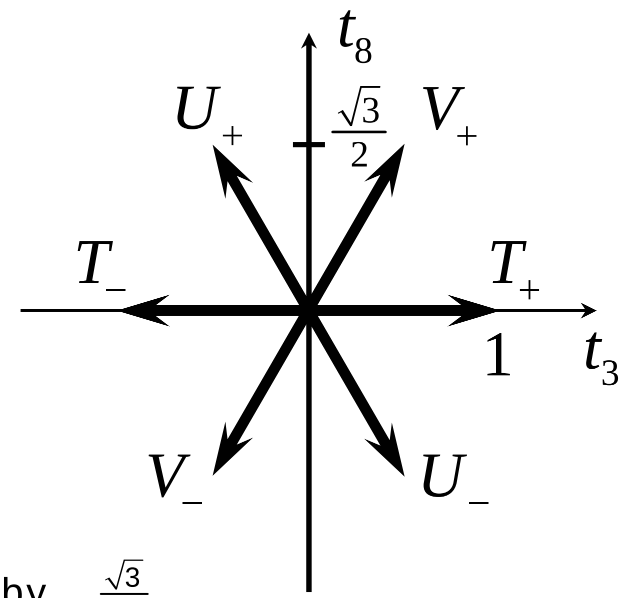
Notes on SU(3)

We obtain the following commutation relations

$$[T_3, T_\pm] = \pm T_\pm \qquad [T_8, T_\pm] = 0$$

$$[T_3, U_\pm] = \mp \tfrac{1}{2} U_\pm \qquad [T_8, U_\pm] = \pm \tfrac{\sqrt{3}}{2} U_\pm$$

$$[T_3, V_\pm] = \pm \tfrac{1}{2} V_\pm \qquad [T_8, V_\pm] = \pm \tfrac{\sqrt{3}}{2} V_\pm$$



Therefore we have

T_+	raises	t_3	by	1	and	does not affect	t_8		
T_-	lowers	t_3	by	1	and	does not affect	t_8		
U_+	lowers	t_3	by	$\frac{1}{2}$	and	raises	t_8	by	$\frac{\sqrt{3}}{2}$
U_-	raises	t_3	by	$\frac{1}{2}$	and	lowers	t_8	by	$\frac{\sqrt{3}}{2}$
V_+	raises	t_3	by	$\frac{1}{2}$	and	raises	t_8	by	$\frac{\sqrt{3}}{2}$
V_-	lowers	t_3	by	$\frac{1}{2}$	and	lowers	t_8	by	$\frac{\sqrt{3}}{2}$

The action of $T_\pm$, $U_\pm$ and $V_\pm$ can be represented as

Notes on SU(3)

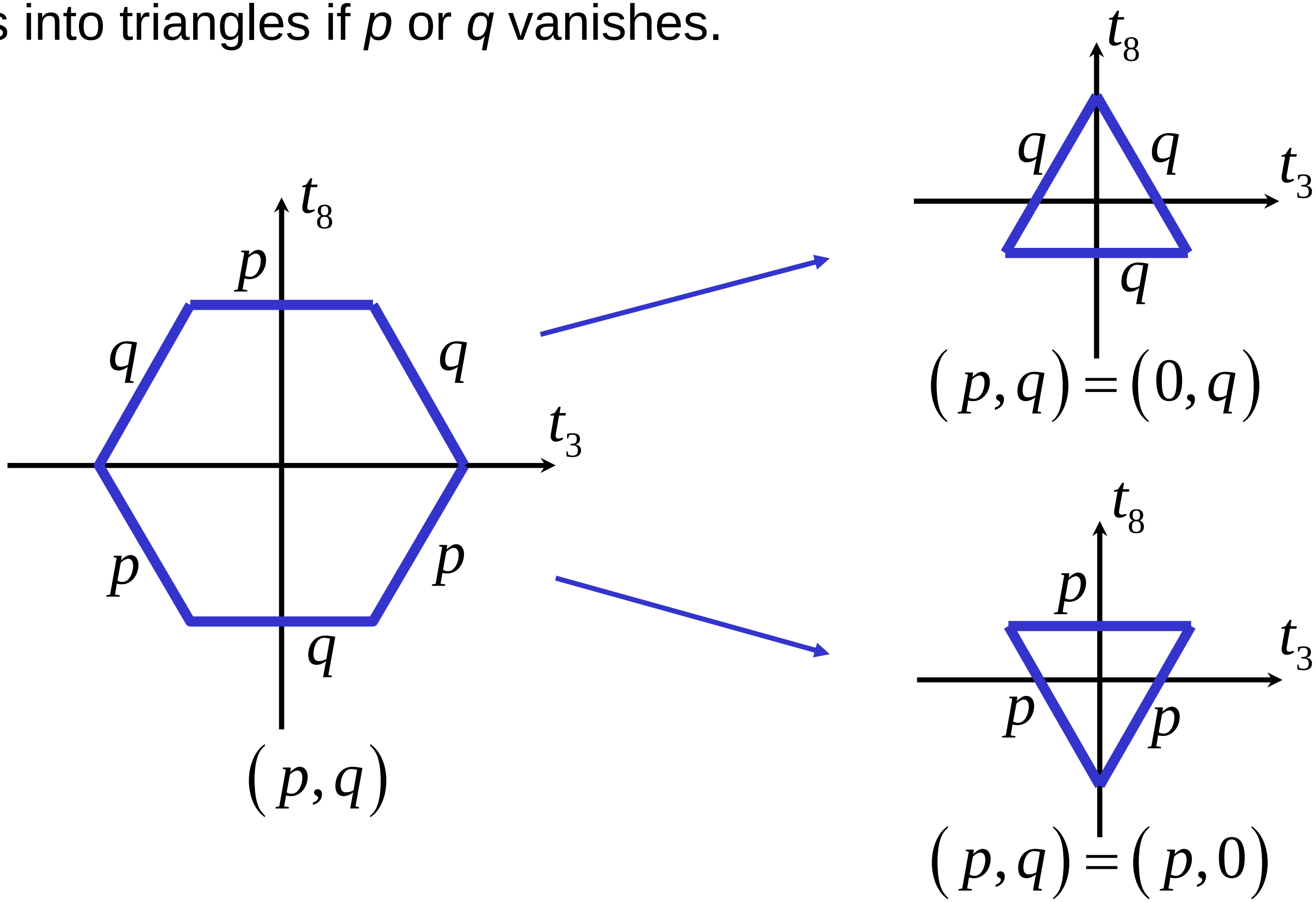
Each irreducible representation of SU(3) can be characterized by a set of two integers (*p*, *q*) which correspond to a multiplet of dimension *d*.

