Introduction to Gauge Theories

- Basics of SU($n$)
- Classical Fields
- U(1) Gauge Invariance
- SU($n$) Gauge Invariance
- The Standard Model
SU($n$) Gauge Invariance

- General Formalism
- Scalar SU($n$) Dynamics
- SU($n$) Dynamics
- Chromodynamics
- Non-Abelian Higgs Model
General Formalism

In 1954 Yang and Mills extended the gauge principle to non-abelian symmetry. We will develop the general formalism necessary to build SU($n$) gauge invariant field theories.

Consider a representation of SU($n$) of dimension $d$. Consider then a complex scalar $d$-plet

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_d \end{pmatrix}$$

and the corresponding free Lagrangian density

$$\mathcal{L}_0 = \left( \partial_\mu \varphi \right)^\dagger \left( \partial^\mu \varphi \right) - m^2 \varphi^\dagger \varphi$$

which is invariant under Poincaré transformations.

Note that $\mathcal{L}_0$ produces a Klein-Gordon equation for each of the $d$ complex components of the $d$-plet $\varphi$. Each $d$-plet component is associated to the same mass.
General Formalism

This Lagrangian density is also invariant under the global SU($n$) phase transformation

$$\varphi \xrightarrow{\varepsilon^a} \varphi' = U_0 \varphi$$

$$U_0 = \exp\left(-iT^a \varepsilon^a\right)$$

where $U_0$ is an element of SU($n$), and $\varepsilon^a$ are real constants. Sum over repeated group generators indices is assumed. Remember that there are $n^2 - 1$ generators $T^a$ of SU($n$); they are hermitian and traceless matrices, here of dimension $d$, and follow the SU($n$) algebra

$$\left[T^a, T^b\right] = if^{abc} T^c$$

where $f^{abc}$ are the structure constants of SU($n$), totally antisymmetric in all three indices.

Consider the local SU($n$) phase transformation

$$\varphi \xrightarrow{\varepsilon^a(x)} \varphi' = U \varphi$$

$$U = \exp\left(-iT^a \varepsilon^a(x)\right)$$

where the $\varepsilon^a$ are now real functions of $x$. We wish to impose local SU($n$) phase, or SU($n$) gauge, invariance to the theory.
## General Formalism

We seek a differentiation operator $D^\mu$ such that

$$D^\mu \stackrel{\varepsilon^a(x)}{\longrightarrow} D'^\mu = UD^\mu U^{-1}$$

which means

$$\left(D^\mu \phi\right) \stackrel{\varepsilon^a(x)}{\longrightarrow} \left(D^\mu \phi\right)' = D'^\mu \phi' = \left(UD^\mu U^{-1}\right) U \phi = U \left(D^\mu \phi\right)$$

In this case the term

$$\left(D^\mu \phi\right)^\dagger \left(D^\mu \phi\right)$$

is invariant under SU($n$) gauge transformations. $D^\mu$ is called the covariant derivative for SU($n$). We try

$$D^\mu = \partial^\mu + igT^a A^a_\mu$$

where $g$ is a real constant, and $A^a_\mu(x)$ are $n^2 - 1$ real gauge fields. The transformation of $A^a_\mu$ under local SU($n$) is defined by

$$A^a_\mu \stackrel{\varepsilon^a(x)}{\longrightarrow} A'^a_\mu$$

$$D'^\mu \equiv \partial^\mu + igT^a A'^a_\mu$$
General Formalism

Therefore

\[ D'_\mu \varphi' = U \left( D_\mu \varphi \right) \]

becomes

\[ \left( \partial_\mu + igT^a A'_\mu^a \right) U \varphi = U \left( \partial_\mu + igT^a A_\mu^a \right) \varphi \]

\[ \left( \partial_\mu U + igT^a A'^a_\mu \right) \varphi = \left( U \partial_\mu + igUT^a A_\mu^a \right) \varphi \]

but

\[ \left( \partial_\mu U - U \partial_\mu \right) \varphi = -ig \left( A'_\mu T^a U - A^a_\mu U T^a \right) \varphi \]

\[ \left( \partial_\mu U - U \partial_\mu \right) \varphi = \partial_\mu U \varphi - U \partial_\mu \varphi = \left( \partial_\mu U \right) \varphi + U \partial_\mu \varphi - U \partial_\mu \varphi = \left( \partial_\mu U \right) \varphi \]

\[ = -iT^a \left( \partial_\mu \epsilon^a \right) U \varphi \]

so

\[ T^a \left( \partial_\mu \epsilon^a \right) U \varphi = g \left( A'^a_\mu T^a - A^a_\mu U T^a U^{-1} \right) U \varphi \]

