# Introduction to Gauge Theories

- Basics of SU(*n*)
- Classical Fields
- U(1) Gauge Invariance
- **SU(*n*) Gauge Invariance**
- The Standard Model

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# SU(*n*) Gauge Invariance

- **General Formalism**
- **Scalar SU(*n*) Dynamics**
- **SU(*n*) Dynamics**
- **Chromodynamics**
- **Non-Abelian Higgs Model**

## General Formalism

In 1954 Yang and Mills extended the gauge principle to non-abelian symmetry. We will develop the general formalism necessary to build SU(*n*) gauge invariant field theories.

Consider a representation of SU(*n*) of dimension *d*. Consider then a complex scalar *d*-plet

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_d \end{pmatrix}$$

and the corresponding free Lagrangian density

$$\mathscr{L}_0 = \left(\partial_\mu \varphi\right)^\dagger \left(\partial^\mu \varphi\right) - m^2 \varphi^\dagger \varphi$$

which is invariant under Poincaré transformations.

Note that $\mathscr{L}_0$ produces a Klein-Gordon equation for each of the *d* complex components of the *d*-plet $\varphi$. Each *d*-plet component is associated to the same mass.

## General Formalism

This Lagrangian density is also invariant under the global SU($n$) phase transformation

$$\varphi \xrightarrow{\varepsilon^a} \varphi' = U_0 \varphi$$

$$U_0 = \exp\left(-iT^a \varepsilon^a\right)$$

where $U_0$ is an element of SU($n$), and $\varepsilon^a$ are real constants. Sum over repeated group generators indices is assumed. Remember that there are $n^2 - 1$ generators $T^a$ of SU($n$); they are hermitian and traceless matrices, here of dimension $d$, and follow the SU($n$) algebra

$$\left[T^a, T^b\right] = if^{abc} T^c$$

where $f^{abc}$ are the structure constants of SU($n$), totally antisymmetric in all three indices.

Consider the local SU($n$) phase transformation

$$\varphi \xrightarrow{\varepsilon^a(x)} \varphi' = U \varphi$$

$$U = \exp\left(-iT^a \varepsilon^a(x)\right)$$

where the $\varepsilon^a$ are now real functions of $x$. We wish to impose local SU($n$) phase, or SU($n$) gauge, invariance to the theory.

## General Formalism

We seek a differentiation operator $D^\mu$ such that

$$D_\mu \xrightarrow{\varepsilon^a(x)} D'_\mu = U D_\mu U^{-1}$$

which means

$$\left(D_\mu \varphi\right) \xrightarrow{\varepsilon^a(x)} \left(D_\mu \varphi\right)' = D'_\mu \varphi' = \left(U D_\mu U^{-1}\right) U \varphi = U \left(D_\mu \varphi\right)$$

In this case the term $\left(D_\mu \varphi\right)^\dagger \left(D^\mu \varphi\right)$

is invariant under SU($n$) gauge transformations. $D^\mu$ is called the covariant derivative for SU($n$). We try

$$D_\mu = \partial_\mu + igT^a A_\mu^a$$

where $g$ is a real constant, and $A_\mu^a(x)$ are $n^2 - 1$ real gauge fields. The transformation of $A_\mu^a$ under local SU($n$) is defined by

$$A_\mu^a \xrightarrow{\varepsilon^a(x)} A_\mu'^a$$

$$D'_\mu \equiv \partial_\mu + igT^a A_\mu'^a$$

## General Formalism

Therefore
$$D'_\mu \varphi' = U\left(D_\mu \varphi\right)$$

becomes
$$\left(\partial_\mu + igT^a A'^a_\mu\right)U\varphi = U\left(\partial_\mu + igT^a A^a_\mu\right)\varphi$$
$$\left(\partial_\mu U + igT^a A'^a_\mu U\right)\varphi = \left(U\partial_\mu + igUT^a A^a_\mu\right)\varphi$$
$$\left(\partial_\mu U - U\partial_\mu\right)\varphi = -ig\left(A'^a_\mu T^a U - A^a_\mu U T^a\right)\varphi$$

but
$$\left(\partial_\mu U - U\partial_\mu\right)\varphi = \partial_\mu U\varphi - U\partial_\mu\varphi = \left(\partial_\mu U\right)\varphi + U\partial_\mu\varphi - U\partial_\mu\varphi = \left(\partial_\mu U\right)\varphi$$
$$= -iT^a\left(\partial_\mu \varepsilon^a\right)U\varphi$$

so
$$T^a\left(\partial_\mu \varepsilon^a\right)U\varphi = g\left(A'^a_\mu T^a - A^a_\mu U T^a U^{-1}\right)U\varphi$$

We finally obtain
$$\boxed{A'^a_\mu T^a = A^a_\mu U T^a U^{-1} + \frac{1}{g} T^a \partial_\mu \varepsilon^a}$$

Setting $T^a \equiv 1$ we obtain the local U(1) case $A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \varepsilon$