$$p = 0, 1, 2, \ldots$$
$$q = 0, 1, 2, \ldots$$
$$d = (1+p)(1+q)\left(1+\tfrac{1}{2}(p+q)\right)$$

q \ p	0	1	2	3	4	5	6	7
0	1	3	6	10	15	21	28	36
1		8	15′	24	35	48		
2			27	42	60			
3				64	90			
4					125			

Notes on SU(3)

Graphically, the irreducible SU(3) multiplets show up as states in the $t_3 - t_8$ plane. The boundaries are hexagons of sides *p* and *q*, which collapses into triangles if *p* or *q* vanishes.



Notes on SU(3)

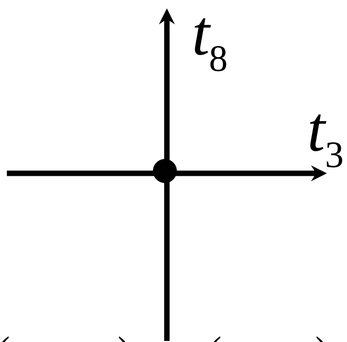
Starting from a (p, q) boundary containing 3(p+q) sites, the representation can be graphically obtained with the following rules:

- the boundary sites are singly occupied;
- the next inward layer is doubly occupied;
- the next inward layer is triply occupied, etc;
- a triangle shaped (or a dot) layer is reached;
- the next layers have the same occupancy as the previous one.

If you did it right the resulting number of states will be

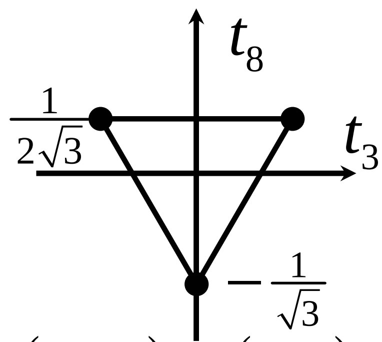
$$d = (1+p)(1+q)\left(1+\tfrac{1}{2}(p+q)\right)$$

Here are the first 3 smallest irreducible representations



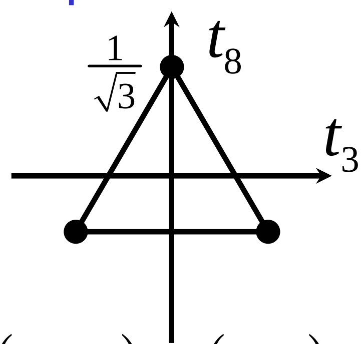
$(p,q)=(0,0)$

1



$(p,q)=(1,0)$

3



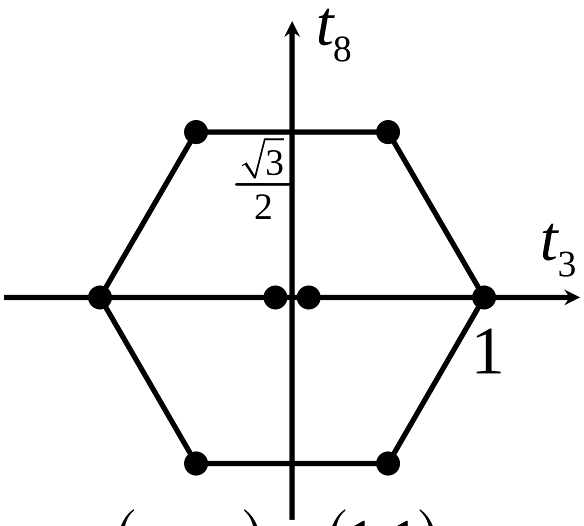
$(p,q)=(0,1)$

$\mathbf{3}^*$

Notice that the triangles for the fundamendal representations are equilateral.

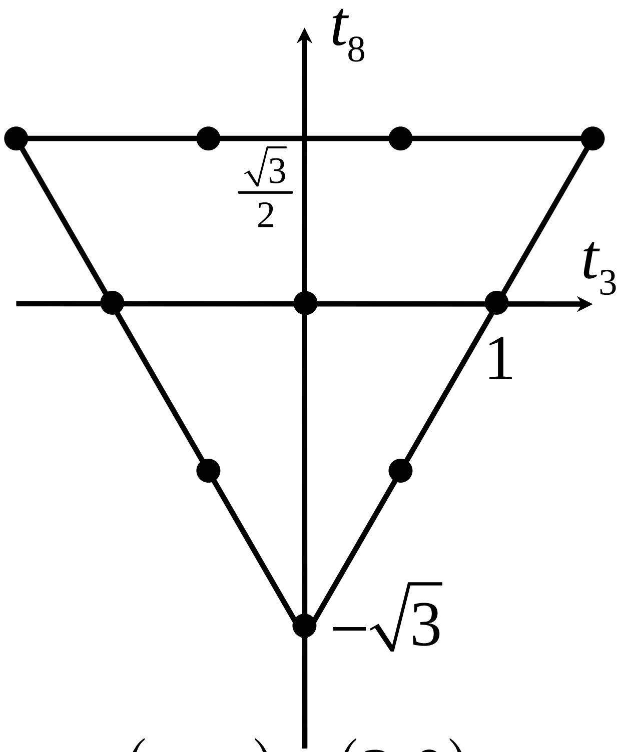
Notes on SU(3)