We finally obtain

\[ A'^a_\mu T^a = A^a_\mu U T^a U^{-1} + \frac{1}{g} T^a \partial_\mu \epsilon^a \]

Setting \( T^a \equiv 1 \) we obtain the local \( U(1) \) case

\[ A'_\mu = A_\mu + \frac{1}{g} \partial_\mu \epsilon \]
**General Formalism**

To obtain the antisymmetric second rank tensor of the gauge field needed to build the gauge field dynamic term, we consider

\[
[D_\mu, D_\nu] \varphi \equiv +igT^a F_{\mu\nu}^a \varphi
\]

Therefore \( F_{\mu\nu}^a(x) \) are antisymmetric tensors by construction. Now

\[
[D_\mu, D_\nu] \varphi = \left[ \partial_\mu, \partial_\nu \right] \varphi + ig \left[ \partial_\mu, T^a A^a_\nu \right] \varphi - ig \left[ \partial_\nu, T^a A^a_\mu \right] \varphi
\]

\[
- g^2 \left[ T^a A^a_\mu, T^b A^b_\nu \right] \varphi
\]

\[
\left[ \partial_\mu, \partial_\nu \right] \varphi = 0
\]

\[
\left[ \partial_\mu, T^a A^a_\nu \right] \varphi = T^a \left[ \partial_\mu, A^a_\nu \right] \varphi = T^a \left( \partial_\mu A^a_\nu - A^a_\nu \partial_\mu \right) \varphi
\]

\[
= T^a \left( \partial_\mu A^a_\nu \varphi - A^a_\nu \partial_\mu \varphi \right)
\]

\[
= T^a \left\{ \left( \partial_\mu A^a_\nu \right) \varphi + A^a_\nu \partial_\mu \varphi - A^a_\nu \partial_\mu \varphi \right\} = T^a \left( \partial_\mu A^a_\nu \right) \varphi
\]

\[
\left[ T^a A^a_\mu, T^b A^b_\nu \right] \varphi = A^a_\mu A^b_\nu \left[ T^a, T^b \right] \varphi = A^a_\mu A^b_\nu i f^{abc} T^c \varphi = A^b_\mu A^c_\nu i f^{abc} T^a \varphi
\]
### General Formalism

Therefore
\[
\left[ D_\mu, D_\nu \right] \varphi = \left[ igT^a \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) - ig^2 A^b_\mu A^c_\nu f^{abc} T^a \right] \varphi
\]
\[
= igT^a \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc} A^b_\mu A^c_\nu \right) \varphi
\]
\[
\equiv igT^a F_{\mu\nu}^a \varphi
\]

We finally obtain
\[
F_{\mu\nu}^a = A^a_\mu - gf^{abc} A^b_\mu A^c_\nu
\]
\[
A^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu
\]

Setting \( T^a \equiv 1 \) and \( f^{abc} \equiv 0 \) we obtain the local U(1) case
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]

We notice that \([D_\mu, D_\nu]\) transforms as \( D_\mu \) under local SU\((n)\)
\[
\left[ D_\mu, D_\nu \right] \xrightarrow{\epsilon^a(x)} \left( \left[ D'_\mu, D'_\nu \right] \right) = \left[ U D_\mu U^{-1}, U D_\nu U^{-1} \right]
\]

Therefore
\[
\left[ D_\mu, D_\nu \right] \varphi \xrightarrow{\epsilon^a(x)} \left( \left[ D_\mu, D_\nu \right] \varphi \right)' = \left( \left[ D_\mu, D_\nu \right] \right)' \varphi' = U \left[ D_\mu, D_\nu \right] \varphi
\]
General Formalism

The transformation properties of $F_{\mu\nu}^a$ under local SU($n$) are defined by

$$F_{\mu\nu}^a \xrightarrow{\varepsilon^a(x)} F'_{\mu\nu}^a$$

$$\left(\left[ D_\mu, D_\nu \right]\right)' \equiv igT^a F'_{\mu\nu}^a$$

Therefore

$$\left(\left[ D_\mu, D_\nu \right]\right)' \phi' = U \left[ D_\mu, D_\nu \right] \phi$$

becomes

$$igT^a F'_{\mu\nu}^a U \phi = igUT^a F_{\mu\nu}^a U^{-1} U \phi$$

Finally

$$F'_{\mu\nu}^a T^a = F_{\mu\nu}^a U T^a U^{-1}$$

Setting $T^a \equiv 1$ and $f^{abc} \equiv 0$ we obtain the local U(1) case

$$F'_{\mu\nu} = F_{\mu\nu}$$

Note that with the case of SU($n$) gauge symmetry, $F_{\mu\nu}^a$ does not transform trivially. We therefore need to verify if $F_{\mu\nu}^a F_{a\mu\nu}^a$ is invariant or not.
General Formalism

