Introduction to Gauge Theories

- Basics of SU(n)
- Classical Fields
- **U(1) Gauge Invariance**
- SU(n) Gauge Invariance
- The Standard Model

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U(1) Gauge Invariance

- **Scalar Electrodynamics**
- **Electrodynamics**
- **Goldstone Model**
- **Higgs Model**

Scalar Electrodynamics

For the Klein-Gordon field, we have obtained

$$\mathscr{L}_{\text{KG}} = \left(\partial_\mu \varphi\right)^* \left(\partial^\mu \varphi\right) - m^2 \varphi^* \varphi$$

which is invariant under the global phase transformation

$$\varphi \xrightarrow{\varepsilon} \varphi' = e^{-i\varepsilon}\varphi \qquad \text{global}$$

Consider the local U(1) phase, or U(1) gauge, transformation

$$\varphi \xrightarrow{\varepsilon(x)} \varphi' = e^{-i\varepsilon(x)}\varphi \qquad \text{local}$$

where ε is now a real function of *x*. We wish to impose U(1) gauge invariance to the theory. To obtain a gauge invariant Lagrangian density, we need to replace the partial derivative ∂_μ by a covariant derivative D_μ such that

$$D_\mu \varphi \xrightarrow{\varepsilon(x)} D'_\mu \varphi' = e^{-i\varepsilon(x)} D_\mu \varphi$$

In this case the term

$$\left(D_\mu \varphi\right)^* \left(D^\mu \varphi\right)$$

is invariant under a U(1) gauge transformation.

Scalar Electrodynamics

We try the form $D_\mu = \partial_\mu + iqA_\mu$

where q is a real constant, and $A_\mu(x)$ is an unknown field. Imposing

$$D_\mu \varphi \xrightarrow{\varepsilon(x)} D'_\mu \varphi' = e^{-i\varepsilon(x)} D_\mu \varphi$$

yields the transformation properties of the field

$$A^\mu \xrightarrow{\varepsilon(x)} A'^\mu = A^\mu + \tfrac{1}{q}\partial^\mu \varepsilon$$

This is precisely the photon field gauge transformation with

$$f(x) = \tfrac{1}{q}\varepsilon(x)$$

From $$F^{\mu\nu}(x) \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

we see that $$\mathscr{L}_{\mathrm{M}} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

does not change if ∂_μ is replace by D_μ.

Scalar Electrodynamics

Combining $\mathscr{L}_{\text{KG}}$ and $\mathscr{L}_{\text{M}}$ while replacing ∂_μ by D_μ, we obtain the Lagrangian density for scalar electrodynamics

$$\mathscr{L} = \left(D_\mu \varphi\right)^* \left(D^\mu \varphi\right) - m^2 \varphi^* \varphi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

which is invariant under Poincaré transformations and under the gauge transformations

$$\varphi \xrightarrow{\varepsilon(x)} \varphi' = e^{-i\varepsilon(x)} \varphi$$

$$A^\mu \xrightarrow{\varepsilon(x)} A'^\mu = A^\mu + \tfrac{1}{q} \partial^\mu \varepsilon$$

The interaction term between the Klein-Gordon and the Maxwell fields is a consequence of this local gauge invariance and is obtained from

$$\mathscr{L} = \mathscr{L}_{\text{KG}} + \mathscr{L}_{\text{M}} + \mathscr{L}_{\text{int}}$$

giving

$$\mathscr{L}_{\text{int}} = -iq\left[\varphi^* \left(\partial^\mu \varphi\right) - \left(\partial^\mu \varphi\right)^* \varphi\right] A_\mu + q^2 A^\mu A_\mu \varphi^* \varphi$$

Scalar Electrodynamics

Since the Lagrangian density for scalar electrodynamics is scale and form invariant under the global phase transformation

$$\varphi \xrightarrow{\varepsilon} \varphi' = e^{-i\varepsilon}\varphi \qquad \text{global}$$

$$A_\mu \xrightarrow{\varepsilon} A'_\mu = A_\mu$$

Noether's theorem yields the continuity equation for the electromagnetic current

$$\partial_\mu j^\mu_{\text{em}} = 0$$

where $j^\mu{}_{\text{em}}$ is proportional to

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)}\varphi - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi^*)}\varphi^* = -\left[\varphi^*\left(\partial^\mu\varphi\right) - \left(\partial^\mu\varphi\right)^*\varphi\right] - 2iqA^\mu\varphi^*\varphi$$