## General Formalism

To obtain the antisymmetric second rank tensor of the gauge field needed to build the gauge field dynamic term, we consider

$$\left[D_\mu, D_\nu\right]\varphi \equiv +igT^a F^a_{\mu\nu}\varphi$$

Therefore $F_{\mu\nu}{}^a(x)$ are antisymmetric tensors by construction. Now

$$\left[D_\mu, D_\nu\right]\varphi = \left[\partial_\mu, \partial_\nu\right]\varphi + ig\left[\partial_\mu, T^a A^a_\nu\right]\varphi - ig\left[\partial_\nu, T^a A^a_\mu\right]\varphi$$
$$-g^2\left[T^a A^a_\mu, T^b A^b_\nu\right]\varphi$$

$$\left[\partial_\mu, \partial_\nu\right]\varphi = 0$$

$$\left[\partial_\mu, T^a A^a_\nu\right]\varphi = T^a\left[\partial_\mu, A^a_\nu\right]\varphi = T^a\left(\partial_\mu A^a_\nu - A^a_\nu \partial_\mu\right)\varphi$$
$$= T^a\left(\partial_\mu A^a_\nu \varphi - A^a_\nu \partial_\mu \varphi\right)$$
$$= T^a\left\{\left(\partial_\mu A^a_\nu\right)\varphi + A^a_\nu \partial_\mu \varphi - A^a_\nu \partial_\mu \varphi\right\} = T^a\left(\partial_\mu A^a_\nu\right)\varphi$$

$$\left[T^a A^a_\mu, T^b A^b_\nu\right]\varphi = A^a_\mu A^b_\nu\left[T^a, T^b\right]\varphi = A^a_\mu A^b_\nu i f^{abc} T^c \varphi = A^b_\mu A^c_\nu i f^{abc} T^a \varphi$$

## General Formalism

Therefore $\left[D_\mu, D_\nu\right]\varphi = \left[igT^a\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) - ig^2 A_\mu^b A_\nu^c f^{abc} T^a\right]\varphi$

$$= igT^a\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - gf^{abc} A_\mu^b A_\nu^c\right)\varphi$$

$$\equiv igT^a F_{\mu\nu}^a \varphi$$

We finally obtain

$$F_{\mu\nu}^a = A_{\mu\nu}^a - gf^{abc} A_\mu^b A_\nu^c$$
$$A_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$$

Setting $T^a \equiv 1$ and $f^{abc} \equiv 0$ we obtain the local U(1) case $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

We notice that $[D_\mu, D_\nu]$ transforms as $D_\mu$ under local SU($n$)

$$\left[D_\mu, D_\nu\right] \xrightarrow{\varepsilon^a(x)} \left(\left[D_\mu, D_\nu\right]\right)' = \left[D'_\mu, D'_\nu\right] = \left[UD_\mu U^{-1}, UD_\nu U^{-1}\right]$$
$$= U\left[D_\mu, D_\nu\right]U^{-1}$$

Therefore

$$\left[D_\mu, D_\nu\right]\varphi \xrightarrow{\varepsilon^a(x)} \left(\left[D_\mu, D_\nu\right]\varphi\right)' = \left(\left[D_\mu, D_\nu\right]\right)'\varphi' = U\left[D_\mu, D_\nu\right]\varphi$$

## General Formalism

The transformation properties of $F_{\mu\nu}^{a}$ under local SU($n$) are defined by

$$F_{\mu\nu}^{a} \xrightarrow{\varepsilon^{a}(x)} F_{\mu\nu}^{\prime a}$$

$$\left(\left[D_{\mu}, D_{\nu}\right]\right)^{\prime} \equiv igT^{a} F_{\mu\nu}^{\prime a}$$

Therefore

$$\left(\left[D_{\mu}, D_{\nu}\right]\right)^{\prime} \varphi^{\prime} = U\left[D_{\mu}, D_{\nu}\right]\varphi$$

becomes

$$igT^{a} F_{\mu\nu}^{\prime a} U\varphi = igUT^{a} F_{\mu\nu}^{a} U^{-1} U\varphi$$

Finally

$$\boxed{F_{\mu\nu}^{\prime a} T^{a} = F_{\mu\nu}^{a} U T^{a} U^{-1}}$$

Setting $T^{a} \equiv 1$ and $f^{abc} \equiv 0$ we obtain the local U(1) case $F_{\mu\nu}^{\prime} = F_{\mu\nu}$

Note that with the case of SU($n$) gauge symmetry, $F_{\mu\nu}^{a}$ does not transform trivially. We therefore need to verify if $F_{\mu\nu}^{a} F^{a\mu\nu}$ is invariant or not.