Consider the quantity \( \left( T^a F^a_{\mu\nu} \right) \left( T^b F^{b\mu\nu} \right) \)

Under local SU\((n)\) it transforms as

\[
\left( T^a F^a_{\mu\nu} \right) \left( T^b F^{b\mu\nu} \right) \rightarrow \varepsilon^a(x) \left( T^a F'^a_{\mu\nu} \right) \left( T^b F'^{b\mu\nu} \right)
\]

\[
\left( T^a F'^a_{\mu\nu} \right) \left( T^b F'^{b\mu\nu} \right) = \left( UT^a U^{-1} F^a_{\mu\nu} \right) \left( UT^b U^{-1} F^{b\mu\nu} \right)
\]

\[
= U \left( T^a F^a_{\mu\nu} \right) \left( T^b F^{b\mu\nu} \right) U^{-1}
\]

Therefore

\[
\text{Tr} \left[ \left( T^a F^a_{\mu\nu} \right) \left( T^b F^{b\mu\nu} \right) \right]
\]

is invariant under SU\((n)\) gauge transformations. We know that

\[
\text{Tr} \left[ \left( T^a F^a_{\mu\nu} \right) \left( T^b F^{b\mu\nu} \right) \right] = F^a_{\mu\nu} F^{b\mu\nu} \text{Tr} \left( T^a T^b \right) = F^a_{\mu\nu} F^{b\mu\nu} \kappa \delta^{ab} = \kappa F^a_{\mu\nu} F^{a\mu\nu}
\]

Therefore

\[
F^a_{\mu\nu} F^{a\mu\nu}
\]

is invariant under SU\((n)\) gauge transformations.
### General Formalism

Using the notation $\varepsilon (x) \equiv \varepsilon^a (x) T^a$

$$A_\mu \equiv A^a_\mu T^a$$

$$F_{\mu \nu} \equiv F^a_{\mu \nu} T^a \quad F_{\mu \nu} = -F_{\nu \mu}$$

where

$$F^a_{\mu \nu} = A^a_{\mu \nu} - gf^{abc} A^b_\mu A^c_\nu$$

$$A^a_{\mu \nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$$

we summarize the results as follows:

$$U = \exp \left( -i \varepsilon (x) \right)$$

$$D_\mu \equiv \partial_\mu + igA_\mu$$

$$\varphi \xrightarrow{\varepsilon (x)} \varphi' = U \varphi$$

$$A_\mu \xrightarrow{\varepsilon (x)} A'_\mu = UA_\mu U^{-1} + \frac{1}{g} \partial_\mu \varepsilon (x)$$

$$F_{\mu \nu} \xrightarrow{\varepsilon (x)} F'_{\mu \nu} = UF_{\mu \nu} U^{-1}$$
**General Formalism**

We can therefore build a pure gauge field Lagrangian density that is invariant under Poincaré transformations and under SU($n$) gauge transformations

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} = -\frac{1}{4 \kappa} \text{Tr} \left[ F_{\mu \nu} F^{\mu \nu} \right]$$

After some algebra, the Euler-Lagrange equations yield

$$\partial_\mu F^{a \mu \nu} - gf^{abc} A_\mu^b F^{c \mu \nu} = 0$$

Note that with

$$\tilde{F}^{a \mu \nu} \equiv \frac{1}{2} \varepsilon^{\mu \nu \rho \sigma} F_{\rho \sigma}^a$$

we also have, using the Jacobi identity,

$$\partial_\mu \tilde{F}^{a \mu \nu} - gf^{abc} A_\mu^b \tilde{F}^{c \mu \nu} = 0$$

Since

$$\left[ A_\mu, F^{\mu \nu} \right] = A_\mu^b F^{c \mu \nu} \left[ T^b, T^c \right] = ig^{abc} T^a A_\mu^b F^{c \mu \nu}$$

We obtain

$$\partial_\mu F^{\mu \nu} + ig \left[ A_\mu, F^{\mu \nu} \right] = 0$$

$$\partial_\mu \tilde{F}^{\mu \nu} + ig \left[ A_\mu, \tilde{F}^{\mu \nu} \right] = 0$$
**General Formalism**