The electromagnetic current can also be obtained by requiring the Euler-Lagrange equation for A^μ to yield Maxwell's equations with a current

$$\partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathscr{L}}{\partial A_\nu} = 0 \Rightarrow \partial_\mu F^{\mu\nu} = j^\nu_{\text{em}}$$

Scalar Electrodynamics

Since
$$\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu A_\nu\right)} = \frac{\partial \mathscr{L}_{\text{M}}}{\partial\left(\partial_\mu A_\nu\right)} = -F^{\mu\nu} \qquad \frac{\partial \mathscr{L}}{\partial A_\nu} = \frac{\partial \mathscr{L}_{\text{int}}}{\partial A_\nu}$$

we obtain
$$j^\mu_{\text{em}} = -\frac{\partial \mathscr{L}_{\text{int}}}{\partial A_\mu}$$

from which we easily obtain
$$j^\mu_{\text{em}} = iq\left[\varphi^*\left(\partial^\mu\varphi\right) - \left(\partial^\mu\varphi\right)^*\varphi\right] - 2q^2 A^\mu\varphi^*\varphi$$

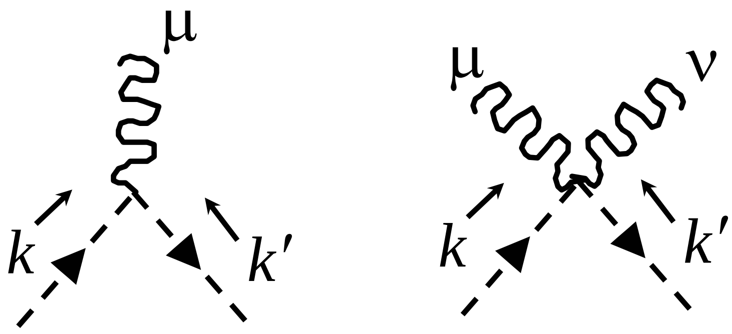
We also have the conserved electric charge
$$Q_{\text{em}} = \int \text{d}V\, j^0_{\text{em}}$$

Note that in the possible gauge $A^0 = 0$, we have
$$Q_{\text{em}} = qQ_{\text{KG}}$$

where Q_{KG} is the Klein-Gordon free field global phase conserved charge. Upon quantization, the Maxwell field will represent photons. The Klein-Gordon field will represent spin 0 particles of charge q and antiparticles of charge $-q$.

Scalar Electrodynamics

The Feynman rules for quantum scalar electrodynamics include the following vertices



$$-iq\left(k_\mu - k'_\mu\right) \qquad\qquad 2iq^2 g_{\mu\nu}$$

which can be obtained visually from

$$j^\mu_{\text{em}} A_\mu = q\left[\varphi^*\left(i\partial^\mu\varphi\right) - \left(i\partial^\mu\varphi^*\right)\varphi\right]A_\mu - 2q^2 g_{\mu\nu} A^\mu A^\nu \varphi^*\varphi$$

Electrodynamics

For the Dirac field, we have obtained

$$\mathscr{L}_{\mathrm{D}} = \bar{\psi}\left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi \quad \text{where} \quad \bar{\psi} \equiv \psi^{\dagger}\gamma^{0}$$

which is invariant under the global phase transformation

$$\psi \xrightarrow{\varepsilon} \psi' = e^{-i\varepsilon}\psi \qquad \text{global}$$

Consider the local U(1) phase, or U(1) gauge, transformation

$$\psi \xrightarrow{\varepsilon(x)} \psi' = e^{-i\varepsilon(x)}\psi \qquad \text{local}$$

where ε is now a real function of *x*. We wish to impose U(1) gauge invariance to the theory. To obtain a gauge invariant Lagrangian density, we need to replace the partial derivative ∂_{μ} by a covariant derivative D_{μ} such that

$$D_{\mu}\psi \xrightarrow{\varepsilon(x)} D'_{\mu}\psi' = e^{-i\varepsilon(x)}D_{\mu}\psi$$

In this case the term

$$\bar{\psi}\gamma^{\mu}D_{\mu}\psi$$

is invariant under a U(1) gauge transformation.