Here are three more examples:



$\left(p,q\right)=\left(1,1\right)$

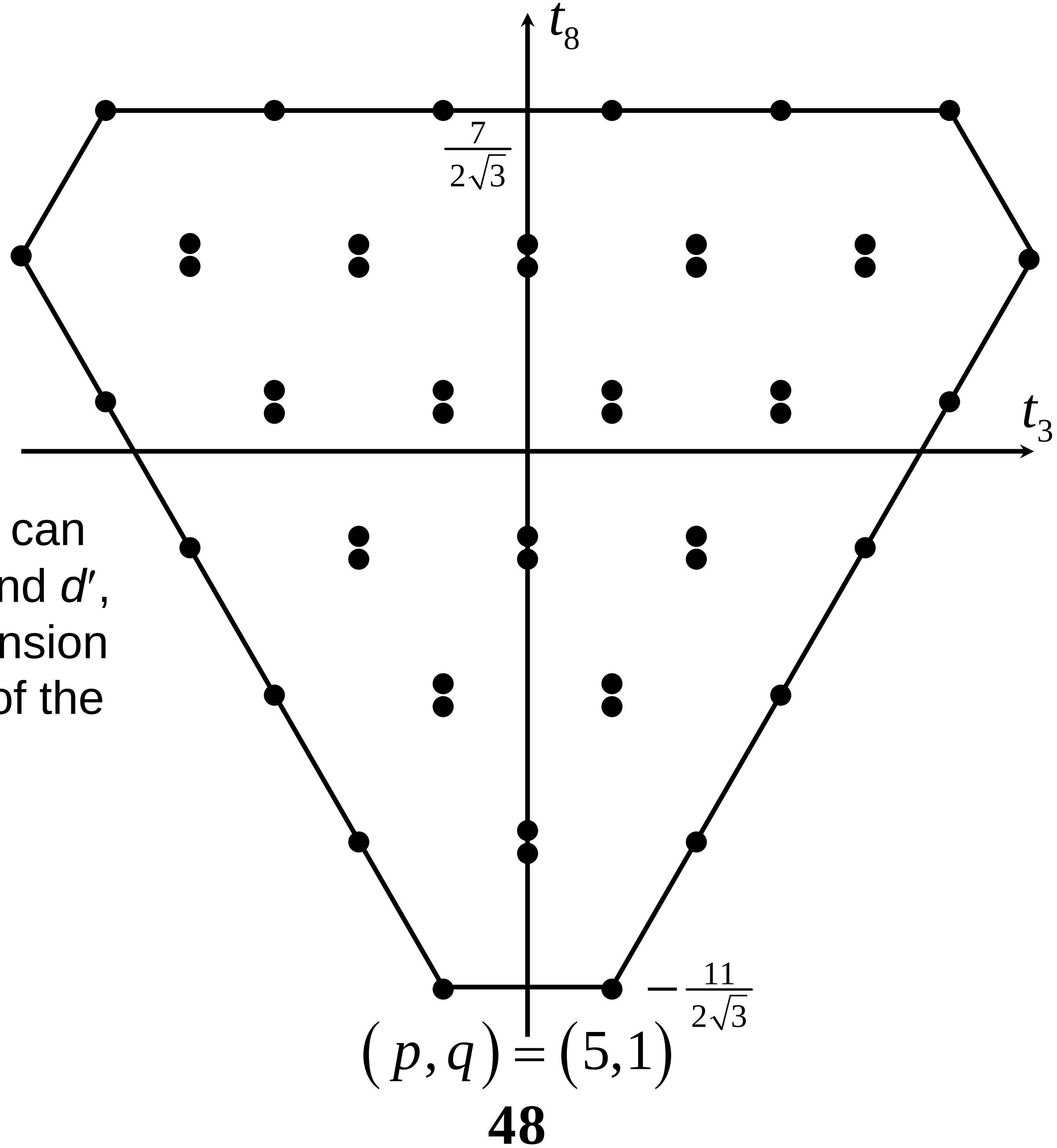
8



$\left(p,q\right)=\left(3,0\right)$

10

Notes on SU(3)



Note that each state can be labeled by t_3, t_8 and d', where d' is the dimension of the sub-multiplet of the SU(2) sub-algebra.

Notes on SU(3)

Note that **d*** is equivalent to **d** only if $p = q$. In particular

$$\mathbf{3} \text{ and } \mathbf{3}^* \text{ are not equivalent}$$

Remember that in general $$T^2 \equiv \vec{T}\cdot\vec{T} = \frac{\left(n^2-1\right)}{d}\sum_{i=1}^{d}\left(t_\alpha\right)_i^2 I$$

where T_α is diagonal with eigenvalues $\left(t_\alpha\right)_i$

Either t_3 or t_8 can be used, as shown in the following example:

$$\mathbf{1} \quad T^2 = \tfrac{8}{1} 0 I = 0$$

$$\mathbf{3} \quad T^2 = \tfrac{8}{3}\left[\left(-\tfrac{1}{2}\right)^2 + \left(0\right)^2 + \left(\tfrac{1}{2}\right)^2\right] I = \tfrac{4}{3} I$$

$$= \tfrac{8}{3}\left[\left(\tfrac{1}{2\sqrt{3}}\right)^2 + \left(-\tfrac{1}{\sqrt{3}}\right)^2 + \left(\tfrac{1}{2\sqrt{3}}\right)^2\right] I = \tfrac{4}{3} I$$