## General Formalism

Consider the quantity $\left(T^a F^a_{\mu\nu}\right)\left(T^b F^{b\mu\nu}\right)$

Under local SU(*n*) it transforms as

$$\left(T^a F^a_{\mu\nu}\right)\left(T^b F^{b\mu\nu}\right) \xrightarrow{\varepsilon^a(x)} \left(T^a F'^a_{\mu\nu}\right)\left(T^b F'^{b\mu\nu}\right)$$

$$\left(T^a F'^a_{\mu\nu}\right)\left(T^b F'^{b\mu\nu}\right) = \left(UT^aU^{-1}F^a_{\mu\nu}\right)\left(UT^bU^{-1}F^{b\mu\nu}\right)$$
$$= U\left(T^a F^a_{\mu\nu}\right)\left(T^b F^{b\mu\nu}\right)U^{-1}$$

Therefore

$$\mathrm{Tr}\left[\left(T^a F^a_{\mu\nu}\right)\left(T^b F^{b\mu\nu}\right)\right]$$

is invariant under SU(*n*) gauge transformations. We know that

$$\mathrm{Tr}\left[\left(T^a F^a_{\mu\nu}\right)\left(T^b F^{b\mu\nu}\right)\right] = F^a_{\mu\nu}F^{b\mu\nu}\mathrm{Tr}\left(T^aT^b\right) = F^a_{\mu\nu}F^{b\mu\nu}\kappa\delta^{ab} = \kappa F^a_{\mu\nu}F^{a\mu\nu}$$

Therefore

$$F^a_{\mu\nu}F^{a\mu\nu}$$

is invariant under SU(*n*) gauge transformations.

## General Formalism

Using the notation $\varepsilon(x) \equiv \varepsilon^a(x) T^a$

$$A_\mu \equiv A_\mu^a T^a$$

$$F_{\mu\nu} \equiv F_{\mu\nu}^a T^a \qquad F_{\mu\nu} = -F_{\nu\mu}$$

where

$$F_{\mu\nu}^a = A_{\mu\nu}^a - g f^{abc} A_\mu^b A_\nu^c$$

$$A_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$$

we summarize the results as follows:

$$U = \exp(-i\varepsilon(x))$$

$$D_\mu \equiv \partial_\mu + igA_\mu$$

$$\varphi \xrightarrow{\varepsilon(x)} \varphi' = U\varphi$$

$$A_\mu \xrightarrow{\varepsilon(x)} A'_\mu = UA_\mu U^{-1} + \tfrac{1}{g}\partial_\mu \varepsilon(x)$$

$$F_{\mu\nu} \xrightarrow{\varepsilon(x)} F'_{\mu\nu} = UF_{\mu\nu}U^{-1}$$

## General Formalism

We can therefore build a pure gauge field Lagrangian density that is invariant under Poincaré transformations and under SU($n$) gauge transformations $\mathscr{L}_{\mathrm{A}} = -\frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu} = -\frac{1}{4\kappa} \mathrm{Tr}\left[ F_{\mu\nu} F^{\mu\nu} \right]$

After some algebra, the Euler-Lagrange equations yield

$$\partial_\mu F^{a\mu\nu} - g f^{abc} A^b_\mu F^{c\mu\nu} = 0$$

Note that with

$$\tilde{F}^{a\mu\nu} \equiv \tfrac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$$

we also have, using the Jacobi identity,

$$\partial_\mu \tilde{F}^{a\mu\nu} - g f^{abc} A^b_\mu \tilde{F}^{c\mu\nu} = 0$$

Since

$$\left[ A_\mu, F^{\mu\nu} \right] = A^b_\mu F^{c\mu\nu} \left[ T^b, T^c \right] = i f^{abc} T^a A^b_\mu F^{c\mu\nu}$$

We obtain

$$\partial_\mu F^{\mu\nu} + ig\left[ A_\mu, F^{\mu\nu} \right] = 0$$

$$\partial_\mu \tilde{F}^{\mu\nu} + ig\left[ A_\mu, \tilde{F}^{\mu\nu} \right] = 0$$

## General Formalism

We can therefore extend the covariant derivative definition to gauge fields

$$D^\mu \equiv \begin{cases} \partial^\mu + igA^\mu & \text{when acting on a fundamental representation field} \\ \partial^\mu + ig\left[ A^\mu , \quad \right] & \text{when acting on a gauge field} \end{cases}$$

The gauge fields equations of motion then take the compact form

$$D_\mu F^{\mu\nu} = 0 \qquad D_\mu \tilde{F}^{\mu\nu} = 0$$

Because of the non-abelian nature of SU(*n*), the gauge fields interact with themselves:

$$\mathscr{L}_{\mathrm{A}} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$$= -\tfrac{1}{4} A^a_{\mu\nu} A^{a\mu\nu} + \tfrac{1}{2} g f^{abc} A^a_{\mu\nu} A^{b\mu} A^{c\nu} - \tfrac{1}{4} g^2 f^{abc} f^{ars} A^b_\mu A^c_\nu A^{r\mu} A^{s\nu}$$