Electrodynamics

We try the form

$$D_\mu = \partial_\mu + iqA_\mu$$

where q is a real constant, and $A_\mu(x)$ is an unknown field. Imposing

$$D_\mu \psi \xrightarrow{\varepsilon(x)} D'_\mu \psi' = e^{-i\varepsilon(x)} D_\mu \psi$$

yields the transformation properties of the field A_μ

$$A^\mu \xrightarrow{\varepsilon(x)} A'^\mu = A^\mu + \tfrac{1}{q} \partial^\mu \varepsilon$$

This is precisely the photon field gauge transformation with

$$f(x) = \tfrac{1}{q} \varepsilon(x)$$

From

$$F^{\mu\nu}(x) \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

we see that

$$\mathscr{L}_\mathrm{M} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

does not change if ∂_μ is replace by D_μ.

Electrodynamics

Combining $\mathscr{L}_{\text{D}}$ and $\mathscr{L}_{\text{M}}$ while replacing ∂_μ by D_μ, we obtain the Lagrangian density for electrodynamics

$$\mathscr{L} = \bar{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

which is invariant under Poincaré transformations and under the gauge transformations

$$\psi \xrightarrow{\varepsilon(x)} \psi' = e^{-i\varepsilon(x)}\psi$$

$$A^\mu \xrightarrow{\varepsilon(x)} A'^\mu = A^\mu + \tfrac{1}{q}\partial^\mu\varepsilon$$

The interaction term between the Dirac and the Maxwell fields is a consequence of this gauge invariance and is obtained from

$$\mathscr{L} = \mathscr{L}_{\text{D}} + \mathscr{L}_{\text{M}} + \mathscr{L}_{\text{int}}$$

giving

$$\mathscr{L}_{\text{int}} = -q\bar{\psi}\gamma^\mu A_\mu\psi$$

Electrodynamics

Since the Lagrangian density for electrodynamics is scale and form invariant under the global phase transformation

$$\psi \xrightarrow{\varepsilon} \psi' = e^{-i\varepsilon}\psi \qquad \text{global}$$

$$A_\mu \xrightarrow{\varepsilon} A'_\mu = A_\mu$$

Noether's theorem yields the continuity equation for the electromagnetic current

$$\partial_\mu j^\mu_{\text{em}} = 0$$

where j^μ_{em} is proportional to

$$\sum_{k=1}^{4}\left[\frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi_k)}\psi_k - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{\psi}_k)}\bar{\psi}_k\right] = \sum_{k=1}^{4}\frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi_k)}\psi_k = i\bar{\psi}\gamma^\mu\psi$$

The electromagnetic current can also be obtained by requiring the Euler-Lagrange equation for A^μ to yield Maxwell's equations with a current

$$\partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathscr{L}}{\partial A_\nu} = 0 \Rightarrow \partial_\mu F^{\mu\nu} = j^\nu_{\text{em}}$$

Electrodynamics

Since
$$\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu A_\nu\right)} = \frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial\left(\partial_\mu A_\nu\right)} = -F^{\mu\nu} \qquad \frac{\partial \mathscr{L}}{\partial A_\nu} = \frac{\partial \mathscr{L}_{\mathrm{int}}}{\partial A_\nu}$$

we obtain
$$j^\mu_{\mathrm{em}} = -\frac{\partial \mathscr{L}_{\mathrm{int}}}{\partial A_\mu}$$

from which we easily obtain
$$j^\mu_{\mathrm{em}} = q\bar{\psi}\gamma^\mu\psi$$

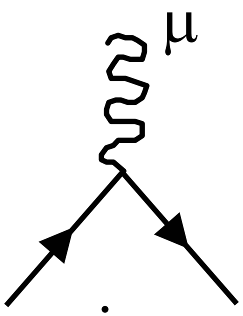
We also have the conserved electric charge
$$Q_{\mathrm{em}} = \int \mathrm{d}V\; j^0_{\mathrm{em}}$$

Note that we have
$$j^\mu_{\mathrm{em}} = q j^\mu_{\mathrm{D}} \qquad Q_{\mathrm{em}} = q Q_{\mathrm{D}}$$

where $j^\mu{}_{\mathrm{D}}$ and Q_{D} are the Dirac free field global phase conserved current and charge. Upon quantization, the Maxwell field will represent photons. The Dirac field will represent spin 1/2 particles of charge *q* and antiparticles of charge -*q*.