We can therefore extend the covariant derivative definition to gauge fields

\[
D_{\mu} \equiv \begin{cases} 
\partial_{\mu} + igA_{\mu} & \text{when acting on a fundamental representation field} \\
\partial_{\mu} + ig[ A_{\mu}, \ ] & \text{when acting on a gauge field}
\end{cases}
\]

The gauge fields equations of motion then take the compact form

\[
D_{\mu} F^{\mu \nu} = 0 \quad D_{\mu} \tilde{F}^{\mu \nu} = 0
\]

Because of the non-abelian nature of SU(\(n\)), the gauge fields interact with themselves:

\[
\mathcal{L}_{A} = -\frac{1}{4} F_{\mu \nu}^{a} F_{a}^{\mu \nu} = -\frac{1}{4} A_{\mu \nu}^{a} A^{a \mu \nu} + \frac{1}{2} g f^{a b c} A_{\mu \nu}^{a} A_{\mu}^{b} A_{\nu}^{c} - \frac{1}{4} g^{2} f^{a b c} f^{a r s} A_{\mu}^{b} A_{\nu}^{c} A^{r \mu} A^{s \nu}
\]

Notice the cubic and quartic terms in \(A_{\mu}^{a}\), which correspond to self-couplings of non-abelian gauge fields. From

\[
\partial_{\mu} F_{\mu \nu}^{a} \equiv j_{A}^{a \mu \nu} \quad \partial_{\nu} j_{A}^{a \mu \nu} = 0
\]

we obtain the conserved current

\[
j_{A}^{a \mu \nu} = g f^{a b c} A_{\mu}^{b} F_{\mu \nu}^{c} = g f^{a b c} A_{\mu}^{b} \left[ A_{\mu}^{c \nu} - g f^{c r s} A_{r \mu}^{b} A_{s \nu}^{c} \right]
\]

\[
= g f^{a b c} A_{\mu}^{b} A_{\mu}^{c \nu} - g^{2} f^{a b c} f^{r s c} A_{\mu}^{b} A_{r \mu}^{b} A_{s \nu}^{c}
\]
General Formalism

The corresponding vertices and Feynman rules can be directly obtained from

\[ j_A^{a\mu} A_\mu^a = -g f^{abc} A_\mu^a A_\nu^b A^{c\mu\nu} + g^2 f^{abc} f^{rsc} A_\mu^a A_\nu^b A^r_\mu A^s_\nu \]

\[ -g f^{abc} \left[ g_{\mu\nu} (k_1 - k_2)_\lambda \right. \]
\[ + g_{\nu\lambda} (k_2 - k_3)_\mu \]
\[ + g_{\lambda\mu} (k_3 - k_1)_\nu \]
Scalar SU(n) Dynamics

Consider the Lagrangian density for scalar local SU(n) dynamics

\[ \mathcal{L} = \left( D_\mu \phi \right)^\dagger \left( D^\mu \phi \right) - m^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \]

where

\[ D_\mu = \partial_\mu + igT^a A_\mu^a \]

\[ F_{\mu\nu}^a = A_{\mu\nu}^a - gf^{abc} A_\mu^b A_\nu^c \]

\[ A_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a \]

which is invariant under Poincaré transformations and under the SU(n) gauge transformations

\[ \phi \xrightarrow{\epsilon^a(x)} \phi' = U \phi \]

\[ A_\mu^a T^a \xrightarrow{\epsilon^a(x)} A'_{\mu}^a T^a = A_{\mu}^a U T^a U^{-1} + \frac{1}{g} T^a \partial_\mu \epsilon^a \]

\[ F_{\mu\nu}^a T^a \xrightarrow{\epsilon^a(x)} F'_{\mu\nu}^a T^a = F_{\mu\nu}^a U T^a U^{-1} \]

where

\[ U = \exp \left( -iT^a \epsilon^a(x) \right) \quad \left[ T^a , T^b \right] = if^{abc} T^c \]

Remember that \( \phi \) is a complex scalar \( d \)-plet.
The \( T^a \) are in a dimension \( d \) representation of SU(n).
Scalar SU(n) Dynamics

We can write
\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_A + \mathcal{L}_{\text{int}} \]

where
\[ \mathcal{L}_0 = \left( \partial_\mu \varphi \right)^\dagger \left( \partial^\mu \varphi \right) - m^2 \varphi^\dagger \varphi \]

The pure gauge field Lagrangian density is as before
\[ \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \]
\[ = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} g f^{abc} A_{\mu\nu}^a A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{abc} f^{ars} A_{\mu}^a A_{\nu}^b A_{r}^c A_{s}^r \]

Notice the cubic and quartic terms in \( A_{\mu}^a \), which correspond to self-couplings of non-abelian gauge fields.