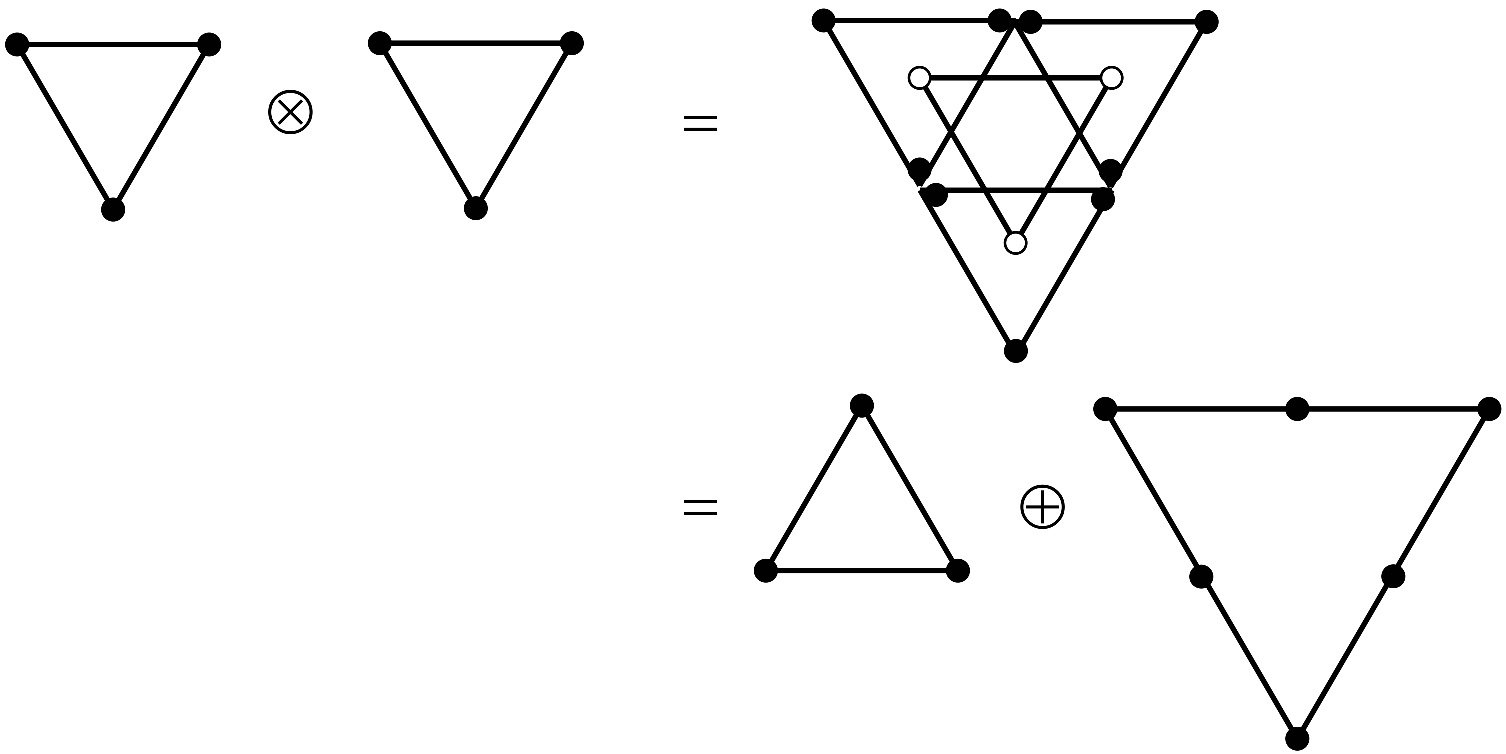
$$\mathbf{3}^* \quad T^2 = \tfrac{8}{3}\left[\left(-\tfrac{1}{2}\right)^2 + \left(0\right)^2 + \left(\tfrac{1}{2}\right)^2\right] I = \tfrac{4}{3} I$$

$$= \tfrac{8}{3}\left[\left(-\tfrac{1}{2\sqrt{3}}\right)^2 + \left(\tfrac{1}{\sqrt{3}}\right)^2 + \left(-\tfrac{1}{2\sqrt{3}}\right)^2\right] I = \tfrac{4}{3} I$$

Notes on SU(3)

The product of irreducible representations can also be done graphically. For example,

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^{*} \oplus \mathbf{6}$$



Notes on SU(3)

We find the following results

$$\mathbf{3}\otimes\mathbf{3}=\mathbf{3}^{*}\oplus\mathbf{6}$$

$$\mathbf{3}\otimes\mathbf{3}^{*}=\mathbf{1}\oplus\mathbf{8}$$

$$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}$$

$$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}=\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{3}\oplus\mathbf{6}^{*}\oplus\mathbf{6}^{*}\oplus\mathbf{15}\oplus\mathbf{15}\oplus\mathbf{15}\oplus\mathbf{15}'$$

and so on. It is remarkable that among the low-lying configurations of quarks, only quark-antiquark and qqq can belong to a colour singlet!

All hadron states and physical observables are colour singlets

It turns out that SU(3) is the only (compact semi-simple) Lie group that satisfies the following requirements:

because there are 3 colours, a quark must be represented by a triplet and

- quark and antiquark states are different;
- quark-antiquark and qqq can be singlets;
- qq and qqqq cannot be singlets.

Colour Factors

We have seen that a quark is a colour SU(3) triplet **3**:

$$|\mathrm{q}\rangle = \sum_{j=1}^{3} c_j |c_j\rangle \qquad \langle \mathrm{q}|\mathrm{q}\rangle = 1 \Rightarrow \sum_{j=1}^{3} |c_j|^2 = 1$$

In matrix notation we have

$$|c_1\rangle = |\text{red}\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |c_2\rangle = |\text{blue}\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |c_3\rangle = |\text{green}\rangle \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

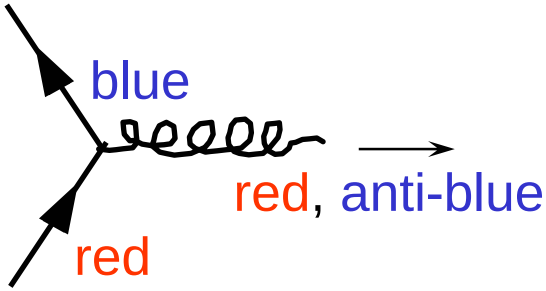
$$|\mathrm{q}\rangle \to c = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 \cdot \text{red} \\ c_2 \cdot \text{blue} \\ c_3 \cdot \text{green} \end{pmatrix} \equiv \begin{pmatrix} \mathrm{R} \\ \mathrm{B} \\ \mathrm{G} \end{pmatrix}$$