Electrodynamics

The Feynman rules for quantum electrodynamics include the following vertex



$$-iq\gamma_\mu$$

which can be obtained visually from

$$j^\mu_{\mathrm{em}} A_\mu = q\bar{\psi}\gamma_\mu \psi A^\mu$$

Note that in the case of electrodynamics, we have the convention

$$q = -e$$

Goldstone Model

We have seen with the Proca equation that adding a mass term ad-hoc spoils gauge invariance. This will turn out to be true in general for all masses in the Standard Model.

Since we wish to consider gauge invariance to generate the interaction between fields, we require a gauge invariant mechanism to generate mass.

This is achieved through hidden symmetry ("spontaneous symmetry breaking").

We will therefore consider models where the equilibrium state is not unique. A choice is made by nature, hiding the invariance of the theory. The equilibrium state is then characterized by all fields being null, except one

$$\varphi(x)\big|_{\text{lowest energy}} \neq 0$$

Since we require the equilibrium state to be invariant under Poincaré transformations, φ must be a scalar field.

The simplest model exhibiting hidden symmetry is the Goldstone model.

Upon quantization, the equilibrium state becomes the vacuum of the theory.

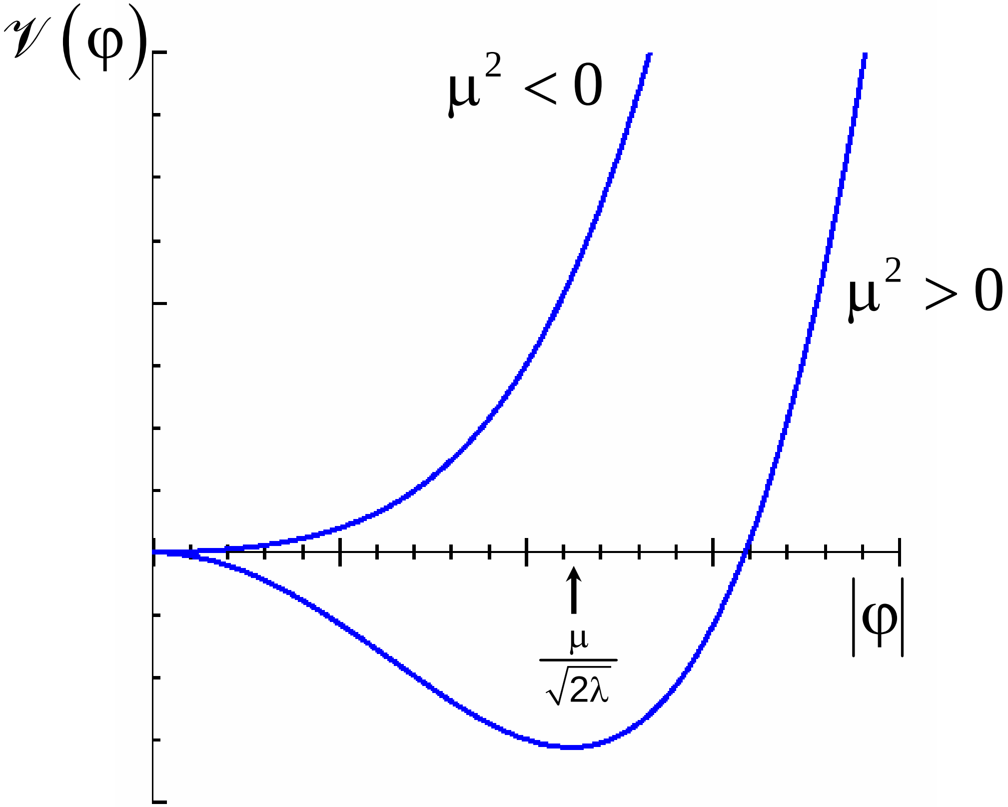
Goldstone Model

Consider the Lagrangian density

$$\mathscr{L} = \left(\partial_\mu \varphi\right)^* \left(\partial^\mu \varphi\right) - \mathscr{V}\left(\varphi\right) \qquad \mathscr{V}\left(\varphi\right) = -\mu^2 \varphi^* \varphi + \lambda \left(\varphi^* \varphi\right)^2 \quad \lambda > 0$$

which is invariant under Poincaré transformations and under the global U(1) phase transformation