The interaction term between the complex scalar d-plet \( \varphi \) and the \( n^2 - 1 \) gauge fields \( A_{\mu}^a \) is governed by
\[ \mathcal{L}_{\text{int}} = -ig \left[ \varphi^\dagger T^a \left( \partial_\mu \varphi \right) - \left( \partial^\mu \varphi \right)^\dagger T^a \varphi \right] A_{\mu}^a + \frac{1}{2} g^2 A_{\mu}^a A^{b\mu} \varphi^\dagger \left[ T^a, T^b \right]_+ \varphi \]

This interaction term is a consequence of the SU(n) gauge invariance.
Scalar SU(n) Dynamics

From requiring
\[ \partial_{\mu} \left( \frac{\partial \mathcal{L}}{\partial \left( \partial_{\mu} A^a_{\nu} \right)} \right) - \frac{\partial \mathcal{L}}{\partial A^a_{\nu}} = 0 \Rightarrow D_{\mu} F^{\mu\nu} = j_{\text{SU}(n)}^v \]
we obtain
\[ j_{\text{SU}(n)}^{a\mu} = -\frac{\partial \mathcal{L}_{\text{int}}}{\partial A^a_{\mu}} = ig \left[ \phi^\dagger T^a \left( \partial^\mu \phi \right) - \left( \partial_{\mu} \phi \right)^\dagger T^a \phi \right] - g^2 A^b_{\mu} \phi^\dagger \left[ T^a, T^b \right]_+ \phi \]

The corresponding vertices and Feynman rules can be directly obtained from
\[ j_{\text{SU}(n)}^{a\mu} A^a_{\mu} \]

\[ -ig T^a \left( k_{\mu} - k'_{\mu} \right) \]

\[ ig^2 g_{\mu\nu} \left[ T^a, T^b \right]_+ \]
SU(n) Dynamics

The formalism can be easily adapted to fermions. Consider the Dirac spinor $d$-plet $\psi$

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^d \end{pmatrix} = \left( \psi^j \right)_k \text{ where } k = 1, 2, 3, 4$$

where each $\psi^j$ is a Dirac spinor with 4 spinor components. This means that, for example,

$$\gamma^\mu \psi = \begin{pmatrix} \gamma^\mu \psi^1 \\ \gamma^\mu \psi^2 \\ \vdots \\ \gamma^\mu \psi^d \end{pmatrix}$$

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = \begin{pmatrix} \psi^1 \gamma^0 \\ \psi^2 \gamma^0 \\ \vdots \\ \psi^d \gamma^0 \end{pmatrix}$$

There are therefore two matrix spaces that don't interfere.
SU(n) Dynamics

Under a SU(n) gauge transformation we have

$$(D_{\mu} \psi) \xrightarrow{\epsilon^a(x)} (D_{\mu} \psi)' = U(D_{\mu} \psi)$$

Then the terms $\bar{\psi} \psi$, $\bar{\psi} D_{\mu} \psi$ are gauge invariant.

Consider the Lagrangian density for local SU(n) dynamics

$$\mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_{\mu} - m \right) \psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

where

$$D_{\mu} = \partial_{\mu} + ig T^a A_{\mu}^a$$
$$F_{\mu\nu}^a = A_{\mu\nu}^a - g f_{abc} A_{\mu}^b A_{\nu}^c$$
$$A_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a$$

which is invariant under Poincaré transformations and under the SU(n) gauge transformations

$$\psi \xrightarrow{\epsilon^a(x)} \psi' = U \psi$$
$$A_{\mu}^a T^a \xrightarrow{\epsilon^a(x)} A_{\mu}^a T^a = A_{\mu}^a U T^a U^{-1} + \frac{1}{g} T^a \partial_{\mu} \epsilon^a$$
$$F_{\mu\nu}^a T^a \xrightarrow{\epsilon^a(x)} F_{\mu\nu}^a T^a = F_{\mu\nu}^a U T^a U^{-1}$$
SU(\(n\)) Dynamics

where 

\[
U = \exp\left(-iT^a \varepsilon^a(x)\right) \quad \text{and} \quad \begin{bmatrix} T^a, T^b \end{bmatrix} = if^{abc}T^c
\]

Remember that \(\psi\) is a Dirac spinor \(d\)-plet and that the \(T^a\) are in a dimension \(d\) representation of SU(\(n\)).