Also, an antiquark is a colour SU(3) triplet **3***

$$|\bar{\mathrm{q}}\rangle = \sum_{j=1}^{3} c_j^* |\bar{c}_j\rangle \qquad |\bar{\mathrm{q}}\rangle \to c^* = \begin{pmatrix} c_1^* \\ c_2^* \\ c_3^* \end{pmatrix} = \begin{pmatrix} c_1^* \cdot \text{anti-red} \\ c_2^* \cdot \text{anti-blue} \\ c_3^* \cdot \text{anti-green} \end{pmatrix} = \begin{pmatrix} \bar{\mathrm{R}} \\ \bar{\mathrm{B}} \\ \bar{\mathrm{G}} \end{pmatrix}$$

Colour Factors

In order to carry the colour force, gluons carry colour and anticolour



colour conservation at vertex

So in principle there are 9 possible colour-anticolour combinations. We can therefore have the following gluon multiplets

$$\mathbf{3}\otimes\mathbf{3}^{*}=\mathbf{1}\oplus\mathbf{8}$$

"If the gluon singlet existed, it could appear as a free particle. It would carry a long range force which would couple to baryons, approximately proportional to mass. No extra contribution to gravity has been found by experiment." This often heard argument is hokey...

Gluons are in the octet multiplet. As we will see later, gauge invariance predicts gluons to be of the adjoint representation of SU(3), which is the octet multiplet. Like quarks, they cannot then be observed as free particle:

$$\left|\mathrm{g}\right\rangle=\sum_{a=1}^{8} z_a\left|\mathrm{g}_a\right\rangle$$

Colour Factors

In matrix notation

$$|\mathrm{g}_1\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad |\mathrm{g}_2\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{etc.} \qquad |\mathrm{g}\rangle \to z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_8 \end{pmatrix}$$

We can built each gluon octet members in the colour basis $|\mathrm{g}_a\rangle \to \frac{1}{\sqrt{2}} c^\dagger \lambda_a c$ which yields

$$|\mathrm{g}_1\rangle = \tfrac{1}{\sqrt{2}}\left(\mathrm{R}\bar{\mathrm{B}} + \mathrm{B}\bar{\mathrm{R}}\right) \qquad |\mathrm{g}_5\rangle = \tfrac{-i}{\sqrt{2}}\left(\mathrm{R}\bar{\mathrm{G}} - \mathrm{G}\bar{\mathrm{R}}\right)$$

$$|\mathrm{g}_2\rangle = \tfrac{-i}{\sqrt{2}}\left(\mathrm{R}\bar{\mathrm{B}} - \mathrm{B}\bar{\mathrm{R}}\right) \qquad |\mathrm{g}_6\rangle = \tfrac{1}{\sqrt{2}}\left(\mathrm{B}\bar{\mathrm{G}} + \mathrm{G}\bar{\mathrm{B}}\right)$$

$$|\mathrm{g}_3\rangle = \tfrac{1}{\sqrt{2}}\left(\mathrm{R}\bar{\mathrm{R}} - \mathrm{B}\bar{\mathrm{B}}\right) \qquad |\mathrm{g}_7\rangle = \tfrac{-i}{\sqrt{2}}\left(\mathrm{B}\bar{\mathrm{G}} - \mathrm{G}\bar{\mathrm{B}}\right)$$

$$|\mathrm{g}_4\rangle = \tfrac{1}{\sqrt{2}}\left(\mathrm{R}\bar{\mathrm{G}} + \mathrm{G}\bar{\mathrm{R}}\right) \qquad |\mathrm{g}_8\rangle = \tfrac{1}{\sqrt{6}}\left(\mathrm{R}\bar{\mathrm{R}} + \mathrm{B}\bar{\mathrm{B}} - 2\mathrm{G}\bar{\mathrm{G}}\right)$$

Since gluons carry colour, they will interact with one another, unlike photons that do not.

Colour Factors

Heaviside-Lorentz system

The strength of the electromagnetic force between charged particles is governed by the coupling constant α, or the fundamental electric charge $g_e = e$:

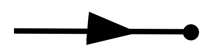
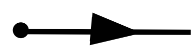
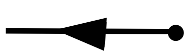
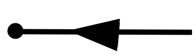
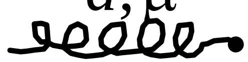
$$g_e^2 = 4\pi\alpha$$

Likewise, the strength of the chromodynamic, or strong, force between colour charged particles is governed by the strong coupling constant α_s, or the fundamental colour charge g_s :

$$g_s^2 = 4\pi\alpha_s$$

We now state the Feynman rules for tree-level diagrams in QCD:

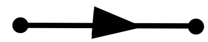
External lines:

- incoming quark  $u(p,s)c$
- outgoing quark  $\bar{u}(p,s)c^\dagger$
- incoming antiquark  $\bar{v}(p,s)c^\dagger$
- outgoing antiquark  $v(p,s)c$
- incoming gluon a,μ  $\varepsilon_\mu(p)z_a$
- outgoing gluon a,μ  $\varepsilon_\mu^*(p)z_a^*$

Colour Factors

Propagators:

- quark-antiquark



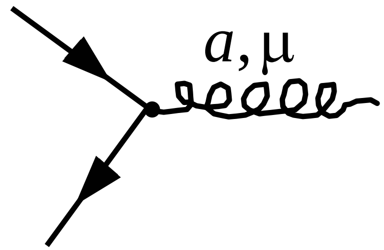
$$\frac{i(\not{q}+m)}{q^2-m^2}$$

- gluon

$$\frac{-ig_{\mu\nu}\delta_{ab}}{q^2}$$

Vertices:

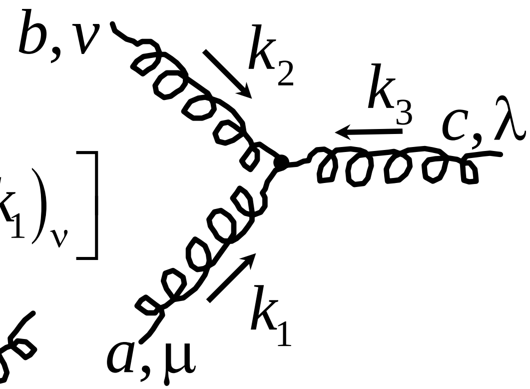
- quark-gluon



$$-ig_s\frac{1}{2}\lambda_a\gamma^\mu$$

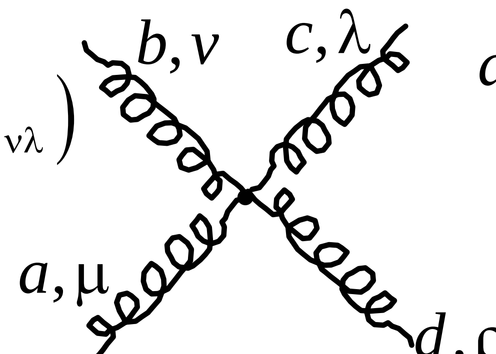
- three-gluon

$$-g_s f_{abc}\left[g_{\mu\nu}(k_1-k_2)_\lambda+g_{\nu\lambda}(k_2-k_3)_\mu+g_{\lambda\mu}(k_3-k_1)_\nu\right]$$



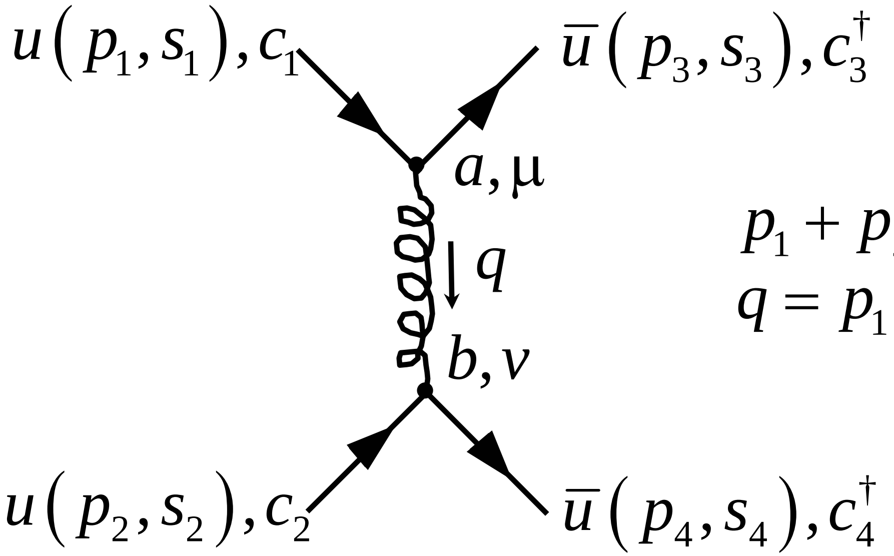
- four-gluon

$$-ig_s^2\Big[f_{abe}f_{cde}\left(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}\right)$$
$$+f_{ace}f_{bde}\left(g_{\mu\nu}g_{\lambda\rho}-g_{\lambda\nu}g_{\mu\rho}\right)$$
$$+f_{ade}f_{cbe}\left(g_{\mu\lambda}g_{\rho\nu}-g_{\rho\lambda}g_{\mu\nu}\right)\Big]$$



Colour Factors

Consider the $q_1 q_2$ = ud QCD interaction



$$p_1 + p_2 = p_3 + p_4$$
$$q = p_1 - p_3 = p_4 - p_2$$

$$-i\mathcal{M} = \left[\bar{u}(3) c_3^\dagger\right]\left[-ig_s \tfrac{1}{2}\lambda_a \gamma^\mu\right]\left[u(1) c_1\right]\left[\frac{-ig_{\mu\nu}\delta_{ab}}{q^2}\right]\left[\bar{u}(4) c_4^\dagger\right]\left[-ig_s \tfrac{1}{2}\lambda_b \gamma^\nu\right]\left[u(2) c_2\right]$$

$$\mathcal{M} = -g_s^2 C_F \frac{\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]}{q^2}$$

where we have the colour factor $C_F = \left[c_3^\dagger \tfrac{1}{2}\lambda_a c_1\right]\left[c_4^\dagger \tfrac{1}{2}\lambda_a c_2\right]$

Colour Factors

Comparing with the QED result for $e^- \mu^-$ scattering we infer the following strong potential energy (in the non-relativistic limit)

$$V_\mathrm{s}(r) = C_F \frac{\alpha_\mathrm{s}}{r} = C_F \frac{g_\mathrm{s}^2}{4\pi r}$$

Since in the electromagnetic case we have

$$V(r) = Q_1 Q_2 \frac{\alpha}{r} = Q_1 Q_2 \frac{g_\mathrm{e}^2}{4\pi r}$$

we see that the colour factor is like the "product of the colour charges" of the two quarks. The colour factor depends in which colour multiplet is the incoming quark pair. Let's use the notation

$$\begin{array}{llll} \mathrm{q}_1(\mathrm{in}) : & 1, p_1, c_1 & \qquad \mathrm{q}_2(\mathrm{in}) : & 2, p_2, c_2 \\ \mathrm{q}_1(\mathrm{out}) : & 3, p_3, c_3 & \qquad \mathrm{q}_2(\mathrm{out}) : & 4, p_4, c_4 \end{array}$$

then
$$|1,2\rangle = |1\rangle |2\rangle = \sum_{jk} c_{1j} c_{2k} \left|c_j\right\rangle_1 \left|c_k\right\rangle_2 = \sum_{jk} \beta_{jk} \left|c_j\right\rangle_1 \left|c_k\right\rangle_2$$

where the β_{jk} are the Clebsch-Gordan coefficients for SU(3) needed to build a given multiplet.