Notice the cubic and quartic terms in $A_\mu{}^a$, which correspond to self-couplings of non-abelian gauge fields. From

$$\partial_\mu F^{a\mu\nu} \equiv j^{a\nu}_A \qquad \partial_\nu j^{a\nu}_A = 0$$

we obtain the conserved current

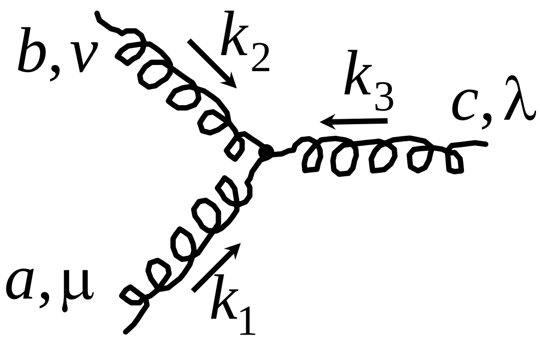
$$j^{a\nu}_A = g f^{abc} A^b_\mu F^{c\mu\nu} = g f^{abc} A^b_\mu \left[ A^{c\mu\nu} - g f^{crs} A^{r\mu} A^{s\nu} \right]$$

$$= g f^{abc} A^b_\mu A^{c\mu\nu} - g^2 f^{abc} f^{rsc} A^b_\mu A^{r\mu} A^{s\nu}$$

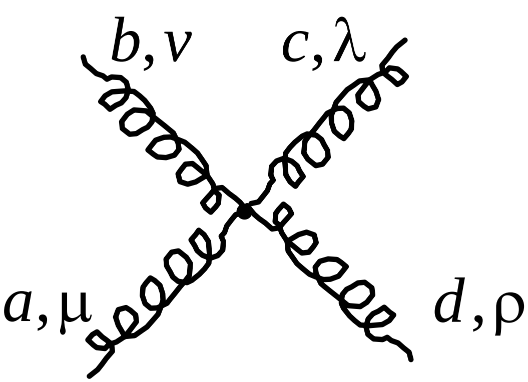
## General Formalism

The corresponding vertices and Feynman rules can be directly obtained from

$$j_A^{a\mu} A_\mu^a = -g f^{abc} A_\mu^a A_\nu^b A^{c\mu\nu} + g^2 f^{abc} f^{rsc} A_\mu^a A_\nu^b A^{r\mu} A^{s\nu}$$




$$-g f^{abc} \Big[ g_{\mu\nu} (k_1 - k_2)_\lambda + g_{\nu\lambda} (k_2 - k_3)_\mu + g_{\lambda\mu} (k_3 - k_1)_\nu \Big]$$




$$-ig^2 \Big[ f^{abe} f^{cde} \left( g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} \right) + f^{ace} f^{bde} \left( g_{\mu\nu} g_{\lambda\rho} - g_{\lambda\nu} g_{\mu\rho} \right) + f^{ade} f^{cbe} \left( g_{\mu\lambda} g_{\rho\nu} - g_{\rho\lambda} g_{\mu\nu} \right) \Big]$$

## Scalar SU($n$) Dynamics

Consider the Lagrangian density for scalar local SU($n$) dynamics

$$\mathcal{L} = \left(D_\mu \varphi\right)^\dagger \left(D^\mu \varphi\right) - m^2 \varphi^\dagger \varphi - \tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

where

$$D_\mu = \partial_\mu + igT^a A^a_\mu$$

$$F^a_{\mu\nu} = A^a_{\mu\nu} - gf^{abc} A^b_\mu A^c_\nu \qquad A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

which is invariant under Poincaré transformations and under the SU($n$) gauge transformations

$$\varphi \xrightarrow{\varepsilon^a(x)} \varphi' = U\varphi$$

$$A^a_\mu T^a \xrightarrow{\varepsilon^a(x)} A'^a_\mu T^a = A^a_\mu U T^a U^{-1} + \tfrac{1}{g} T^a \partial_\mu \varepsilon^a$$

$$F^a_{\mu\nu} T^a \xrightarrow{\varepsilon^a(x)} F'^a_{\mu\nu} T^a = F^a_{\mu\nu} U T^a U^{-1}$$

where

$$U = \exp\left(-iT^a \varepsilon^a(x)\right) \qquad \left[T^a, T^b\right] = if^{abc} T^c$$

Remember that $\varphi$ is a complex scalar $d$-plet.

The $T^a$ are in a dimension $d$ representation of SU($n$).

## Scalar SU($n$) Dynamics

We can write $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_A + \mathscr{L}_{int}$

where $\mathscr{L}_0 = \left(\partial_\mu \varphi\right)^\dagger \left(\partial^\mu \varphi\right) - m^2 \varphi^\dagger \varphi$

The pure gauge field Lagrangian density is as before

$$\mathscr{L}_A = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$
$$= -\tfrac{1}{4} A^a_{\mu\nu} A^{a\mu\nu} + \tfrac{1}{2} g f^{abc} A^a_{\mu\nu} A^{b\mu} A^{c\nu} - \tfrac{1}{4} g^2 f^{abc} f^{ars} A^b_\mu A^c_\nu A^{r\mu} A^{s\nu}$$

Notice the cubic and quartic terms in $A_\mu{}^a$, which correspond to self-couplings of non-abelian gauge fields.