We can write 

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_A + \mathcal{L}_{\text{int}}
\]

where 

\[
\mathcal{L}_0 = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi
\]

yields \(d\) Dirac equations for each one of the \(\psi\) \(d\)-plet components. They are all associated to the same mass. The pure gauge field Lagrangian density is, as before

\[
\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}
\]

\[
= -\frac{1}{4} A^a_{\mu\nu} A^{a\mu\nu} + \frac{1}{2} gf^{abc} A^a_{\mu\nu} A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{ars} A^b_{\mu} A^c_{\nu} A^{r\mu} A^{s\nu}
\]

Notice the cubic and quartic terms in \(A^a_{\mu}\), which correspond to self-couplings of non-abelian gauge fields.

The interaction term between the Dirac spinor \(d\)-plet \(\psi\) and the \(n^2 - 1\) gauge fields \(A^{a}_{\mu}\) is governed by

\[
\mathcal{L}_{\text{int}} = -g \bar{\psi}\gamma^\mu T^a \psi A^a_{\mu}
\]

This interaction term is a consequence of the SU(\(n\)) gauge invariance.
SU(n) Dynamics

From requiring

\[ \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu A_v^a \right)} \right) - \frac{\partial \mathcal{L}}{\partial A_v^a} = 0 \implies D_\mu F^{\mu \nu} = j_{SU(n)}^\nu \]

we obtain

\[ j^{a\mu}_{SU(n)} = -\frac{\partial \mathcal{L}_{int}}{\partial A_\mu^a} = g\bar{\psi}\gamma^\mu T^a \psi \]

The corresponding vertex and Feynman rule can be directly obtained from

\[ j^{a\mu}_{SU(n)} A_\mu^a \]

\[ -igT^a \gamma_\mu \]
Chromodynamics

Consider the SU(3) dynamics, or chromodynamics. We can then consider the triplet of quarks

\[ \psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} = \begin{pmatrix} \psi^\text{red} \\ \psi^\text{blue} \\ \psi^\text{green} \end{pmatrix} \]

We therefore use the fundamental representation of SU(3) of dimension \( d = n = 3 \) with the \( n^2 - 1 = 8 \) group generators

\[ T^a = \frac{1}{2} \lambda^a \quad , \quad a = 1, 2, \ldots, 8 \]

where \( \lambda^a \) are the Gell-Mann matrices. Remember the SU(3) group structure constants

\[ f_{123} = 1 \quad , \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2} \]

\[ f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \]

The 8 gauge fields \( A^a_{\mu}(x) \) are the gluon fields.
Since quantum chromodynamics is the theory of the strong interaction, we use \( g = g_s \).
■ Chromodynamics

The Lagrangian density for chromodynamics is then
\[ \mathcal{L} = \bar{\psi} \left( i \gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} = \mathcal{L}_0 + \mathcal{L}_A + \mathcal{L}_{\text{int}} \]
where we have

pure quarks
\[ \mathcal{L}_0 = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi \]
where colours do not mix

pure gluons
\[ \mathcal{L}_A = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} \]
\[ = -\frac{1}{4} A_{\mu \nu}^a A^{a \mu \nu} + \frac{1}{2} g_s f^{abc} A_{\mu \nu}^a A_{\nu}^b A_{\mu}^c - \frac{1}{4} g_s^2 f^{abc} f^{ars} A_{\mu}^a A_{\nu}^b A_{\mu}^c A_{\nu}^s \]
where the non-abelian nature of SU(3) leads to self-couplings between gluons.

quark-gluon interaction
\[ \mathcal{L}_{\text{int}} = -g_s \bar{\psi} \gamma^\mu \frac{1}{2} \lambda^a \psi A_\mu^a \]
which is a consequence of gauge invariance.
Non-Abelian Higgs Model

The Higgs model can be extended to non-abelian theories. For the simplest case, consider the complex scalar doublet

\[ \phi = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) \]

and consider the Lagrangian density

\[ \mathcal{L} = \left( D_\mu \phi \right)^\dagger \left( D^\mu \phi \right) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \mathcal{V}(\phi) \]

where

\[ D_\mu = \partial_\mu + ig \frac{1}{2} \sigma^a A^a_\mu \]

\[ F^a_{\mu\nu} = A^a_{\mu\nu} - g \varepsilon^{abc} A^b_\mu A^c_\nu \]

\[ A^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \]

\[ \mathcal{V}(\phi) = -\mu^2 \phi^\dagger \phi + \lambda \left( \phi^\dagger \phi \right)^2 \quad \lambda > 0 \]