Colour Factors

Now, $$|3,4\rangle = |3\rangle|4\rangle = \sum_{jk} c_{3j}c_{4k}\left|c_j\right\rangle_1\left|c_k\right\rangle_2 = \sum_{jk}\beta_{jk}\left|c_j\right\rangle_1\left|c_k\right\rangle_2$$

Since the incoming and outgoing quark pairs are necessarily in the same multiplet, we have used $c_{1j}c_{2k} = c_{3j}c_{4k}$

Therefore $\langle 1,2|1,2\rangle = \langle 3,4|3,4\rangle = 1 \Rightarrow \sum_{jk}\beta_{jk}{}^2 = 1$

implies $\langle 1,2|3,4\rangle = 1$

Now consider the total colour operator

$$\vec{T}(1,2) = \vec{T}(3,4) \equiv \vec{T} \qquad \vec{T}(1) = \vec{T}(3) \equiv \vec{T}^{(1)} \qquad \vec{T}(2) = \vec{T}(4) \equiv \vec{T}^{(2)}$$

Then $\vec{T} = \vec{T}^{(1)} + \vec{T}^{(2)}$

$$T^2 = \left(T^{(1)}\right)^2 + \left(T^{(2)}\right)^2 + 2\vec{T}^{(1)}\cdot\vec{T}^{(2)}$$

therefore $\vec{T}^{(1)}\cdot\vec{T}^{(2)}$ is proportional to *I*.

Colour Factors

Now,

$$\left\langle 3,4\left|T^2\right|1,2\right\rangle = T^2 = \left(T^{(1)}\right)^2 + \left(T^{(2)}\right)^2 + 2\left\langle 3\left|\vec{T}^{(1)}\right|1\right\rangle\cdot\left\langle 4\left|\vec{T}^{(2)}\right|2\right\rangle$$

but $\left\langle 3\left|T_a^{(1)}\right|1\right\rangle = \sum_{jk} c_{3j}^{*} c_{1k}\left\langle c_j\left|T_a^{(1)}\right|c_k\right\rangle_1 = I\sum_{jk} c_{3j}^{*} c_{1k}\left(\tfrac{1}{2}\lambda_a\right)_{jk}$

$$= I c_3^{\dagger}\tfrac{1}{2}\lambda_a c_1$$

Likewise $\left\langle 4\left|T_a^{(2)}\right|2\right\rangle = I c_4^{\dagger}\tfrac{1}{2}\lambda_a c_2$

Therefore we obtain

$$\vec{T}^{(1)}\cdot\vec{T}^{(2)} = \left\langle 3,4\left|\vec{T}^{(1)}\cdot\vec{T}^{(2)}\right|1,2\right\rangle = I\left[c_3^{\dagger}\tfrac{1}{2}\lambda_a c_1\right]\left[c_4^{\dagger}\tfrac{1}{2}\lambda_a c_2\right]$$

$$= I C_F = \tfrac{1}{2}\left[T^2 - \left(T^{(1)}\right)^2 - \left(T^{(2)}\right)^2\right]$$

Since each quark is in the colour triplet **3** state, we have

$$\left(T^{(1)}\right)^2 = \left(T^{(2)}\right)^2 = T^2(\mathbf{3}) = \tfrac{4}{3} I$$

Colour Factors

The possible multiplets for the quark pair is given by $\mathbf{3}\otimes\mathbf{3}=\mathbf{3}^*\oplus\mathbf{6}$

In the triplet **3*** case, we have $T^2=\frac{4}{3}I$

Therefore $C_F\left(\mathbf{3}^*\right)=\frac{1}{2}\left[\frac{4}{3}-\frac{4}{3}-\frac{4}{3}\right]=-\frac{2}{3}$

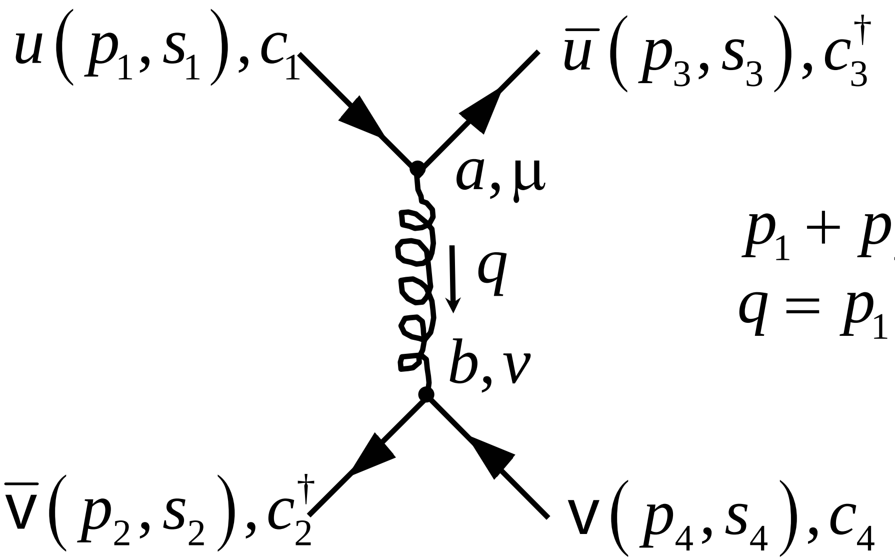
In the sextet **6** case, we have $T^2=\frac{10}{3}I$

Therefore $C_F\left(\mathbf{6}\right)=\frac{1}{2}\left[\frac{10}{3}-\frac{4}{3}-\frac{4}{3}\right]=\frac{1}{3}$

We conclude that in the triplet state the quarks attract each other, while they repel each other in the sextet state. Neither state can be observed free in nature, but pairs of quarks occur in baryons, which are singlet totally antisymmetric states. This means that pairs of quarks in baryons must be in an antisymmetric state, which is the case when they are in the triplet state, ie when they attract each other! This is not a proof, but we see that the colour potential is favorable for binding when 3 quarks are in a singlet configuration.