The interaction term between the complex scalar $d$-plet $\varphi$ and the $n^2 - 1$ gauge fields $A_\mu{}^a$ is governed by

$$\mathscr{L}_{int} = -ig\left[\varphi^\dagger T^a \left(\partial^\mu \varphi\right) - \left(\partial^\mu \varphi\right)^\dagger T^a \varphi\right] A^a_\mu + \tfrac{1}{2} g^2 A^a_\mu A^{b\mu} \varphi^\dagger \left[T^a, T^b\right]_+ \varphi$$

This interaction term is a consequence of the SU($n$) gauge invariance.

## Scalar SU($n$) Dynamics

From requiring

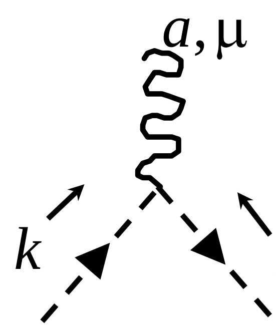
$$\partial_\mu \left( \frac{\partial \mathscr{L}}{\partial \left( \partial_\mu A_\nu^a \right)} \right) - \frac{\partial \mathscr{L}}{\partial A_\nu^a} = 0 \Rightarrow D_\mu F^{\mu\nu} = j^\nu_{\mathrm{SU}(n)}$$

we obtain

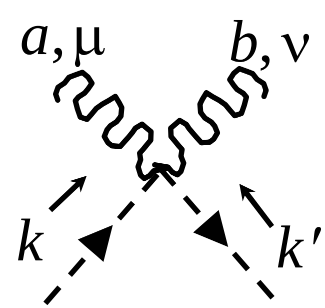
$$j^{a\mu}_{\mathrm{SU}(n)} = -\frac{\partial \mathscr{L}_{\mathrm{int}}}{\partial A^a_\mu} = ig\left[ \varphi^\dagger T^a \left(\partial^\mu \varphi\right) - \left(\partial^\mu \varphi\right)^\dagger T^a \varphi \right] - g^2 A^{b\mu} \varphi^\dagger \left[ T^a, T^b \right]_+ \varphi$$

The corresponding vertices and Feynman rules can be directly obtained from

$$j^{a\mu}_{\mathrm{SU}(n)} A^a_\mu$$




$$-igT^a \left( k_\mu - k'_\mu \right)$$




$$ig^2 g_{\mu\nu} \left[ T^a, T^b \right]_+$$

## SU($n$) Dynamics

The formalism can be easily adapted to fermions.

Consider the Dirac spinor $d$-plet $\psi$

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^d \end{pmatrix} \qquad \left(\psi^j\right)_k \text{ where } k = 1,2,3,4$$

where each $\psi^j$ is a Dirac spinor with 4 spinor components. This means that, for example,

$$\gamma^\mu \psi = \begin{pmatrix} \gamma^\mu \psi^1 \\ \gamma^\mu \psi^2 \\ \vdots \\ \gamma^\mu \psi^d \end{pmatrix}$$

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = \begin{pmatrix} \psi^{1\dagger}\gamma^0 & \psi^{2\dagger}\gamma^0 & \cdots & \psi^{d\dagger}\gamma^0 \end{pmatrix}$$

There are therefore two matrix spaces that don't interfere.

## SU($n$) Dynamics

Under a SU($n$) gauge transformation we have

$$\left(D_\mu \psi\right) \xrightarrow{\varepsilon^a(x)} \left(D_\mu \psi\right)' = U\left(D_\mu \psi\right)$$

Then the terms $\bar{\psi}\psi$ , $\bar{\psi}D_\mu\psi$ are gauge invariant.

Consider the Lagrangian density for local SU($n$) dynamics

$$\mathscr{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \qquad \text{where} \qquad \begin{aligned} D_\mu &= \partial_\mu + igT^a A^a_\mu \\ F^a_{\mu\nu} &= A^a_{\mu\nu} - gf^{abc}A^b_\mu A^c_\nu \\ A^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \end{aligned}$$

which is invariant under Poincaré transformations and under the SU($n$) gauge transformations

$$\psi \xrightarrow{\varepsilon^a(x)} \psi' = U\psi$$

$$A^a_\mu T^a \xrightarrow{\varepsilon^a(x)} A'^a_\mu T^a = A^a_\mu U T^a U^{-1} + \frac{1}{g}T^a \partial_\mu \varepsilon^a$$

$$F^a_{\mu\nu} T^a \xrightarrow{\varepsilon^a(x)} F'^a_{\mu\nu} T^a = F^a_{\mu\nu} U T^a U^{-1}$$

## SU($n$) Dynamics

where $U = \exp\left(-iT^a \varepsilon^a(x)\right)$ and $\left[T^a, T^b\right] = if^{abc}T^c$

Remember that $\psi$ is a Dirac spinor $d$-plet and that the $T^a$ are in a dimension $d$ representation of SU($n$).