$$\varphi \to \varphi' = e^{-i\varepsilon}\varphi$$



Goldstone Model

We note two cases:

a) $\mu^2 < 0$

Then
$$\mathscr{V}(\varphi)\Big|_{\min} = 0 \Rightarrow |\varphi| = 0$$
$$\mathscr{L} = \left(\partial_\mu \varphi\right)^* \left(\partial^\mu \varphi\right) - m^2 \varphi^* \varphi - \mathscr{U}(\varphi)$$
$$\mathscr{U}(\varphi) = \lambda\left(\varphi^* \varphi\right)^2 \qquad \lambda > 0$$

where $m^2 = -\mu^2$ is the mass associated to the complex Klein-Gordon field.

b) $\mu^2 > 0$

Then
$$\mathscr{V}(\varphi)\Big|_{\min} = -\frac{\mu^2 v^2}{4} \Rightarrow |\varphi|^2 = |\varphi_0|^2 = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} > 0$$

and a local maximum at $|\varphi| = 0$.

The equilibrium is then characterised by
$$\varphi_0 = \frac{v}{\sqrt{2}} e^{i\theta}$$

Only one point is required. Nature spontaneously chooses one, say
$$\theta = 0 \rightarrow \varphi_0 = \frac{v}{\sqrt{2}} > 0$$

This is always possible because of global U(1) phase invariance.

Goldstone Model

We can then write: $\varphi(x) = \frac{1}{\sqrt{2}}\left[v + \sigma(x) + i\eta(x) \right]$

where $\sigma(x)$ and $\eta(x)$ measure the deviation of $\varphi(x)$ from equilibrium. A bit of algebra yields

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \mu^2\sigma^2 + \tfrac{1}{2}\left(\partial_\mu \eta\right)\left(\partial^\mu \eta\right) + \mathscr{L}_{\text{int}}$$

$$\mathscr{L}_{\text{int}} = -\lambda v \sigma\left(\sigma^2 + \eta^2\right) - \tfrac{1}{4}\lambda\left(\sigma^2 + \eta^2\right)^2$$

where we have not included a constant term that does not affect the theory. Note that there are no quadratic terms that couple the fields $\sigma(x)$ and $\eta(x)$.

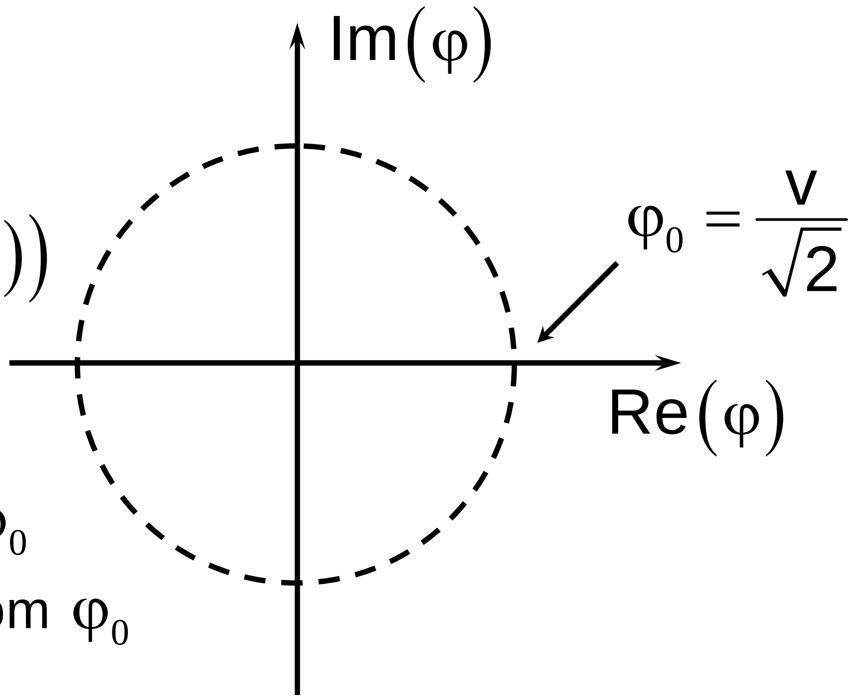
We can then interpret

$$\sigma \to \text{real Klein-Gordon field} \qquad \tfrac{1}{2}m^2 = \mu^2$$

$$\eta \to \text{real Klein-Gordon field} \qquad m_\eta = 0$$

Goldstone Model

$$\varphi(x) = \tfrac{1}{\sqrt{2}}\left(\mathrm{v} + \sigma(x) + i\eta(x)\right)$$