The 3 Pauli matrices follow the SU(2) algebra

\[ \left[ \frac{1}{2} \sigma^a, \frac{1}{2} \sigma^b \right] = i \varepsilon^{abc} \frac{1}{2} \sigma^c \quad \left[ \frac{1}{2} \sigma^a, \frac{1}{2} \sigma^b \right]_+ = \frac{1}{2} \delta^{ab} \]
Non-Abelian Higgs Model

This Lagrangian density is invariant under Poincaré transformation and under the SU(2) gauge transformations

\[ \varphi \xrightarrow{\varepsilon^a(x)} \varphi' = U \varphi \]

\[ A_a^\mu \sigma^a \xrightarrow{\varepsilon^a(x)} A'^a_\mu \sigma^a = A_a^\mu U \sigma^a U^{-1} + \frac{1}{g} \sigma^a \partial_\mu \varepsilon^a \]

\[ F_{\mu \nu}^a \sigma^a \xrightarrow{\varepsilon^a(x)} F'_{\mu \nu}^a \sigma^a = F_{\mu \nu}^a U \sigma^a U^{-1} \]

where

\[ U = \exp \left( -i \frac{1}{2} \sigma^a \varepsilon^a (x) \right) \]

is a member of the SU(2) group.

To insure Poincaré invariance, \( A_\mu^a \) must vanish for the equilibrium state. Therefore, the equilibrium state corresponds to \( \mathcal{V}(\varphi)_{\text{min}} \).
Non-Abelian Higgs Model

As for the abelian theory, we have two cases:

a) $\mu^2 < 0$

Then

$$\mathcal{L} = \left( D_\mu \phi \right)^\dagger \left( D^\mu \phi \right) - m^2 \phi^\dagger \phi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \lambda \left( \phi^\dagger \phi \right)^2$$

and no symmetry hiding occurs. The Lagrangian density becomes that of scalar SU(2) dynamics with an extra quartic self interaction term

where $m^2 = -\mu^2$ is the mass associated to the complex scalar doublet $\phi$.

b) $\mu^2 > 0$

Then

$$\mathcal{L} = -\frac{1}{4} \mu^2 \nu^2 \Rightarrow \phi^\dagger \phi = \phi_0^\dagger \phi_0 = \frac{\mu^2}{2\lambda} = \frac{1}{2} \nu^2 > 0$$

The equilibrium state is degenerate and can be characterised by

$$\begin{pmatrix} \phi_0 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} \phi_{01} \\ \phi_{02} \end{pmatrix}$$

$$\begin{pmatrix} \phi_{02} \\ \phi_{01} \end{pmatrix} = \tan \theta_3$$

$$|\phi_{01}|^2 + |\phi_{02}|^2 = \frac{1}{2} \nu^2$$

$$\nu = \sqrt{2}$$
Non-Abelian Higgs Model

Nature spontaneously chooses one equilibrium point, say
\[ \theta_1 = \theta_2 = 0 \quad \theta_3 = \frac{1}{2} \pi \quad \varphi_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \]

which is always possible since the theory is also globally SU(2) invariant. With
\[
\varphi_1'(x) = \frac{1}{\sqrt{2}} \left[ \eta_1(x) + i \eta_2(x) \right] \\
\varphi_2'(x) = \frac{1}{\sqrt{2}} \left[ \sigma(x) + i \eta_3(x) \right]
\]

where \( \sigma \) and \( \eta_j \) are real functions of \( x \), we can write
\[
\varphi(x) = \varphi'(x) + \varphi_0 = \begin{pmatrix} \varphi_1'(x) \\ \varphi_2'(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_1(x) + i \eta_2(x) \\ v + \sigma(x) + i \eta_3(x) \end{pmatrix}
\]