We can write $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{A}} + \mathscr{L}_{\text{int}}$

where $\mathscr{L}_0 = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$

yields $d$ Dirac equations for each one of the $\psi$ $d$-plet components. They are all associated to the same mass. The pure gauge field Lagrangian density is, as before

$$\begin{aligned}\mathscr{L}_{\text{A}} &= -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \\ &= -\tfrac{1}{4} A^a_{\mu\nu} A^{a\mu\nu} + \tfrac{1}{2} g f^{abc} A^a_{\mu\nu} A^{b\mu} A^{c\nu} - \tfrac{1}{4} g^2 f^{abc} f^{ars} A^b_\mu A^c_\nu A^{r\mu} A^{s\nu}\end{aligned}$$

Notice the cubic and quartic terms in $A_\mu{}^a$, which correspond to self-couplings of non-abelian gauge fields.

The interaction term between the Dirac spinor $d$-plet $\psi$ and the $n^2 - 1$ gauge fields $A_\mu{}^a$ is governed by

$$\mathscr{L}_{\text{int}} = -g\bar{\psi}\gamma^\mu T^a \psi A^a_\mu$$

This interaction term is a consequence of the SU($n$) gauge invariance.

## SU($n$) Dynamics

From requiring

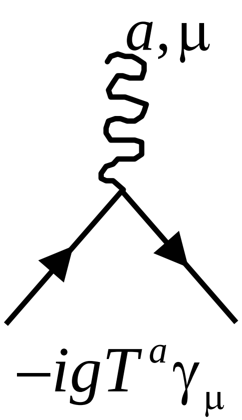
$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu A_\nu^a\right)}\right)-\frac{\partial\mathscr{L}}{\partial A_\nu^a}=0\Rightarrow D_\mu F^{\mu\nu}=j^\nu_{\mathrm{SU}(n)}$$

we obtain

$$j^{a\mu}_{\mathrm{SU}(n)}=-\frac{\partial\mathscr{L}_{\mathrm{int}}}{\partial A_\mu^a}=g\bar{\psi}\gamma^\mu T^a\psi$$

The corresponding vertex and Feynman rule can be directly obtained from

$$j^{a\mu}_{\mathrm{SU}(n)}A_\mu^a$$



## Chromodynamics

Consider the SU(3) dynamics, or chromodynamics. We can then consider the triplet of quarks

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} = \begin{pmatrix} \psi^{\text{red}} \\ \psi^{\text{blue}} \\ \psi^{\text{green}} \end{pmatrix}$$

We therefore use the fundamental representation of SU(3) of dimension $d = n = 3$ with the $n^2 - 1 = 8$ group generators

$$T^a = \tfrac{1}{2}\lambda^a \quad , \ a = 1, 2, \ldots, 8 \qquad\qquad \left[\tfrac{1}{2}\lambda^a, \tfrac{1}{2}\lambda^b\right] = i f^{abc}\, \tfrac{1}{2}\lambda^c$$

where $\lambda^a$ are the Gell-Mann matrices. Remember the SU(3) group structure constants

$$f_{123} = 1 \qquad\qquad f_{458} = f_{678} = \tfrac{\sqrt{3}}{2}$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \tfrac{1}{2}$$

The 8 gauge fields $A_\mu^a(x)$ are the gluon fields.

Since quantum chromodynamics is the theory of the strong interaction, we use $g = g_s$.

## Chromodynamics

The Lagrangian density for chromodynamics is then

$$\mathscr{L} = \bar{\psi}\left(i\gamma^{\mu}D_{\mu} - m\right)\psi - \tfrac{1}{4}F^{a}_{\mu\nu}F^{a\mu\nu} = \mathscr{L}_{0} + \mathscr{L}_{\mathrm{A}} + \mathscr{L}_{\mathrm{int}}$$

where we have

pure quarks

$$\mathscr{L}_{0} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

where colours do not mix

pure gluons

$$\mathscr{L}_{\mathrm{A}} = -\tfrac{1}{4}F^{a}_{\mu\nu}F^{a\mu\nu}$$

$$= -\tfrac{1}{4}A^{a}_{\mu\nu}A^{a\mu\nu} + \tfrac{1}{2}g_{\mathrm{s}}f^{abc}A^{a}_{\mu\nu}A^{b\mu}A^{c\nu} - \tfrac{1}{4}g_{\mathrm{s}}{}^{2}f^{abc}f^{ars}A^{b}_{\mu}A^{c}_{\nu}A^{r\mu}A^{s\nu}$$

where the non-abelian nature of SU(3) leads to self-couplings between gluons.

quark-gluon interaction

$$\mathscr{L}_{\mathrm{int}} = -g_{\mathrm{s}}\bar{\psi}\gamma^{\mu}\tfrac{1}{2}\lambda^{a}\psi A^{a}_{\mu}$$

which is a consequence of gauge invariance.