Colour Factors

Similarly we can consider the $q\bar{q} = u\bar{d}$ QCD interaction



$$p_1 + p_2 = p_3 + p_4$$
$$q = p_1 - p_3 = p_4 - p_2$$

$$-i\mathcal{M} = \left[\bar{u}(3)c_3^\dagger\right]\left[-ig_s \tfrac{1}{2}\lambda_a\gamma^\mu\right]\left[u(1)c_1\right]\left[\frac{-ig_{\mu\nu}\delta_{ab}}{q^2}\right]\left[\bar{v}(2)c_2^\dagger\right]\left[-ig_s \tfrac{1}{2}\lambda_b\gamma^\nu\right]\left[v(4)c_4\right]$$

$$\mathcal{M} = -g_s^2 C_F \frac{\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{v}(2)\gamma_\mu v(4)\right]}{q^2}$$

where we have the colour factor $C_F = \left[c_3^\dagger \tfrac{1}{2}\lambda_a c_1\right]\left[c_2^\dagger \tfrac{1}{2}\lambda_a c_4\right]$

Colour Factors

Comparing with the QED result for $e^- \mu^+$ scattering we infer the following strong potential energy (in the non-relativistic limit)

$$V_s(r) = -C_F \frac{\alpha_s}{r} = -C_F \frac{g_s^2}{4\pi r}$$

The negative sign comes from the fact that the vertex (both in QED and QCD) does not carry the sign of the charge (electric or colour).

Again, the colour factor depends in which colour multiplet is the incoming quark pair. Let's use the notation

$$\begin{array}{llll} \mathrm{q}_1(\mathrm{in}) : 1, p_1, c_1 & & \bar{\mathrm{q}}_2(\mathrm{in}) : 2, p_2, c_2 \\ \mathrm{q}_1(\mathrm{out}) : 3, p_3, c_3 & & \bar{\mathrm{q}}_2(\mathrm{out}) : 4, p_4, c_4 \end{array}$$

then

$$|1,2\rangle = |1\rangle|2\rangle = \sum_{jk} c_{1j} c_{2k}^* |c_j\rangle_1 |\bar{c}_k\rangle_2 = \sum_{jk} \beta_{jk} |c_j\rangle_1 |\bar{c}_k\rangle_2$$

$$|3,4\rangle = |3\rangle|4\rangle = \sum_{jk} c_{3j} c_{4k}^* |c_j\rangle_1 |\bar{c}_k\rangle_2 = \sum_{jk} \beta_{jk} |c_j\rangle_1 |\bar{c}_k\rangle_2$$

Colour Factors

To obtain the colour factor in terms of the total colour operator, we proceed as for the quark-quark case, but care is needed when dealing with the anticolour basis representation of the SU(3) generators.

As before, for the quark triplet **3** we obtain

$$\left\langle 3\left|T_a^{(1)}\right|1\right\rangle = I c_3^\dagger \tfrac{1}{2}\lambda_a c_1$$

But for the antiquark triplet **3*** we must use the generators of the complex conjugate representation

$$\left\langle 4\left|T_a^{(2)}\right|2\right\rangle = \sum_{jk} c_{4j} c_{2k}^* \left\langle \bar{c}_j\left|T_a^{(2)}\right|\bar{c}_k\right\rangle_2 = I\sum_{jk} c_{4j} c_{2k}^* \left(-\tfrac{1}{2}\lambda_a^*\right)_{jk}$$

But $\lambda_a^\dagger = \lambda_a \Rightarrow \lambda_a^* = \lambda_a^\mathsf{T}$

therefore $\left\langle 4\left|T_a^{(2)}\right|2\right\rangle = -I\sum_{jk} c_{4j} c_{2k}^* \left(\tfrac{1}{2}\lambda_a\right)_{kj} = -I c_2^\dagger \tfrac{1}{2}\lambda_a c_4$

so we finally obtain

$$\vec{T}^{(1)}\cdot\vec{T}^{(2)} = \left\langle 3,4\left|\vec{T}^{(1)}\cdot\vec{T}^{2}\right|1,2\right\rangle = -I\left[c_3^\dagger \tfrac{1}{2}\lambda_a c_1\right]\left[c_2^\dagger \tfrac{1}{2}\lambda_a c_4\right]$$

$$= -IC_F = \tfrac{1}{2}\left[T^2 - \left(T^{(1)}\right)^2 - \left(T^{(2)}\right)^2\right]$$

Colour Factors

The quark is in the triplet **3** state, and the antiquark in the triplet **3*** state

$$\left(T^{(1)}\right)^2 = T^2\left(\mathbf{3}\right) = \tfrac{4}{3} I \qquad \left(T^{(2)}\right)^2 = T^2\left(\mathbf{3}^*\right) = \tfrac{4}{3} I$$

The possible multiplets for the q-qbar pair is given by $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}$

In the singlet **1** case, we have $T^2 = 0$: $\quad C_F\left(\mathbf{1}\right) = -\tfrac{1}{2}\left[0 - \tfrac{4}{3} - \tfrac{4}{3}\right] = \tfrac{4}{3}$

In the singlet **8** case, we have $T^2 = 3I$: $\quad C_F\left(\mathbf{8}\right) = -\tfrac{1}{2}\left[3 - \tfrac{4}{3} - \tfrac{4}{3}\right] = -\tfrac{1}{6}$

We conclude that the force is attractive in the colour singlet case, but repulsive for the octet.

Again, this is not a proof, but it shows that the colour potential is favorable for binding a quark and an antiquark in the colour singlet configuration (which corresponds to colour singlet mesons found in nature). It is not favorable to the existence of coloured mesons.

We notice that the strong potential energy operator can be written as

$$\hat{V}_s\left(r\right) = \vec{T}^{(1)} \cdot \vec{T}^{(2)} \frac{\alpha_s}{r}$$

both for the quark-quark and quark-antiquark interactions. This is the analogue of the spin-spin interaction, or the isospin-isospin interaction.