We see that

$\sigma(x) \rightarrow$ radial displacement from φ_0

$\eta(x) \rightarrow$ tangential displacement from φ_0

Therefore $\eta(x)$ represents displacement along the constant

$$\mathscr{V}(\varphi)\big|_{\min} = -\frac{\mu^2 \mathrm{v}^2}{4}$$

U(1) global degeneracy of the equilibrium $\Rightarrow m_\eta = 0$

$\eta(x)$ is the Goldstone boson field.

Goldstone Model

In summary, the Goldstone Model exhibits hidden symmetry in that the equilibrium state is degenerate. One component of the original complex scalar field acquires mass, while the other one (the component along the field equilibrium) is the massless Goldstone boson.

$$\begin{array}{llcl} & & & \text{n.d.f} \\ \text{Initially}: & \text{complex } \varphi & \rightarrow & 2 \\ \text{After}: & \text{real massive } \sigma & \rightarrow & 1 \\ & \text{real massless } \eta & \rightarrow & 1 \quad \leftarrow \text{ Goldstone boson} \end{array}$$

The Goldstone theorem states that the number of massless spin zero Goldstone bosons will be equal to the number of spontaneously hidden symmetry generators. In the example above, we have one corresponding to the one generator of the spontaneously hidden U(1) symmetry.

No truly massless Goldstone bosons are observed in nature. But, in the massless u and d quark limit, the pions $\pi^{\pm}$ and π^0 can be viewed as the Goldstone bosons corresponding to a chiral SU(2) symmetry that is spontaneously hidden into a $SU(2)_V$ (isospin) symmetry.

We need a hidden symmetry mechanism that does not generate Goldstone bosons.

Higgs Model

We can generalize the Goldstone model to be invariant under U(1) gauge transformations by the substitution

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

We obtain the Higgs model Lagrangian density

$$\mathscr{L} = \left(D_\mu \varphi\right)^* \left(D^\mu \varphi\right) - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \mathscr{V}\left(\varphi\right)$$

where

$$F^{\mu\nu}\left(x\right) = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\mathscr{V}\left(\varphi\right) = -\mu^2 \varphi^* \varphi + \lambda \left(\varphi^* \varphi\right)^2 \qquad \lambda > 0$$

This Lagrangian density is invariant under Poincaré transformations and under the U(1) gauge transformations

$$\varphi \xrightarrow{\varepsilon(x)} \varphi' = e^{-i\varepsilon(x)} \varphi$$

$$A^\mu \xrightarrow{\varepsilon(x)} A'^\mu = A^\mu + \tfrac{1}{q} \partial^\mu \varepsilon$$

To ensure Poincaré invariance, A^μ must vanish for the equilibrium state. Therefore the equilibrium state corresponds to $\mathscr{V}(\varphi)|_{min}$.

Higgs Model

As for the Goldstone model, we have two cases:

a) $\mu^2 < 0$

Then $\mathscr{V}(\varphi)\big|_{\text{min}} = 0 \Rightarrow |\varphi| = 0$

and no symmetry hiding occurs. The Lagrangian density becomes that of scalar electrodynamics with an extra quartic self interaction term

$$\mathscr{L} = \left(D_\mu\varphi\right)^*\left(D^\mu\varphi\right) - m^2\varphi^*\varphi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \lambda\left(\varphi^*\varphi\right)^2$$

where $m^2 = -\mu^2$ is the mass associated to the complex Klein-Gordon field.

b) $\mu^2 > 0$

Then $\mathscr{V}(\varphi)\big|_{\text{min}} = -\dfrac{\mu^2 v^2}{4} \Rightarrow |\varphi|^2 = |\varphi_0|^2 = \dfrac{\mu^2}{2\lambda} \equiv \dfrac{v^2}{2} > 0$

The equilibrium state is degenerate and characterized by $\varphi_0 = \dfrac{v}{\sqrt{2}}e^{i\theta}$

Nature spontaneously chooses one equilibrium point, say

$$\theta = 0 \rightarrow \varphi_0 = \frac{v}{\sqrt{2}} > 0$$

which is always possible since the theory is also globally U(1) phase invariant.