## Non-Abelian Higgs Model

The Higgs model can be extended to non-abelian theories. For the simplest case, consider the complex scalar doublet

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

and consider the Lagrangian density

$$\mathscr{L} = \left(D_\mu \varphi\right)^\dagger \left(D^\mu \varphi\right) - \tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \mathscr{V}\left(\varphi\right)$$

where

$$D_\mu = \partial_\mu + ig\,\tfrac{1}{2}\sigma^a A^a_\mu$$

$$F^a_{\mu\nu} = A^a_{\mu\nu} - g\varepsilon^{abc} A^b_\mu A^c_\nu$$

$$A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

$$\mathscr{V}\left(\varphi\right) = -\mu^2 \varphi^\dagger \varphi + \lambda\left(\varphi^\dagger \varphi\right)^2 \qquad \lambda > 0$$

The 3 Pauli matrices follow the SU(2) algebra

$$\left[\tfrac{1}{2}\sigma^a, \tfrac{1}{2}\sigma^b\right] = i\varepsilon^{abc}\,\tfrac{1}{2}\sigma^c \qquad\qquad \left[\tfrac{1}{2}\sigma^a, \tfrac{1}{2}\sigma^b\right]_+ = \tfrac{1}{2}\delta^{ab}$$

## Non-Abelian Higgs Model

This Lagrangian density is invariant under Poincaré transformation and under the SU(2) gauge transformations

$$\varphi \xrightarrow{\varepsilon^a(x)} \varphi' = U\varphi$$

$$A_\mu^a \sigma^a \xrightarrow{\varepsilon^a(x)} A_\mu'^a \sigma^a = A_\mu^a U \sigma^a U^{-1} + \frac{1}{g}\sigma^a \partial_\mu \varepsilon^a$$

$$F_{\mu\nu}^a \sigma^a \xrightarrow{\varepsilon^a(x)} F_{\mu\nu}'^a \sigma^a = F_{\mu\nu}^a U \sigma^a U^{-1}$$

where

$$U = \exp\left(-i\,\tfrac{1}{2}\sigma^a \varepsilon^a(x)\right)$$

is a member of the SU(2) group.

To insure Poincaré invariance, $A_\mu^a$ must vanish for the equilibrium state. Therefore, the equilibrium state corresponds to $\mathscr{V}(\varphi)_{min}$.

## Non-Abelian Higgs Model

As for the abelian theory, we have two cases:

a) $\mu^2 < 0$

Then $\mathscr{V}(\varphi)\big|_{\min} = 0 \Rightarrow \varphi^\dagger\varphi = \varphi_1^*\varphi_1 + \varphi_2^*\varphi_2 = 0 \Rightarrow |\varphi_1| = |\varphi_2| = 0$

and no symmetry hiding occurs. The Lagrangian density becomes that of scalar SU(2) dynamics with an extra quartic self interaction term

$$\mathscr{L} = \left(D_\mu\varphi\right)^\dagger\left(D^\mu\varphi\right) - m^2\varphi^\dagger\varphi - \tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \lambda\left(\varphi^\dagger\varphi\right)^2$$

where $m^2 = -\mu^2$ is the mass associated to the complex scalar doublet $\varphi$.

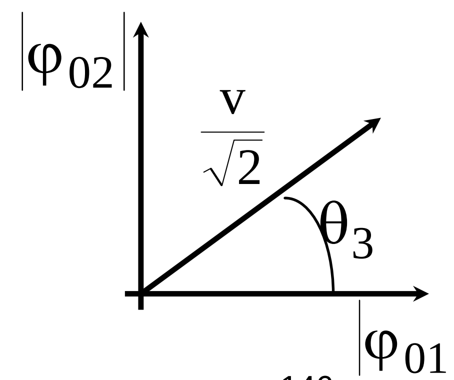
b) $\mu^2 > 0$

Then $\mathscr{V}(\varphi)\big|_{\min} = -\tfrac{1}{4}\mu^2 v^2 \Rightarrow \varphi^\dagger\varphi = \varphi_0^\dagger\varphi_0 = \dfrac{\mu^2}{2\lambda} = \tfrac{1}{2}v^2 > 0$

The equilibrium state is degenerate and can be characterised by

$$\varphi_0 = \begin{pmatrix}\varphi_{01}\\ \varphi_{02}\end{pmatrix} \qquad \varphi_{01} = |\varphi_{01}|e^{i\theta_1} \qquad \varphi_{02} = |\varphi_{02}|e^{i\theta_2}$$

$$\frac{|\varphi_{02}|}{|\varphi_{01}|} = \tan\theta_3 \qquad |\varphi_{01}|^2 + |\varphi_{02}|^2 = \tfrac{1}{2}v^2$$



## Non-Abelian Higgs Model

Nature spontaneously chooses one equilibrium point, say

$$\theta_1 = \theta_2 = 0 \qquad \theta_3 = \tfrac{1}{2}\pi \qquad \varphi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}$$

which is always possible since the theory is also globally SU(2) invariant.
With

$$\varphi_1'(x) = \tfrac{1}{\sqrt{2}}\left[\eta_1(x) + i\eta_2(x)\right]$$
$$\varphi_2'(x) = \tfrac{1}{\sqrt{2}}\left[\sigma(x) + i\eta_3(x)\right]$$

$$\varphi' = \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} \eta_1 + i\eta_2 \\ \sigma + i\eta_3 \end{pmatrix}$$

where $\sigma$ and $\eta_j$ are real functions of $x$, we can write

$$\varphi(x) = \varphi'(x) + \varphi_0 = \begin{pmatrix} \varphi_1'(x) \\ \frac{v}{\sqrt{2}} + \varphi_2'(x) \end{pmatrix} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} \eta_1(x) + i\eta_2(x) \\ v + \sigma(x) + i\eta_3(x) \end{pmatrix}$$