So \( \varphi'(x) \), and hence \( \sigma(x) \) and \( \eta_j(x) \), measures the deviation of \( \varphi(x) \) from equilibrium. With
\[
\sigma^a A^a_\mu = \begin{pmatrix} A^3_\mu \\ \sqrt{2} A^-_\mu \\ \sqrt{2} A^+_\mu \\ - A^3_\mu \end{pmatrix} \quad A^\pm_\mu \equiv \frac{1}{\sqrt{2}} \left( A^1_\mu \pm i A^2_\mu \right) \quad A^-_\mu = \left( A^+_\mu \right)^* \\
\eta^\pm \equiv \frac{1}{\sqrt{2}} \left( \eta_1 \pm i \eta_2 \right) \quad \eta^- = \left( \eta^+ \right)^*
\]
Non-Abelian Higgs Model

and after some effort we obtain

\[ L = \frac{1}{2} \left( \partial_{\mu} \sigma \right) \left( \partial^{\mu} \sigma \right) - \mu^{2} \sigma^{2} + \sum_{j=1}^{3} \frac{1}{2} \left( \partial_{\mu} \eta_{j} \right) \left( \partial^{\mu} \eta_{j} \right) \]

\[ - \frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu} + \frac{1}{2} \left( \frac{1}{2} g v \right)^{2} A_{\mu}^{a} A^{a \mu} \]

\[ - \frac{1}{2} i g v \left( A_{\mu}^{+} \partial^{\mu} \eta^{-} - A_{\mu}^{-} \partial^{\mu} \eta^{-} - i A_{\mu}^{3} \partial^{\mu} \eta_{3} \right) \]

where \( L'_{\text{int}} \) contains terms cubic and quartic in the fields \( \sigma, \eta_{j} \) and \( A_{\mu}^{a} \). An insignificant constant has been discarded. We can interpret

\( \sigma \rightarrow \text{real Klein-Gordon field} \quad \frac{1}{2} m_{\sigma}^{2} = \mu^{2} \)

but the interpretation

\( \eta_{j} \rightarrow \text{real Klein-Gordon fields} \quad m_{\eta} = 0 \)

\( A_{\mu}^{a} \rightarrow \text{real Proca fields} \quad M_{A} = \frac{1}{2} g v \)

is not possible because of the quadratic terms

\[ - \frac{1}{2} i g v \left( A_{\mu}^{+} \partial^{\mu} \eta^{+} - A_{\mu}^{-} \partial^{\mu} \eta^{-} - i A_{\mu}^{3} \partial^{\mu} \eta_{3} \right) \]
Non-Abelian Higgs Model

The $\eta_j$ fields are the would-be-Goldstone boson fields. They are unphysical and can be eliminated through a SU(2) gauge transformation yielding the form

$$\varphi(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v + \sigma(x) \end{array} \right)$$

This is called the unitary gauge. In this gauge, the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} \left( \partial_{\mu} \sigma \right) \left( \partial_{\mu} \sigma \right) - \mu^2 \sigma^2 - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} \left( \frac{1}{2} g v \right)^2 A_{\mu}^a A^{a\mu} + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = -\lambda v \sigma^3 - \frac{1}{4} \lambda \sigma^4 + \frac{1}{8} g^2 A_{\mu}^a A^{a\mu} \left( 2 v \sigma + \sigma^2 \right)$$

Since $\mathcal{L}_{\text{int}}$ contains no quadratic terms in the fields we can interpret

$$\sigma \rightarrow \text{real Klein-Gordon field} \quad \frac{1}{2} m^2 = \mu^2$$

$$A_{\mu}^a \rightarrow \text{real Proca fields} \quad M_A = \frac{1}{2} g v$$
Non-Abelian Higgs Model

We can verify the number of degrees of freedom

\[
\begin{aligned}
\text{Initially:} & \quad \left\{ \begin{array}{l}
\text{complex } \varphi \text{ doublet } \rightarrow 4 \\
\text{real massless } A^a_\mu \rightarrow 6 
\end{array} \right\} \rightarrow 10 \\
\text{After:} & \quad \left\{ \begin{array}{l}
\text{real massive } \sigma \rightarrow 1 \\
\text{real massive } A^a_\mu \rightarrow 9 
\end{array} \right\} \rightarrow 10
\end{aligned}
\]

The physical content of the theory is independent of the gauge. Therefore our theory contains 10 physical degrees of freedom.

The massless would-be-Goldstone boson fields \( \eta_j \) have disappeared from the theory, and have allowed the gauge fields \( A^a_\mu \) to acquire mass. The massive scalar field \( \sigma \) is a Higgs boson field.

Note that gauge invariance has also given us the way the Higgs boson field self couples and the way it couples to the massive \( A^a_\mu \).

In general, after symmetry hiding, the number of massive Higgs bosons, the number of massive gauge fields, and the number of remaining massless gauge fields depend on the pattern of symmetry hiding.