Higgs Model

We can then write: $\varphi(x) = \frac{1}{\sqrt{2}}\left[\text{v} + \sigma(x) + i\eta(x)\right]$

where $\sigma(x)$ and $\eta(x)$ measure the deviation of $\varphi(x)$ from equilibrium. A bit of algebra yields

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu\sigma\right)\left(\partial^\mu\sigma\right) - \mu^2\sigma^2 + \tfrac{1}{2}\left(\partial_\mu\eta\right)\left(\partial^\mu\eta\right) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\left(q\text{v}\right)^2 A^\mu A_\mu$$
$$+q\text{v}\left(\partial_\mu\eta\right)A^\mu + \mathscr{L}'_{\text{int}}$$

where $\mathscr{L}'_{\text{int}}$ contains interaction terms cubic or quartic in the fields $\sigma(x)$, $\eta(x)$ and $A^\mu(x)$. An insignificant constant has been discarded. We can then interpret

$\sigma \rightarrow$ real Klein-Gordon field $\quad \tfrac{1}{2}m^2 = \mu^2$

but the interpretation

$\eta \rightarrow$ real Klein-Gordon field $\quad m_\eta = 0$

$A^\mu \rightarrow$ real Proca field $\quad M_A = q\text{v}$

is not possible because of the quadratic term

$$q\text{v}\left(\partial_\mu\eta\right)A^\mu$$

Higgs Model

Also, the number of degrees of freedom would not add up:

n.d.f

$$\text{Initially:} \quad \left.\begin{cases} \text{complex } \varphi & \to \ 2 \\ \text{real massless } A^{\mu} & \to \ 2 \end{cases}\right\} \to \ 4$$

$$\text{After} \quad : \quad \left.\begin{cases} \text{real massive } \ \sigma & \to \ 1 \\ \text{real massless } \eta & \to \ 1 \\ \text{real massive } A^{\mu} & \to \ 3 \end{cases}\right\} \to \ 5$$

We conclude that the Lagrangian density after spontaneous symmetry hiding contains an unphysical field.

The field $\eta(x)$ can be eliminated through a gauge transformation yielding the form

$$\varphi(x) = \tfrac{1}{\sqrt{2}}\left[\text{v} + \sigma(x)\right]$$

This is called the unitary gauge.

The $\eta(x)$ field is called a would-be-Goldstone boson field.

Higgs Model

In this gauge we have

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \mu^2\sigma^2 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\left(q\mathrm{v}\right)^2 A^\mu A_\mu + \mathscr{L}_{\text{int}}$$

$$\mathscr{L}_{\text{int}} = -\lambda \mathrm{v}\sigma^3 - \tfrac{1}{4}\lambda\sigma^4 + \tfrac{1}{2}q^2 A^\mu A_\mu \left(2\mathrm{v}\sigma + \sigma^2\right)$$

Since $\mathscr{L}_{\text{int}}$ contains no quadratic terms in the fields, we can interpret

$$\sigma \to \text{real Klein-Gordon field} \qquad \tfrac{1}{2}m^2 = \mu^2$$

$$A^\mu \to \text{real Proca field} \qquad M_A = q\mathrm{v}$$

and the number of degrees of freedom do add up: n.d.f

$$\text{Initially:} \quad \left.\begin{cases} \text{complex } \varphi & \to \ 2 \\ \text{real massless } A^\mu & \to \ 2 \end{cases}\right\} \to \ 4$$

$$\text{After} \quad : \quad \left.\begin{cases} \text{real massive } \sigma & \to \ 1 \\ \text{real massive } A^\mu & \to \ 3 \end{cases}\right\} \to \ 4$$

Since the initial Lagrangian density is gauge invariant, our theory contains 4 physical degrees of freedom taken by a real massive scalar particle and a real massive vector particle.

Higgs Model

The massless Goldstone boson field $\eta(x)$ has disappeared from the theory, and has allowed the $A^\mu(x)$ field to acquire mass!

vector boson acquires mass without spoiling gauge invariance	$\Rightarrow$	Higgs mechanism

The massive scalar field is a **Higgs boson** field.

Note that gauge invariance has also given us the way the Higgs boson self couples and the way it couples to the massive $A^\mu(x)$.

The Standard Model requires the Higgs mechanism to be applied to a non-abelian gauge invariant theory.