So $\varphi'(x)$, and hence $\sigma(x)$ and $\eta_j(x)$ , measures the deviation of $\varphi(x)$ from equilibrium. With

$$\sigma^a A_\mu^a = \begin{pmatrix} A_\mu^3 & \sqrt{2}A_\mu^- \\ \sqrt{2}A_\mu^+ & -A_\mu^3 \end{pmatrix}$$

$$A_\mu^\pm \equiv \tfrac{1}{\sqrt{2}}\left(A_\mu^1 \pm iA_\mu^2\right) \qquad A_\mu^- = \left(A_\mu^+\right)^*$$

$$\eta^\pm \equiv \tfrac{1}{\sqrt{2}}\left(\eta_1 \pm i\eta_2\right) \qquad \eta^- = \left(\eta^+\right)^*$$

## Non-Abelian Higgs Model

and after some effort we obtain

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \mu^2\sigma^2 + \sum_{j=1}^{3} \tfrac{1}{2}\left(\partial_\mu \eta_j\right)\left(\partial^\mu \eta_j\right)$$

$$-\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \tfrac{1}{2}\left(\tfrac{1}{2} g v\right)^2 A^a_\mu A^{a\mu}$$

$$-\tfrac{1}{2} i g v \left( A^+_\mu \partial^\mu \eta^+ - A^-_\mu \partial^\mu \eta^- - i A^3_\mu \partial^\mu \eta_3 \right)$$

$$+\mathscr{L}'_{\text{int}}$$

where $\mathscr{L}'_{\text{int}}$ contains terms cubic and quartic in the fields $\sigma$, $\eta_j$ and $A_\mu{}^a$. An insignificant constant has been discarded. We can interpret

$$\sigma \to \text{real Klein-Gordon field} \quad \tfrac{1}{2} m^2 = \mu^2$$

but the interpretation

$$\eta_j \to \text{real Klein-Gordon fields} \quad m_\eta = 0$$

$$A^a_\mu \to \text{real Proca fields} \quad M_A = \tfrac{1}{2} g v$$

is not possible because of the quadratic terms

$$-\tfrac{1}{2} i g v \left( A^+_\mu \partial^\mu \eta^+ - A^-_\mu \partial^\mu \eta^- - i A^3_\mu \partial^\mu \eta_3 \right)$$

## Non-Abelian Higgs Model

The $\eta_j$ fields are the would-be-Goldstone boson fields. They are unphysical and can be eliminated through a SU(2) gauge transformation yielding the form

$$\varphi(x) = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \sigma(x) \end{pmatrix}$$

This is called the unitary gauge. In this gauge, the Lagrangian density becomes

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) - \mu^2 \sigma^2 - \tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \tfrac{1}{2}\left(\tfrac{1}{2} g v\right)^2 A^a_\mu A^{a\mu} + \mathscr{L}_{\text{int}}$$

$$\mathscr{L}_{\text{int}} = -\lambda v \sigma^3 - \tfrac{1}{4} \lambda \sigma^4 + \tfrac{1}{8} g^2 A^a_\mu A^{a\mu} \left(2 v \sigma + \sigma^2\right)$$

Since $\mathscr{L}_{\text{int}}$ contains no quadratic terms in the fields we can interpret

$$\sigma \rightarrow \text{real Klein-Gordon field} \qquad \tfrac{1}{2} m^2 = \mu^2$$

$$A^a_\mu \rightarrow \text{real Proca fields} \qquad M_A = \tfrac{1}{2} g v$$

## Non-Abelian Higgs Model

We can verify the number of degrees of freedom

$$
\begin{array}{lll}
 & & \text{n.d.f} \\
\text{Initially:} & \left\{ \begin{array}{ll} \text{complex } \varphi \text{ doublet} & \rightarrow \; 4 \\ \text{real massless } A_\mu^a & \rightarrow \; 6 \end{array} \right\} & \rightarrow \; 10 \\
\text{After :} & \left\{ \begin{array}{ll} \text{real massive } \sigma & \rightarrow \; 1 \\ \text{real massive } A_\mu^a & \rightarrow \; 9 \end{array} \right\} & \rightarrow \; 10
\end{array}
$$

The physical content of the theory is independent of the gauge. Therefore our theory contains 10 physical degrees of freedom.

The massless would-be-Goldstone boson fields $\eta_j$ have disappeared from the theory, and have allowed the gauge fields $A_\mu{}^a$ to acquire mass.

The massive scalar field $\sigma$ is a Higgs boson field.

Note that gauge invariance has also given us the way the Higgs boson field self couples and the way it couples to the massive $A_\mu{}^a$.

In general, after symmetry hiding, the number of massive Higgs bosons, the number of massive gauge fields, and the number of remaining massless gauge fields depend on the pattern of symmetry hiding.